5.8 Transport Equations for Residual Directly Observable, an n-ary Radiant Energy

In this section we shall prepare the way for the extension of the concept of the natural solution of the equation of transfer to the radiant energy field in an optical medium. We shall derive from the time-dependent equations of transfer for the n-ary radiances the corresponding time-dependent transport equations for n-ary radiant energy. We shall eventually find that the latter equations are completely solvable in terms of simple closed algebraic forms in all homogeneous optical media. This fact will allow an important insight into the structure of the associated time-dependent radiance field in the same medium, and thereby shed further light on the difficult optical ringing problem in natural optical media, introduced in Secs. 5.6 and 5.7. We begin with a discuss-ion and solution of the transport equation for zero-order radiant energy (or alternatively, the residual radiant energy) in an optical medium with an arbitrary source. Then the transport equations for nth order radiant energy will be derived along with the transport equations for directly observable radiant energy. Throughout this section the optical medium will be homogeneous with arbitrary sources of radiant flux distributed throughout. The volume scattering function is to be arbitrary but of fixed directional dependence, and unless otherwise specified the scattering-attenuation ratio
p is also arbitrary but fixed, with $4 < p < 1$. Residual Radiant Energy
In order to help fix the main ideas in the present discussion, let the optical medium X under consideration be depicted as in Fig. S.10, that is, as an extensive region X
with a boundary Y on each point y of which is incident a radiance distribution N_o (y, •) which may be extended into X to obtain initial radiance distributions No (x, •) at each point x in X, after the manner of (1) of Sec. 5.1. In the terminology of Sec. 3.10 (see, e.g. , (4) of Sec. 3.10) NQ(x,E) is the transmitted (or residual) radiance at x in the direction ~. The alternative term "residual radiance" will be particularly appropriate in the context of the present discussion, and so is singled out for special use.

Suppose now that sources of radiant flux are present within X. This is a relatively new condition since (except for the brief discussion of example 3 of Sec. 3.9), no systematic explicit use of internal sources was required. We have now arrived at a point in our developments where the advent of the special radiometric concept needed for the description of internal sources takes place naturally. We therefore hypothesize the existence .of an emission radiance function N_n, defined for each time t in some time period and at each point x in X, and direction t in ... The dimensions of N_n are precisely those of N_* (radiance per unit length)

FIG. 5.1© Computing residual radiant energy in medium x.

and the use of N may be best understood by keeping this equality of dimensions in mind. Physically, N_n (x, & , t) is intended to describe the radiance emitted at x and time t per unit length in the direction E. We envision N, (x,E,t) to be generated by some radiant emission mechanism in y. This mechanism generally takes two distinct forms, which may be in operation singly or simultaneously. These forms are described in Sec. 19 of Ref_. (251] and therefore need not be repeated at length here. It suffices for our present purposes to observe that the radiance N_n (x,~ , t) arises generally either through scattering by change in frequency from an arbitrary frequency to the one under consideration, or through the emission processes of conversion of nonradiant energy to radiant energy.

When internal sources, characterized by means of an emission radiance function N_n. are present throughout a medium x, the initial radiance function No is defined throughout as follows. We write

"N°(z,&,t)" for N°(x,&,t-r/v)Tr(x,0

r

N_n (x' ~ ~ , $_t$') $_{Tr- r 1}$ (x' , 0 dr'

0

This definition takes place in the same general geometrical setting of (2) of Sec. 3.10 and reduces to (2) of Sec. 3.10-when X is source-free and the light field is in the steady state. Here as usual z = x + Er , and t'= t-r' /v . A slightly more general definition can be written if X itself has changing inherent optical properties. Also, if scattering with change of frequency is to be explicitly taken into account, we may replace N_n by the true emission function N_e . The details of this more general definition of N° may be found in Sec. 22 of Ref. [M]. Such generality will not be required in any of our discussions, and so in the interests of simplicity of exposition, the present definition will be retained. Immediately forthcoming from (1) is the equation of transfer for initial radiance in the presence of internal sources:

0

1 aN ~ . D ° N

v I t +

This is obtained by taking the lagrangian derivative of the definitional identity which (1) implies. That is, while following in imagination a photon packet along a natural path through X, we differentiate the right side of (1) , by adapting the general procedure used to obtain equation (3) of Sec. 3.15 from equation C'1) of that section. Now, we use D/Dt instead of d/dr, where D/Dt is defined in (5) of Sec. 3.15. Equation (2) is a direct generalization of (2) of Sec. 5.2.

We are now ready to define the notion of residual radiant energy and to establish its various analytical representations. By setting n = 0 in the definitions (16) and (17) of Sec. 5.1 we obtain the definitional identity:

U$^\circ$(Xt t)

X

N°(x,&,t) dil(E)
dV(X) (3)
U°(X,t) is the residual (or reduced or unattenuated) radiant energy in X at time t. When X is understood and fixed throughout a discussion (as in the present one) its name may be dropped from the notation and we will write "UO(t)" for
the residual radiant energy. The term "residual" is particularly well adapted to the photon interpretation of light. For in that interpretation, U°(t) is simply the radiant

energy content of X at time t associated with photons which have not been scattered or absorbed relative to the incident and omission sources of flux on X. Thus the photons making up U^{O}M are those left over and in their original unscattered state after t units of time have elapsed since the external sources over X (represented by N_{o}) and the internal sources over X (represented by N_{n}) have been turned on,

Transport Equation for Residual Radiant Energy

The transport equation for residual radiant energy can be obtained directly from (2) by applying the integral operations occurring in (3) to each side of (2). Thus, integrating (2) term by term, the time derivative term becomes

0
N^{o}(x, t) dn(t) dV(x) au. (t) (4) at

Next, we observe that the spatial derivative term may be written as

since ~ is a variable independent of location on X. Then we observe that the integral
& N^{o} (x,&,t) dS1(C)
M
w
defines the residual radiance counterpart to the vector irradiance function H, as developed in Sec. 2,8. If we write "H^{o} (x, t) " for the preceding integral, we can then on to perform the remaining integration, as required by to obtain

• H^{o}(x,t) dV(x)

which by the divergence theorem may be written as a surface integral of H over the boundary Y of x; thus

o . Ha(X,t) dV (X)
. _ 1
X Y
H^{0} (x, t) • n (x) dA(x)

where n (x) is the unit inward normal to x at each x on Y, and A is the area measure of Y. Suppose we write
rtTro ($_{t}$) it
or

1113^{1}–(Y, t) tr

for

jy

H^{0}(x,t) ' n(x) dV(x) (6)

Thus IT$^{\circ}$(Y,t) is the net inward flux to x across the boundary Y of x. Finally we write topn M to or lipn (X,,t)" for

dv (X)

~ 7)

Thus P_n (X,t) is the input radiant flux over x at time t. Assembling the results summarized in (4)-(7), equation (2) becomes:

where we have written:

"$_T$ ~~ a

far

1 va

Equation (8) is the requisite transport equation for residual radiant energy in medium y at time t.

The Attenuation Time Constant

The quantity T defined in (9) and which has the dimension of time, is the attenuation time constant for X. The significance of T_a will become apparent as the discussions of this section proceed. However, a preliminary insight into its significance can be obtained as follows. Imagine all of E, to be an infinite homogeneous three-dimensional optical medium about the origin 0. Let the initial radiant energy content of E_3 be zero. Let the sources in E_3 be confined to a point source at 0 which is turned on at time t = 0 and which pours radiant flux out into x at a constant rate P_n (i.e., P_n (t) is independent of t, t > 0). At any finite time t > 0 the spherical wave front traveling outward from 0 is of radius vt. For every t > 0, let Y' be any given sphere of radius r(>vt), and let x' be the medium bounded by Y', as in Fig. 5.10.

Under these conditions we have in particular 1^{70}(t) = 0 for every t, 0 < t < r/v, and (8) reduces to

dU°tt Uo t * Pn a

(10)

with initial condition

Uo (0) = 0 (11)

The solution of (10), subject to (11), is

Uo (t3 = Uo (m) (1-e-t/Ta)

(12)

over the time interval (Q, r/v), and where we have written:

for P_nT_a

The significance of T_a now springs into view if we recall a well-known result of elementary circuit analysis concerning the charging of a simple capacitance-resistance

DC circuit such as that depicted in Fig. 5.11. When switch S is closed at time t a 4, battery B of voltage V pumps electrons along the circuit A which has resistance R, until the capacitor of capacitance C (initially discharged) is fully charged. The amount q (t) of charge on the capacitor

FIG. 5.11 The analogy between an electric circuit and an optical medium,

at time t > 4 is given by the equation

q (t) q(°°) (.1-e-t/RQ where we have written:

(14)

With the strong structural resemblance between (12) and (14) in mind, we can make the following pairings between the radiative transfer concepts and the electrical circuit concepts

In the Optical Medium	In the Electrical Circuit
The medium X	The circuit A
The Source Point D	The battery B V/R
n	
U°M	q (t)
1/v	C 1/a
T_a (attenuation time	RC (circuit time constant) constant]

Hence the buildup of residual radiant energy in an extensive homogeneous medium X is analogous to the charging of a capacitor in a simple DC capacitor-resistance circuit. The internal source of radiant flux P_n is analogous to the basic current associated with the battery voltage V and circuit resistance R. The capacitance of the circuit is, for given geometry, dependent on the materials of the plates. Thus the smaller the speed of propagation in the material, the larger the capacitance, and the larger the steady state charge q(00). Analogously, the smaller the speed of propagation .v in the optical medium, all other things being equal, the larger the steady state stored energy Uo (**) . On this basis (which is not, however, logically compelling) we pair 1/v with C. Furthermore, the less dense the conducting material of the circuit, the smaller is the conductance 1/R; similarly, the less dense the material of the optical medium the smaller is a. On this basis we pair 1/a with R. The standard circuit time constant RC then pairs off with T_a. This pairing of time constants is relatively strongly suggested by direct comparison of (12) and (14), whereas the suggested pairings of 1/v with C and 1/a with R are not as strong and, indeed, the pairings may be switched without affecting the important pairing of T_a with RC, the pairing of principal interest at the moment. However, the indicated optical counterparts to R and C are quite interesting to contemplate, particularly when it appears that the analogy between the medium x and the circuit A can be extended quite far by establishing a link with the analogies summarized in the closing paragraph-of Sec. 5.6. Apparently, if these analogies can be extended far enough, then with sufficient care and ingenuity, some of the time-dependent radiative transfer problems can possibly be solved by electrical for even acoustical) analog methods in which the time-dependent electrical (or reverberating acoustical) field replaces the radiant field,

Just as in the electrical case, the attenuation time constant T_a is the time required for the residual radiant energy to attain 63 percent of its steady $_o$ state value, Below is given a table for the values of U (t) /U (=) for various values of t in terms of multiples of T_a

TABLE 1

Values of U^o (t) /U^o (°°) for various values of t in terms of multiples of T_a

t nT_a	U° (t) n^o (°°)
T_a	0.63
ZT	9.86
3T_a	0.95
4T a	0.98
5T_a	0.99

General Representation of Residual Radiant Energy

The solution (1Z) of the residual radiant energy is a solution differential equation special case of the more for general

t

u ^{o}(t) Uo(0)eat/Ttx + e(t'-t)/Ta P (tt) dt' (16)

0

v

where we have written:

"P (t) {t
for

I° (t) +P (t,) (17)

o

n

The solution (15) represents the residual radiant energy in a general homogeneous optical medium X with known combined internal and external source flux function P_o, as given by (17)*

Transport Equation for n-ary Radiant Energy

We derive next the transport equation for the second main radiometric concept of this section, the n-ary radiant energy Um(t). The definition of U^n(t) was given in steady state form in Sec. 5.1. Thus we have for every nonnegative integer n,

U^n (X, t)
1 v

N^n(x,&,t) do(g)

We shall write $1^1$0(t)" for U^n (X, t) whenever X is understood.
Starting with the time-dependent radiance field in X we apply to (5) of Sec, 5.7 the lagrangian derivative operator D/Dt in exactly the way d/dr was applied to (11) of Sec. 5.1 to yield (1) df Sec. 5.2. We have, as a consequence, for every integer n, n > 1:

1 a Nn + . o Nn - aN^n + N n v ~t
(19)

which is te time-dependent equation of transfer for n-ary radiance Nn , , and which is to be compared to (2) above and (1) of Sec. 5.2. Applying the integral operations in (18) to each member of each side of (19), we find that:
We write
and

1 v
inn (x, t) it
or "15'n(Y,t)" for

M w
n

dV (x) $^{3}\underline{1}_{a}$111 Is (2v)

&N^{n}(x.&,t) dsl(E) (21)

H^{n}(x,t) • n(x)dA(x) (22)

Y

wnere n(x) is defined as in (6). Finally we observe that

N* (x, ~ , t) d2 (~)

dv(x) a s Un-lM (23)

With the results (20) through (23) in mind, (19) yields up the following transport equation for n-ary radiant energy:

dU^{n} t ,~ - U^{n} t + U^{n-1} t +

Ton(t)

a s

(24)

for every integer n > 1. The main details of derivation of (24) thus proceed as'in the case of the residual radiant energy (8). Here we have written:

"Ts" for 1/vs (25)

In equation (24), TF^{n}(t) is the net inward radiant, tux across the boundary Y of X at time t. The radiant flux

(t)

has scattering order n relative to that of Vo (t) . A term by term interpretation of (24) is instructive: the time rate of change of n-ary radiant energy in X at time t is the sum of a growth term Un- (t) /T_s (which is the rate of conversion of (n-1) -ary scattered energy into n-ary scattered energy), a decay term U (t) /T_a (which is the rate of conversion of n-ary energy into (n+I)ary energy and nonradiant energy), and finally a general net rate of growth term giving the net balance of influx and efflux o f n -ary radiant energy across the boundary of X. The quantity T_s is the scattering time constant for the medium X. It is a concept which helps write (24) in a uniform manner in terms of the fundamental time-like quantities T_a and T_s e

Transport Equation for Directly Observable Radiant Energy

The radiant energy U associated with directly observable radiance N, using a standard radiance meter is called the directly observable radiant energy. This energy is to be held both in conceptual and empirical contrast t o the n-ary radiant energy Un, n > 1, which is not directly observable in practice. (The residual radiant energy is indirectly observable using techniques alluded to in Sec. 3.10 and Sec. 16 o f Ref. [M].) We now derive the transport equation for U (t) . We begin with the definitional identity:

0

X
N (x,, t) df2(E)
M
dv(x) (26)

based on (2) and (12) of Sec. 2.7. As usual we shall drop reference to X, when X is understood.

Starting with the time-dependent radiance equation (4) of Sec. 3.15, we now apply the integral operations in (26) to each side of the transfer equation and obtain, in a manner analogous to that culminating in (8) and (24) above, the result:

du (t - U t + g(t) + P (t) t a n
(27)

This is the transport equation for directly observable radiant energy. In the equation we have written:

"T a" for
1 va

and where a in turn is the value of the constant volume absorption function in X. Furthermore, we have written:

117(t) it
or "g(Y,t)" for I x(x,t) • n(x)dA(x) (28)
Y

The unit vector n (x) is defined as in Fig. 5.10, and so P7(t) is the net inward radiant flux into X over the boundary Y of X.

The Natural Solution for Directly Observable Radiant Energy

It is a relatively easy matter to verify (using (Sa) of. Sec. 5.7) that

U(X,t) U^1 (X,t) (29) j~0

holds for.every t > 0, where U (x,t) is defined as, in (26) and the UJ (X,t) ari defined as in (18) . Thus, once each UJ (X,t) , j > 0, is known, U(X,t) is known and computable. Equation (23"x) represents the natural solution of the directly observable radiant energy.

In the case of radiant energy the natural solution procedure is not as vitally essential in the solution of U (t) as in the natural solution procedure for the case of radiance in

Secs. 5.6 and 5.7. Indeed, the solution of (27) is written down quite readily, assuming 7(t) and P_n(t).given. Thus, writing,

"P(t)" for g(t) +

SEC. 5.8 TRANSPORT SOLUTIONS 83 we have, analogously to (16)

t

U (t) = U(O)e^- t/Ta + $_e$ (t' -t) IT_a p (t +) dt' (31) .10

The quantity T_a is the absorption time constant for y and is related to T_a and T_s as follows

1 _ 1 + 1 (32) a a s

The natural solution procedure for radiant energy is, however, quite useful in throwing light on the inner workings of time-dependent light fields, for the solutions of the transport equations for Un are readily obtained in simple closed forms which are quite amenable to all manners of explicit, rearrangements and manipulations. Some of the properties of time dependent radiant energy fields will be explored in the next few sections.

We conclude this section with an important observation which will facilitate the studies below. This concerns the connection between the net fluxes T^I(t), n > 0 occurring in

(8) and (24), and the net flux 17(t) occurring in (27). This connection is established by means of the natural solution representation of the directly observable radiant energy U(t) as given in (29). Thus, by summing over all n > 1 in (24)

00 00 00 00

U^n (t) - 1 U^n (t) + 1 Un

$_{Ir}$n $_{(t)}$

$^{-1}$ tt3 +

n=1 a n=l ^{fi}s n=1 n=1

and adding to these terms the corresponding terms of (8). we obtain

00

dU(t

t I I'^{ll} (t) + P~(t) ~ a n=0

comparing this with (27) we conclude that 00

I r^n(t) .(33) n=0