5,10 Properties o f Time-Dependent n-ary Radiant Energy Fields an Related Fields

We now turn to examine in detail some of the more intuitively interesting properties of time-dependent radiant energy fields. In order to present the properties in their simplest forms, we shall adopt for study throughout this section a light field evolving under either standard growth or decay conditions or optical reverberation conditions in an

optical medium X over a time interval (O,tl) (Sec. 5.9). It will be clear from_ the results stated below how analogous or complementary statements and properties can be formulated under still more general conditions. We begin with a study of some of the fine-structure properties of n-ary radiant energy fields and then go on to a formulation of the various representations of related radiant energy quantities.

Some Fine-Structure Properties of n-ary Radiant Energy

Property 1. Let t be a fixed time in (O,t_i). Then the sequence UO(t), $U^1(t)$, $_{*WS}$, $U^n(t)$, ego of n-ary radiant energies at time t is a monotonic decreasing sequence with limit 0. The proof of this property is based on (14) of Sec. 5.9. By (13) of Sec. 5.9 we see that:

$$1 \text{ i } m_n F_n (t /T a) = 1$$

Hence by noting that 0 < p < 1, we see that 1 $im_n U^n$ (,,) a 0

so that

$$1 \text{ } im_n U^n (t) = 0$$

for t in (0 , tl) . As for the monotonicity of the sequence, i suffices to note that:

t

Un+l (t) 1-F_{n+l} (t/T_a)

$u^n(t)$ 1_$F_n(t/T_a)$

and that F (t/Ta,) increases monotonically, with n, to This may A seen by verifying that unity.

$$0 < 1 - F_{n+l} (t/T_a) < 1 - F_n(t/T_o) < 1$$

for every n > 0 and every positive t. The limit part of property 1 follows also from (2) by using the ratio test convergent infinite series.

for

Property 2. Under standard growth conditions,

du^n (t) t ^{n} e-t/Ta > 0

t ~ pn T n. s

for every t in (D, tj) .The proof is immediate. For example, one may use (14) of Sec. 5.9 directly with the calculus, or one may use algebra with the fact that dUn (t) /dt is the

difference given in (24) of Sec, 5.8, with $1^{5'n}$(t) = 0. Property 2 shows in particular that each n-ary radiant energy component increases monotonically with time. Property 2 is to be compared with:

Property 3. Under standard decay conditions

dU^n (t t

$t^n e^{-t/Ta}$
r_n 0
5
for every t in (o, t 1) , The proof is immediately obtainable from (15) of Sec. 5.9. Hence the rates of growth and decay of n-ary radiant energy under standard conditions are, to within a constant multiplicative factor, identical in structure within a given space,

Property 4. Under standard growth conditions,

Un+k t k $U^n(t)$

for every t in (0,t1) and positive integers n, k. This follows from property 2 and (24) of Sec. 5.8 with Pn (t) = 0. The inequality is reversed under standard decay conditions.

Property 5. In the steady state o f the standard growth process,

Un(°°) = Pnuo(Go)

for every n ? 0. Hence

U^{n+k} ~~ k

U^n (,,) P

for every pair n, k of nonnegative integers.

Property 6. In the optical reverberation case (equation (10) o f Sec. 5.9) we have the ratio:

$U^n(t) /tj^{n-1}(t)$ = vts = t/nTs

for n ? I and t in (O, t_1). Thus, the ratio of successive n-ary radiant energy contents increases Zine ar Zy with in-creasing time and decreases hyperbolically with increasing n.

Property 7. In the optical reverberation case with point source (equation (10) of Sec. 5.9) $U^n(t)$, for a given scattering order, attains a maximum when the radius of the wave front is n times the attenuation Length I/a. Further, for any given total volume scattering value s and time t in (0,t1), that component Un(t) is maximal whose order n makes
the absolute value of

1 = (t/nTS) - 1

a minimum. The geometric content of properties 6 and 7 are summarized in part (a) of Figure 5.12.

Property 8. In the optical reverberation case, the directly observable radiant energy U(t) is given by

U (t)
$= U_n$
e-t/Ta

The proof rests on (10) of Sec. 5.9 and (29) of Sec. 5.8 and the simple calculation:

(t
j=0 n e- t/Tot j=0
U $e^{-t/Ta}$ • et/Ts = U e t/Ta n n

in which (32) of Sec. 5.8 was used. It follows immediately from property 8 that, in optical media with no absorption,
i . e ., for which a = D , U (t) is independent of t in the reverberation case. Part (b) of Figure 5.12 gives plots of Un(t) for the first four scattering orders in the optical reverberation case in which a = 4 and U_n = 1. In the figure we have

FIG. 5.12(a) The geometric version of property 7 of scattered radiant energy.

OPTICAL REVERBERATION CASE (10) of sec. 5, 9

0.6

0.5

0.4

If

`r 0.3

C 0.2

0.1

I 2
3 4 5 8 7 8' 9 10 r = t/
TQ

FIG. 5,12(b) The geometric version of property 7 of scattered radiant energy:--Concluded.

written "T" for t/T_s . Thus the medium is a nonabsorbing medium (p = 0) with conserved directly observable energy. Note how the scattering order components of U(t) well up one after another, reaching their maxima, as described by property 7. Finally, according to property 8., the sum of the ordinates of all the curves at each T should add up to unity,

Scattered, Absorbed, and Attenuated Radiant Energies

We now round out the roster o f the types of radiant energy fields most commonly encountered in theoretical discussions of time-dependent light fields. Until further notice, source conditions are arbitrary and with 7(t) = 0.

So far we have introduced the residual radiant energy ((3) of Sec. 5.8), the nary radiant energy ((19) of Sec. 5.8), and the directly observable radiant energy ((26) of

Sec. 5.8) with its natural representation ((29) of Sec. S.9). By writing:

I'U* (t) t' for U^3 (t) (3) ~=1

we define the scattered (or diffuse) radiant energy (in at time t. We then have from (29) of Sec. 5.8 the following radiant energy counterpart to the time-dependent integral. equation of transfer (cf. (4) of sec. 5.4)

U (t) = Uo (t) + U* (t) (4)

Using the emission radiant flux function P_n and recalling that we have set 7(t) = D for t in (0, t_i)., let us write

"U(t *a)" for

P_n (t') dt' - U^o (t)

for t in (o, tl). The , meaning of this new radiant energy becomes clear when it' is recalled that Uo (t) is the residual (i.e., the unattenuated) radiant en-ergy. Therefore, since the integral gives the total radiant energy input to the medium, the difference in (~} must be all the energy present at time t that has 'undergone attenuation (absorption or at least one scattering operation). We call U(t ;a) the attenuated radiant energy (in the medium X) at time t. Only part of U(t;a) is detectable. In fact, the detectable part of U(t;a) is precisely U* (t) . Thus let us write

"U (t; a)" for U(t;a) - U(t;s) (6)

where, for uniformity of notation and heuristic purposes, we have agreed momentarily to write

"U(t;s)" for U* (t) (7)

Then from (b) ire have

U(t;a) = U(t;a)

a formula remarkably similar in structure to the basic *relation:

a = a + s

derived from (4) of Sec. 4.2. We call U(t;a) the absorbed radiant energy (in X) at time t. The absorbed radiant energy is radiant energy that has disappeared from the present radiometric scene via absorption processes.

Representations of U (t ; a) , U(t;s), and U(t;a)

The transport equations for the three auxiliarly radiant energies and their solutions are relatively easy to obtain. We shall illustrate the power of the natural solution procedure by basing the derivations of these equations and representations directly on the knowledge of the n-ary radiant energies developed so far.

We begin with the derivation of the differential equation for attenuated radiant energy U(t ;a) . From the definition (5) we have

dU(t;a) = P (t) - dU°(t) t n C1 G

From (8) of Sec. 5.8 we obtain

dU(t;a) = UO(t) t

recalling that the condition Vn (t) = D is in force for every n > o (hence FO(t) = D, in particular, holds). This elegant formula for the growth rate of U (t ; oc) shows perhaps most clearly the reservoir source of U (t ; a) (namely, U (t)) and the main line which taps the reservoir (namely, T_a, i. e. , attenuation). At standard steady state (9) shows that:

(1D)

Thus in the steady state attained under standard growth conditions the rate of increase of U (t ; a) is precisely the input rate P,,, S'0 that attenuated radiant energy in the medium increases as fast as it is put into the medium by the source.

Next we consider the scattered radiant energy U (t -Is), or "U* (t)" as we would call it ordinarily. The representation (3) of U(t;s) gives rise to the associated differential equation for U(t;s) by computing (with the help of (24) of Sec. 5.8) the following derivative

dU t;s) ~ dU3(t)

t . 7 ~ 1]c t

UJ (t) + Uj (t)

a s

0

1 + 1 JU(t;s) + U (t) 7 77T_ -

Hence

au t;s = _ u t;s a

Here we begin to see some of the utility of the various time constants Tat T , T_a. They serve to remind one of the correct dimensions of each term in an equation or representation, and they serve also to show the physical mechanism associated with that term. Thus we see at a glance from (11) that the rate of growth of U (t ; s) - -the

scattered radiant energy--is augmented by scattering of residual radiant energy Uo(t) and decreased by absorption of scattered radiant energy U(t;s) .
There is no need to solve (11) since we need only sum the, representations of the UJ (t) in (3) to obtain the desired representation of U (t ; s) . Thus J, under standard growth conditions ((14) of Sec. 5.9):
U t; s) = ~ U^k (t) = cc U^k [1- F (t lTa)$]_k$ =1 k1
=1

k (t1Ta)J -t/T e a W0 J
Hence

U(t;s) T U0(°°) 1 (1-e-t/Ta) _ 1^7;

(12)
An alternate representation of U(t;s)is obtained by distributing T_aU (w) throughout the preceding representation. The result is
a Uo 1-e-t/Ta - Uo t
U(t;s) T-- ( ) ( )) a
(13)
From this we obtain immediately the representation for the directly observable radiant energy. For, by (4) and (13), we have

U (t) ^{T}a o „ I-a-t/Ta a
(14)
which is clearly a solution of (27) of Sec. 5.8 under standard growth conditions.
Finally the absorbed radiant energy is represented most simply as

U (t ; a) = P_n t - U (t)
(. 5)
under standard growth conditions. This representation follows from (4) , (5) , and (8) . A representation under more general growth conditions is obtained by retaining the integral in (5) , The differential equation for U(t;a) under standard growth conditions is readily obtained:
dU t;a ^ dU t;a _ dU(t;s tt
U^o t _ _ U t;s) + Uo t to T a ~s
U(t;s) + U° (t) T
a
Hence
dUt;a =Ut (16) t
a
We have made a point of deriving the differential equation for U(t;a) so as to make possible the comparison between it and (9). The comparison lends valuable insight into the general roles of scattering and absorption in radiative transfer phenomena. Thus, in the case of (16), the reservoir source for U(t;a) is

the directly observable radiant energy and the energy is tapped via the process of absorption.