5.14 Operator-Theoretic Basis for the Natural Solution procedure

We close the present chapter with an overview of the theoretical activities of the chapter. As in the earlier general discussions of the canonical equations (Sec. 4.7) the present discussion will perhaps not so much increase our ability to solve specific problems of applied radiative transfer as i t will deepen insight into the essential structure of the natural solution procedure, and therefore radiative transfer theory. In particular the general results below will show how radiative transfer theory, via the integral form of the equation of transfer, is connected to those parts of the main stream of. mathematical physics which share with the present field certain operator equations whose mode of solution coincides,, on the abstract level, with the natural mode of solution studied in this chapter. The discussion is intended to he intuitive, as far as the material will allow.

Let L be a general (not necessarily linear) operator defined on a domain -V of functions such that Lf is in ~3 whenever f is in JD .Thus L naps elements of Z into *E) . Next suppose J9 has a "distance function" d defined, on it such that i f f and g are in f3, then d (f , g) is a nonnegative real number with the Properties:

(i) d(f,g) = 0 if and only if f = g

(ii) d(f,g) = d(g f f)

(iii) d(f,h) L d(f,g) + d(g,h)

The function d i s called a metric for , and as can b e seen, it has the three main properties of ordinary distance relation of everyday life. Ve summarize all this by saying that the pair ($3 ,d) is a metric space.

Now the connection between d) and the radiative transfer setting of this chapter is quite easily made. Let be an optical medium with initial radiance N and let 51 be the operator in (5) of Sec. 5.7. Then write

for N^o + (.) S^1

and we have an example of the operator L above, where SB is now the set of all radiance functions on X. Thus i f N i s a radiance function on X (i.e., N has the dimensions of radiance) then certainly

$N^o + NS^1$

is again radiance function on X. We are not asserting at the moment that N is a solution of the equation of transfer, but merely making an observation that the function displayed above has the dimensions of radiance, and that is all at the moment that is required for admission into JD. Hence L as defined in (1) maps elements of 0 back into AD .

Next we show that there is a very natural counterpart in-radiative transfer theory to the abstract metric d for each fixed time t and bounded optical medium X. Let us write

'[1]d (f , g)'! for

X

lf(xt) - g(x, ~t) ds2(~)

N

It is easy to verify that if f = g, then d(f, g) = 4,

and that i f d (f , g) = 0, then f = g except on sets o f directions & and points x of zero measure. This exception can be smoothed over by advanced technical devices,* and we henceforth can assume condition (i) for a metric to be satisfied. Next one can verify conditions (ii) and (iii) with ease and the verification is left to interested readers. We call the metric function d as defined in (2), the radiometric. By various standard techniques (e.g., averaging) (2) can readily be extended to unbounded media. An alternate choice of metric can also be made by writing

"d(f,g)" for sup if(xo~tt) - g(x,&,t) 1 (2a) x,~

where

Ifsup h (x, x, E

*In particular, this can be done by means of equivalence classes of functions, an equivalence class being the set of all radiance functions on a domain Y which differ from one another at most on subsets of Y of zero measure. Then we go on to work with equivalence sets of functions rather than individual functions. However, for the present we work directly with the radiance functions, with no essential loss of rigor.

means the supremum (the maximum) of the values of h (x, &) as X,& vary over all permissible values in the domain of h. The function d in (2a) also satisfies all the properties (i) to (iii) of a metric. We shall call d in (2a) the supremum metric,

We summarize what has been done so far: The operator (1) associated with the integral equation of transfer of classical radiative transfer theory may be viewed as a special case of an abstract operator L on a metric space (.9 Al d] , the particular classical form of the operator being given in (1). with Z being the class of all radiance functions on X, and with d the radiometric as defined in (2) or the supremum metric as given in (2a). In what follows we allow.6 to contain negative valued radiance functions as well as nonnegative valued radiance functions. Of course in physically meaningful applications we shall always work with the latter; however, for mathematical purposes it is convenient also to have the former.

We now come to a key property of the radiative transfer operator S which can be abstracted from the setting of the present chapter and carried out far into the reaches of abstract operator theory, where its general utility can be more easily discerned. In Sec. 5.7 we showed that if IT is an upper bound for supremum) of a radiance function, then (cf. (7) of Sec: 5.7)

(NS ^{1}) (x,&,t) < TTp(1-e-t/Ta)

for every x in X, & in and t in (0,t) , where "T_a" stands for 1/va. From this we are led to deduce that for every pair f , g of radiance functions, and with the supremum metric (2a) o

d(Lf,Lg) < c d(f,g) (3)

where c is a number which depends only on t, p and Ta, i.e., where we have written:

for p (I-e- t/Ta) .

In all normal optical media (i.e., for which 0 < p < 1], we have 0 < c < 1 whenever t > 0. The proof of (3) is immediate, using the definitions (1) and (2a). Whenever an operator L on a general metric space (9 ,d) has property (3) , we say that L is a contraction mapping or that it has the contraction property. Hence our particular classical radiative transfer operator L given in (1) is a contraction mapping, relative to (2a). The reader may show that (3) also holds under suitable conditions, relative to (2) .

To summarize our findings so far: The operator L associated with the time-dependent integral equation of transfer may be viewed as a special case of a contraction mapping L on a metric space (.fJ,d) .

We now have developed enough abstract machinery to illustrate the essential activity of the natural solution procedure, on a very general level--a level which is in contact

with the general representations of widely different natural phenomena in modern physics. Let us choose any function f I in 0 and write

"f (1) i, for Lf(°)

Thus we operate on f (o) in b with L to obtain f (1) in ~S. W repeat this operation a finite number of times to obtain f(nl where we have written

"f (n)" for Lf (n-1)

In this way we obtain a sequence

f(°) , f(1) , ... , f(n), ...

of functions in SD. As in the case of Sec. 5.1, we can define iterates Ln of L so that (cf., e.g., (11) of Sec. 5.1):

f

Of (o)

Before going on, I i i reader could verify that if we 1and NO for f then f Y is simply

use L in

j

i.e. , the sum. of the n- ary race an up to order n. Since L is a contraction mapping, we have, for

d (f (n) f f (m)) = d (Lnf (o) Lmf (o))

< cn d(f~ (n) 2 LM_ 11 f (a))

+ d(f(m_n-1) ff(m-n'J

c

m-n-1

< cn (Cd(f(°) f(1)] + d(f(1),f (2 +. . .

< cn d(f (o) , f (1)) 1 + c + cZ + MOO + cn u(f(o) f (1)~

Since c is less than 1, cn is arbitrarily, small for sufficiently large n. Thus the sequence

f(O), f(I)) Jn)

constructed above is a Cauchy sequence (in the sense of modern calculus) .

By establishing this feature of the sequence we have reached the penultimate step in our general discussion of. the natural solution procedure.

The significance of the Cauchy sequence feature of fm f(n) . s e e)

is this: In all physically meaningful settings for the metric space d) , it is possible to arrange matters so that, whenever a sequence

o f elements in 5D is a Cauchy sequence in the sense of (4), then that sequence has a limit in 'b . In general, whenever a metric space (O , d) has this property, we say that (O, d) is complete. It is easy to show that all physically meaningful radiative transfer settings always can be represented by complete metric spaces (f), d] . Let us assume therefore for the remainder of the discussion that (JD , d) is complete.

Taking up the thread of the argument at (4) we now can assert the existence of a limit function f to the sequence constructed above. Thus let us write:

i t fit

for lim f(n) n

We now show that f has two very important properties

(i) f satisfies the operator equation f = Lf

(ii) f is the only function in ~D for which (i) holds, i.e., if g = Lg and f = Lf, then f = g.

Property$_{(n}$(il$_{)}$ follows readily by noting that, by definition, f Lf Hence applying the limit operation to each side of this identity, the result follows by observing that L is a continuous mapping* (so that the limit opration can - be pushed past L and made to act directly on f n-l] . Property (ii) follows from (i) and the contraction property of L:

d(f,g) = d(Lf,Lg) < c d(f,g)

From this (since c < 1) we must have d(f,g) = 4, so that f = g.

Let us now make the final summary of what has been done so far in this section: The natural mode of solution in radiative transfer theory has been found to take its place as a special case of a very general operator technique in modern functional analysis. This technique is based on the following theorem (cf., e.g., (140]):

Theorem (Principle of Contraction Mappings). Every contraction mapping L on a complete metric space (D, d) generates one and only one solution o f the equation f.= L f.

*A point which is readily established in functional analysis texts (cf., e.g., [140)),

 The classical radiative transfer setting entities are paired off with the abstract setting entities of the preceding theorem as follows

In Radiative Transfer Theory In the Theorem

- a) Set .9 of all radiance functions on an optical medium X

b) The radiometric d, as in (2) or d . (2a)

c) The operator L, as in (1) L

We will make one final remark on the existence of the solution f of the general operator equation f = Lf. This is the observation that the solution f defined in (5) is independent of the initial function f (O) starting the chain of iterations Lnf (off. This fact becomes clear, at least logically by noting the, uniqueness property' (i i) above. For if f oand g are two distinct initial functions, then construction of their iteration sequences yields f and g such that property (i) holds for each.