6.2 Abstract Spherical Harmonic Method

The motivation and prerequisites of the abstract spherical harmonic method having been dispatched in Sec. 6.1, we turn directly to the method itself, now applied to the general time-dependent equation of transfer with source term ((14) of Sec. 3.15) :

1 .~--- + • ON = aN + N * + N 13

v t n

where. N is defined on a general optical medium y which may be finite or infinite, generally inhomogeneous, but isotropic. We assume furthermore that there exists an orthonormal family {00$_9$ ~i , ~2, .6's.} of functions on E which has the completeness property,

The completeness property of {~a , 01P 02P .. . } applied to the radiance distribution N(x, •) at x in x yields

N(x,&,t) f - (xvt)

7 J

where we have written

'If J (x, t) " for [N* (x , % , t) , ~~] (3) Thus f~ (x, t) is the scalar obtained by performing the integration:

N(x,E,t) T.-(t) dn(E) .

M V

In a similar manner we obtain:

N_n (x 1 & 0 t) a , I ^{f}n ~ 7 (x ' t) ~ $_7$ M (4) ~=0

as the representation of the emission function

have written

N_n ,

where we

1 1 f n . J (x , t) " for [N_n (x, • , t) 1 9 ~ $_i$ I (s)

The representation of the volume scattering function a is next. Since o uses two directional variables,, we use the completeness property twice. First we obtain:

v(x;E' ;E;t) '~ Q J

~- (x;C ;t) M (6)

j=0

where we have written:

+Q_j ($_{x;E}$ 1 ; t) ' 1 for [a(x;&' ; • ;t) , ~~ (7) Next we obtain

Q_j (x;E' ;t) _ v~$_k$(x,t) T O " (8) k=0

where we have written:

Ilajk(x;t)" for $^{[o}$ j (x; • ;t) , ik] (9) Combining these representations, we have:

C D M

o(x;&' ;& it) " 1 1 a_{jk}(x;t) Tk(" ~ i M j=0 k=0

(10)

The reason for introducing the conjugates of the ~k into (10) will become clear shortly.

Now the whole purpose of the spherical harmonic method, as we have seen in Sec. 6.1, is to effectively separate the spatial variables from the directional variables in the equation of transfer so that the latter may be contained in a system of simple, directly integrable differential equations involving spatial variables only. We now apply the abstract harmonic representations of N, N_n, and a to the equation of transfer (1), and effect such a separation of variables. On

the right side of (1) we have N_n already represented. Then for the term N* (the summations all go from 0 to W):

N_* (x,g,t) £^{f}i(^{x,t}) f i (&') ~j a j k (x ; t) T k ("f j (C) d i d [E '

1 ~

M

zfj(xtt)I (~1 j a j k (X ' t) -fk(C do (C ') J

J

$_i$ (X,t) I ~J^{cr}j k ($^{X; t}$) ~j

If i(x,t)~~Q3k(x;t)~j (~) aik I

1

^{f}i (x, t) i zCF_{ji}(x - 1 t) y &)

~fi(x·t)a~$_i$(X;t)I ~~M

(11)

Since the medium X is assumed isotropic, the volume attenuation - function values a(x;&) are independent of E, and so a need not be represented by a series of the complete family {00$_t$ #1$_1$ ~z, *00)0 By means of (4), (10). and (11) we can therefore represent the right side of (1) in the form:

0 0

- a (x) f (x, t) + I f i (x: t) aj i (x; t) + $f_{n, j}$(x, t) ~$_j$ (E) jU0

i=o

(12)

Attention is now directed to the left side of (1) . The time derivative term is directly treated to yield:

1 af. (x, t)

jio

The spatial derivative term becomes

• QN (x, , t) ' 0 i I f $_j$ (x I t) f_j (E) i .v

• of . (x,t) - M (14)

3

Combining (12), (13), and (14) according to (1) , we have:

1 af.(x,t)

.~ v t + • Of_j (x,t) + a(x) f_j (x,t) - .~ fi(x,t)vji(x;t)

15)

If it weren't for the spatial derivative term the contents of the square bracket would have been free of the variable &, and a system of equations would have been obtained by setting each bracketed j th term to zero. At any rate we can eliminate the presence of ~ by an integration over 'E . The orthonormality property of {~o , 01, 02, . . . ? is available for use in this task. Thus multiplying each side of (15) by jk(~) and integrating over E, the orthonormality property immediately yields

$_1$ af_k(x,t)
v at
• of j (x, t) dSI (&)
j=0
M
a(x)f_k(x,t)

v_{jk}(x,t) + $f_{n,k}$(x,t)(16)

If we now write

"$_{DJ\ .k}$~,
for
o(T_k(t) dS1(E) (1'7)

then we obtain, at last, the spherical harmonic analysis of $_{(1)}$

1 a f_k
v , ,_.T + . ~ f j D j k ~-v
a f_k + I fj ajk + f~~k j=0
(18)
k

This is the requisite abstract spherical harmonic system of partial differential equations for the family {fo, fi, f2, ... of functions, the abstract harmonic coefficient functions of the radiance distribution N (x, •) . Knowledge of these fj

allows construction of N(x, •) according to (2). The heart of the abstract harmonic method of solving the equation of transfer thus resides in (18).

Finite Forms of the Abstract Harmonic Equations

An inspection of the system (18) of abstract harmonic equations governing the harmonic coefficient functions fk shows two infinite series involved in the system. The presence of these infinite series could occasionally negate the practical utility of the system, for example in numerical solution work. It is interesting to observe, however, that these infinite series may be rigorously removed and replaced by finite sums under the combined action-of two very general conditions, one physical, the other mathematical. The mathematical condition simplifies the differential operator series; the physical condition simplifies the scattering term series. We shall now briefly indicate the nature of these conditions.

We shall say that the family { O o , 0 1 , 02.1 . . -. } of funct i o n s o n ? has the finite recurrence property of degree v if for every element ' in F and every 0 - in the family, there exist v constants AAk and v elements i_a,, ..., OCIof {0a,

0 1 , 0 2 , o . . . } such t atv
V
~J ~ ~ Ajk Oak (0 kul
(19)
holds for every E in ~. The motivation for this property arises in an attempt to simplify the form of the operators D- and to reduce to a finite series the infinite series involving them in (18). For example, in an orthogonal,-three-dimensional coordinate frame in which x = (X I , X 2 , x 3) , we have

We use this form in (17) to obtain the representation
D. = a.ᵃ + b. ᵃ + c. ᵃ (20)
~k ~k lxt' jk ax_z ~k Z X 3 '
where we have written:
I l a j k t y for I ~ • f 0~ (9) k~) dQ~) (21)
M r

VI C k}t

By postulating a finite recurrence property of degree v for {~a, *1, 6 , ...}, if follows that a'k= Q whenever the indices k and j differ by a sufficiently large amount: indeed al k m 4 for all but at most v terms. Similarly with b -k and
c k• This means that for fixed k D -= U whenever j i sufficiently large and so the number ~~ terms on the left of 18 become finite in the present case, It turns out that any orthonormal family obtained from suitable nth order ordinary differential equations (a rich source of orthonormal families by means of Sturm-Liouville theory) will possess a finite recurrence property of degree v.
Finally, the physical condition which simplifies the abstract harmonic equations is that of isotropy of the medium. In the present case the isotropy reduces the general functional dependence of a on the independent variables g' and to the special dependence of a on the scalar product 9'
of the directions, This simplified structure of a in turn manifests itself in a simplification of the representation (10) to the form:
a(x;E' ;E;t) M Q_j (x;t) T_j (E') 0_j M (24) j=o
We shall. not go into the derivation details of this relation in the present abstract case. It suffices to note that this form can be obtained when the members of the orthonormal family (0o, 01, 02, ...) obey a general type of addition theorem often valid for functions arising in Sturm-Liouville theory. Examples of addition theorems for such functions are, e, g,,, in[318] . [See (12) and (15) of Sec. 6.3.)
The simplifying effect of (24) becomes evident when we recalculate N_* (x,g ,t) after the manner of (11)

J a 3 (X ; t) T) (V) Y E) I d0(9')
Jf_i(x,t) z y x ; t) T j (E ') f j M
i I

M
dQ (4 ')
Z^{f}i($^{x \cdot}$p^t)I zai(x;t)~i(0 (&') I dil(C)

$f_i(x,t)$ a J - (x;t) ~. M ai .
$_1$ J J J
s $f_i(x,t)$ $o_f(x;t)$ O_iM (25) 1

By combining the preceding two 'Conditions, the total effect on (18) is a complete finitization of each equation in the system of equations, thereby rendering them more effective for numerical computations. We may summarize these constructions as follows:

Let X be an arbitrary isotropic, inhomogeneous op tica2 medium with internal emission radiance function N_n and general time-dependent radiance field N as governed by the equation o f transfer (1). Let (09, 01, 0_7., . . .) be an orthonormal family o f functions defined on the unit sphere ' such that: the family (a) possesses the completeness property (see (19) of Sec. 6.1); (b) possesses the finite recurrence property (2.); (c) satisfies an addition theorem (24). Then each member o f the general abstract harmonic system o f partial differential equations (18) reduces to the following finite form: For some positive integer v: