L~uJ 6,3 Classical Spherical Harmonic Method: General Media The general theory of the abstract harmonic method developed in the preceding section will now be illustrated for the classical case in which the orthonormal family is constructed from families of associated Legendre functions of the first kind and circular (trigonometric) functions. 'The optical medium X will be generally inhomogeneous and isotropic, with time varying inherent optical properties, and given internal sources'.

The Orthonormal Family

We begin by observing that the classical spherical harmonic method customarily uses the ordered pair of numbers to specify a point ~ in E, where we have written "u" for cos e, and where (8, ~) are the two angles customarily used to specify & in H (see Sec. 2,4 and also example 14 of Sec. 2.11 for an earlier use of U in con unction with Legendre polynomials) . The range of the variable u is thus the interval[-I,1], and the range of 0, [0,2w]. Every C in E determines a unique (8,~), that is a unique u in [-1,1] and a unique 0 in [0,2n].

Conversely, any pair (u, o) in [-1,1]x [0,21r] determines a unique ~ in

The values of associated Legendre functions are usually denoted by "P_n(u) ". The integer n is nonnegative, i.e. , n > 0 and the integer m satisfies the inequalities: -n < m < n. The general relations in the theory of Legendre polynomials we shall use below may be found fully developed, e.g., in [318], [289]. and [119]. In particular we shall note that

where "Pn" denotes the Legendre function of the first kind and of degree n. For our present purposes, we note that the associated Legendre function P_n Zs a real valued function with domain [-1,1] and defined for all integers m,n such that n is nonnegative and lml < n.The associated Legendre functions include, by (2), the Legendre polynomials as special cases. Any functions Pn arising in the subsequent discussions for which n < 0, are to be zero-valued functions. In view of (1) and (2) only PR with n+l nonnegative indices m need be tabulated.

The orthogonality property of the family of associated Legendre functions takes the form:

1 0.9 , whenever n f r P^m(u) Pr(u) du =

2 ($_n$+$_m$ t

W1 7n+1 Tn-m) whenever n = r

The integral properties of the family of circular functions needed here are summarized by the equations

where m is confined to integral values. These properties can be succinctly summarized by using complex variables, Thus, all three equations in (4) may be expressed by writing

2 zr

imp

e d~ = 21dom

v
where aom is an instance of the general Kronecker delta 6--, The. use of complex variables will considerably facilitate our work in this section, and so they will be retained throughout. One can always return to the real number setting by finding and considering separately the real and imaginary parts of a complex term.

The details of the construction of the requisite orthonormal family on .27-are clearly indicated by considering (3) and (5). Thus to an arbitrary ~ in , (to which corresponds a unique air (V, ~)) and to every pair of integers m, n, with n,> 0, m n we assign the complex: number in[~ where we have written

it $_o$m( )" for A^m Pm(1~) eimo (6) n n n

where in turn we have written

11$_A$ mil n

for

1

we can limit tabulations of A to nonnegative indices m. Furthermore, by recalling (1), the complex conjugate of $4^m{}_n$(E) may be expressed as follows

Am P^m( ) e imp n n n

On over The orthonormality property of the family of functions may now be verified. For example:

+1

Pn (~) Pr (p)

T

^{6}nr

The remaining case where the upper indices of On may differ is straightforward using (s) . Hence we have

A~m fb dot = d 6 9

n a mb n a

for every n, a } U and b, m such that JbI Vi a, Iml < n.

An exact one-to-one correspondence can be established between the abstract family (0o, 01, 02, . . . of Sec. 6.2 and the spherical harmonic family presently under consideration.

Thus to 0] of the earlier discussion we pair 0n, where j = nz + M.+ n. This correspondence arises when one contemplates Fig. 6.1 in which each dot in the figure is paired with the integer couple (m,n) , n > 4 , I ml < n, corresponding to the indices of f_n. Then counting each row of dots by reading from left to right and counting rows from bottom to top, each dot is given a single index j. For example the dot in the first row, corresponding to (0,0) is given the index U . The dot corresponding to (-1,1) is given the index 1, (0 ,1) the index 2, (--3,4) the index 17, etc. In general

n

t m , n 3 = 1 , 3 } 0

o u t -4 -3 -2 - I 1 0 I 2 3 4 - -

FIG, 6.1 Scheme for establishing the correspondence between the abstract and classical spherical harmonic method.

(m,n) is paired with the index j = n' + m + n (10)

and

4) n is paired with 11) J

Observe that the pairings are unique: given (m,n) there is precisely one j > 0 corresponding to this pair; given j > 0, there is precisely one pair (m,n) on the array corresponding to j and is readily obtained under the conditions on m,n described above.

Properties of the Orthonormal Family

We shall now show that the family of spherical harmonics On on possesses the three main properties sufficient to insure a reduction of the general abstract harmonic system

(18) of Sec. 6.2 to its finite version (26) of Sec. 6.2. (The proof o f the orthonormal i ty o f 'the family of spherical harmonics was outlined in the discussion leading to (9).)

The completeness property of the set of spherical harmonics holds. However, the property depends on some relatively advanced arguments, and the interested reader is referred to Chapter 7 of [47] for the general theory of completeness of families of functions arising from nth order differential equations'.

The addition theorem for Legendre functions holds and takes the form (see, e.g., [119

n

$P_n($ $P_n(u)$ $P_n(u')$ + 2

m= 1

Jv) P_n (u') Co s m (0- ~_')

(12)

where & and ~' are any two directions in -t and

are their corresponding angular representations. Using (1), (2), the evenness of cosine, and the oddness of sine, (12) may be compactly written as

(n m I P^M (u)PM(u') $_e$im(0-0') n+m n n

(13)

The argument o f P in (13) is the scalar product of &

and . This scalar p ro4uct is reminiscent of the isotropy condition for an optical medium. We now show how the isotropy condition leads in the present case to the representation of a in the form of (24) of Sec. 6.2. When isotropy

holds, the value of a (for a fixed x and t) is known once
• ~' is known, i.e., once a number V a & • E' in the interval [-1,1] is specified. This value of a under isotropy conditions will be denoted by "Q(x; • ' ;t)". Therefore, the family of Legendre polynomials P_n being complete (a fact also supplied by the general theory in [47] cited above) , we may express a (x; E • E ; t) as follows

j=0 2 J (X;t) P3

where we have written

via i (x;t)" for 2n

v (x; u ; t) Pi (p) du

(is)

Using (13) to represent P_j (&• E') in (14), we have

J

a x~ . ~ .t ,~ 12'+1 Q.(x.t) '-m ! Pm) Pm(V,)eim(O-o')

77r j=01 j m W -~ ij+m)!J

J

a i (x; t) ~mC

j=0 J m -~ J 7

(16)

This is reducible to the form of (24) of Sec. 6.2 as may be seen by using the correspondence between ~j and on established above. (To show the correspondence in complete detail, let a - (x;t) be denoted ad hoc as "a (x;t)" and require it to have value a_3 j(x;t) for m in the range - j < m < j.)

In this way we see how the addition theorem for the Pn and the isotropy condition on scattering combine to form the extremely useful representation (16). The reader may now extend this idea to still other complete orthonormal families of functions defined on [-1,1] provided an addition theorem of the kind (13) is available for the, family.

Next, we observe that the orthonormal family of functions 0 satisfies the finite recurrence property of degree 2. This observation is based on the following three well-known

recurrence properties of associated Legendre functions (see, e.g., [2891, [119]) :.

m (n+a,) Pn-1(u3 + (n+1 1-m) Pn+1 (u)

u P n (u) _ (17) 2n+1

P^{m}+^{1}(u) - Pm±1(u)

sin e PM n [0) = n 1 n 1 (2n+1)

(18)

m (n-m +^{2}) (m-n-1) $P^{m}_{+}{}^{1}$(u) + (n+m-1) (n+m) P m-1 (u)

n1 n1

sineP_n(u) 10

(2n+1)

(19)

As an example of how these recurrence relations give rise to instances of the general recurrence property (19) of Sec. 6.2, consider (17). Here we recall that "u" denotes

C •k ; k is the unit vector along the positive z -axis. Hence in (19) off Sim 6.2 is now k. Next ,;multiply each side of
(17) by e Applying the general definition (6) and making so e algebraic rearrangements'. the net result is
~ •k 0n (9) C'(n,m) On_ 1(&) + C(n+l.,m) fn +1 (E) (20)
where we have written
"t~ n-m ~+m 1/2
C(n,m) far (21) $_n$- 'n+
Hence in (19) of Sec. 6.2, we have v 2, and the A M are now in the form of C(j,k) , with j = n2 + m + n, and a, (n-1)'
+ m + (n-1). a_x (n+1) ^{2} + m + (n+l) . The specific representation of • k On (&) in (20) is now used in (20) of Sec. 6.2 to effect an evaluation of the number cjk, and hence the sum:

$c._k$ $_a$ ...x .. (22)
jiO
which forms part of the operation:
v
f $_j D_{jk}$ (23)
j=0

in (26) of Sec. 6.2. To see how (22) is evaluated, let us represent N (x, ~, t) by means of the functions fin:
N x t ^{co} ^{n} F^m(x t) 41^mM (24) n=0 m = -n
where we have written:
"Fn (x, t) ~` for

N (x, ~ , t) fn (t) dtl (t) (25)
Thus Fn in the present context corresponds to f - in the abstract context of Sec. 6.2, just as on corresponds to o . Furthermore, the correspondence o f j . ~n fl with the pair
~f indices (m,n) of Fm is once again that established above. (See Fig. 6.1 and ~10) , (11).)
Returning to (22), we 'consider it in the context of (18) of Sec. 6.2, but now using the present family {0_n1 of orthonormal functions. We therefore are to consider:
r
n
in which k = a' + b + a.
Thus the infinite sum of z-derivatives in (18) of Sec. 6.2 is reduced to a sum of two such derivatives,
The general procedure should now be clear: by placing the recurrence relations (18) and (19) into their appropriate counterparts of (20). the numbers ajk and b .k in (21), (22)
of Sec. 6.2 are readily evaluated. Then the sums:
V af. V ~f .

a k - , b k
j i o > > j=D 3
are evaluated analogously to the manner displayed in (26). These details may be left to the reader.

General Equations for Spherical Harmonic Method

The net result of the reduction calculations on (26) outlined above may be written in the form:

$aF^{b}(x,t)$ a

(27)

where we have written

~~h b f ~ a+b a+b-1 1/2
(a,) for r I + (2g) a

and where C(a,b) is defined generally in (21), Furthermore, we have written:

t 1Fb (x,t) i t for n,a

N (X t) 4_b d (29) n a

analogously to (25), so that N_n has the representation:

n
N_n (x! ~ 3 t) = I I $F_{n\ ,n}(x,t)$ t_o (~) (30) n=0 m = -n

The set of equations (27) forms: a cooled infinite system of equations in the unknown functions F_a, a .n 0,1,2, .,., Jbi < a. The functions F are generally complex valued, according to their define construction (25) , and such that N (x, ~ , t) is real valued, according to (24) . The general initial conditions for the system (27) are:

Fb $_a$(x,©) " N^o (xs' ~ 0 0) 0 a b ( ) dil ~ ~ 9 (31) M
w

for every x in X, and where No i s the given initial radiance function on X x E at t a 0. For steady state versions o (27), the time derivative term is zero. The functions Ft_a then have domain X and (31) is replaced by:

$F^b(x)$ $N^o(x\ _opt)$ 0^b dn() (32)
a o a w

for x_o over some appropriate subset of the boundary of X (cf. , e.g. , (2b) of Sec. 6.4) . The system (27) is of sufficient generality to solve such problems as point source , beam source, and general internal source Problems in the sea; natural light field problems in lakes, harbors, and the sea. Observe that the inherent optical properties in the form of $a_,$ and a_a may be quite general, and that the term $F_{g\ a}$ provides for internal sources of radiant flux, such as artificial light sources (laser beams, searchlights, submerged incandescent point sources, etc.) or natural light sources (phosphorescence, animal sources, etc.). The general methods of solution of (27) and its manifold variants are well known and may be implemented by programmed machine procedures. I f the model is sufficiently simple (as, e.g., in the illustration of Sec. 6.4) the associated simplified form of system (27) may be solved by hand and evaluated numerically or even used for general theoretical reasoning.