6.4 Classical Spherical Harmonic Method: Plane-Parallel Media

The classical spherical harmonic method developed in the preceding section for general media will now be illustrated in a setting of primary importance in hydrologic (and meteorologic) optics: the plane-parallel optical medium. Throughout this section, then, we shall assume that X is a plane-parallel medium of arbitrary (finite or infinite) depth. The incident light field and the optical properties of X are assumed to be in the steady state and independent of the x and y coordinates throughout X, thus establishing a stratified medium and stratified steady radiance field throughout X.

Under the present conditions on the medium X, the general system of equations (27) of Sec. 6.3 reduces to:

aFb aFh+

C(a,b) a·- 1 + C(a+l,b) (-a + a) Fb + Fb

a z az a a n,a (1) a = 0,1,2, ...; JbI < a

Here we have adopted the terrestrially based coordinate system for hydrologic optics (Sec. 2.4) wherein depth z is measured os it ive downwards from the iai r-water boundary. Thus "-z" in now replaces "x_3" in the general formula (27) of Sec. 6.3 and "x" and "Y" " " ~' replace xi" and x2 , respectively. The functions a and c $_a$ may vary with depth,

The first few equations of system (1). written out in groups for each value of a, are: n,0

$3F^0$

2 n,2

Thus the group of equations for a = © consists of one equation; the group for a = 1 consists of three equations; the group for a = 2 consists of five equations. In general the group for a = n consists of 2n + 1 equations, Some of the derivative terms are missing in the displayed system above because of the conditions Fu laced on the indices at the outset of the discussion. Thus -- © if a < 0 or a < b I . A convenient auxiliary rule to observe in this respect is that: whenever a-b = fl or a+b = a , then C(a,b) = 0.

A Formal Solution Procedure

The system (1), which rep resents tithe system of equations for the spherical harmonic method in a plane-parallel setting, displays an interesting type of coupling among the functions Fu. Observe how the upper index b is fixed in each equation of the system. We shall now show how this feature permits a simplification of the general solution procedure of the system. The manner of simplification may be easily seen by means of the diagram in Fig. 6.2.

Each dot in Fig. 6.2 represents an ordered pair (b , a) of indices corresponding to F_a. The effect of the rather weak coupling among the unknown functions Fa of system (1) is such that we can partition the set of unknown functions into subsets, corresponding to the vertical columns of dots, and

0
autonomous columns
s • • 3
I I 2 I•I
I

FIG. 6.2 A way of grouping the functions Fb into autonomous fmilies for solution purposes.

solve for the unknown functions 9sso±ciated$_{th}$solely with each column. That is, the unknowns F in;the b column can be obtained independently of the unknowns in the other columns of of the array. This observation can be put into a convenient mathematical form as follows. Let us write

"Fb" for (F^b F^b + , F^b + , JbI jbj 1 lbl 2

and

Thus,	" e,ga,	$_{ip}$b" n	for	(Fb n,IbI	,	F^b n,
I P O	$_{tF}$D U f	F^0 1	F^0 2	. , .		
F 1 =	(F 1 , 1	F^1, 2	p^i, 3	*a.~	r	1
$!^2$	(F22,	F^2 3,	F^2 4,	a . .)		2

, Fb , bl+I nvlbl+2
4

and so on. With this notation, we see that the part of system (1) corresponding to an arbitrary fixed index b may be written succinctly in vector form as

d (fib fi^b) _ Ir^b + pb n

where we have written:

for

0 C(Ibl+l,b) 0 0 0 a,.

C(f bl+1,b)	0	C(jbj+2,b)	0	0 ...
0	C(b l+2,b)	0	C(I b I +3,b)	0 . . a
0	0	C(Ibl+3, b}	0	C(Ibl+3,b) „a
0	0	0	C(Ibl+4,b)	0 ...
0	0	0	0	C(Ib l+5,b) ...

 and where we have written:

"a," for

-a + 00	0		0	...
0	_ a + Qa	0	0	*..
0	0	- a + az	...	...
a . .	. 0 .	. a .	...	...
A . .	• . .			

The system (4) may be rearranged into the form:

d1P^{b} ,, b b b F (3 + Gn

b 0, + 1$_{t}$ + 2s

where we have written

for d(Cb)_ 1 (8)

for 7b(eb) 1 n

and where " (Cb) "'01„ denotes the formal inverse of Z°b .

The formal solution procedure for (1) is now seen to be reduced to that associated with (7) and thereby becomes relatively straight forward on either the numerical or manual levels.

Of course, in practice, when numerical solutions are desired, the system (7) must be truncated too a finite system along with the number of components of , and the formal inversion

inversion of 4° must be reduced to a workable procedure. Before going on to consider such truncations, we can place the system into a standard form occasionally useful for formal theoretical considerations and which also show's the general overall structure of the system (1). Thus we first agree to write
'IF" for "Gn" for and finally:
G2 ni

r rr for diag (. .. P J^{-2}
wh re "diag" denotes a diagonal block matrix with 63a as the it block matrix along the diagonal, Then the system (7) takes the generic form:

This is the desired vectorial version of the system (1) , showing the overall linear form of the system, a form reminiscent of the equation of transfer without the path function term. Thus we see from still another vantage point that the net effect of the spherical harmonic method is the removal of the complex directional dependence of the radiance field generated by the presence of the path function term N_* in the general equation of transfer.

A Truncated Solution Procedure

As an illustration of the use of (7) in practice, we consider the case of an arbitrarily stratified source-free plane-parallel medium. Thus in (7) we set

for every integer b, I b I > 0. This is a commonly occurring radiometric situation inmost natural media in geophysical optics, so that the present illustration retains a wide range of applicability. The effect of condition (11) is rather far reaching. To see this effect, observe that by the definition of C(a,b) we have
C(a, -b) = C(a,b)
From this it follows that, formally
e-b - e^b and so -b -- Ob (12) Thus we need only consider:

Now the truncation procedure which we intend to apply to (13) may best be described by returning to the original system (1) and keeping in mind the diagram of Fig. 6.2. This return to (1) is also desirable, so as to bypass the formal inversion procedure leading to (13). It is clear from the diagram in Fig. 6.2 that a truncation may take p I ce at the m row, in the sense that no unknown functions F_a will be

allowed in the system which have indices a > m. Then the truncated autonomous system of equations associated with b = O is:
aC (m , 0) 1-1 i (-a+Qm) F 0 m
(14)
The effect of the truncation becomes clear on inspection of the equation corresponding to the case a = m. The derivative of Fm_+1 is omitted from the equation for this
case. Thus in the system displayed above there are (m+l) differential equations and m+l unknown functions: FQ
i, j = O,

The truncated autonomous system of equations
with b = 1 is
Here the system associated with b = 1 consists of m
d
*Ge_t M, differential equations in m unknown functions: In F~ , ' = 1,
1 the system associated with b were b < m,
consists of m 1 b differential g+nea equations in the m+l -b unknown
functions Fj , ' = b b+l... , mThus for the case b = m-1, ~

we have two equations
a
a

m'-1: BF M-1
C gym, m- l^M
-=

m: a F^{m-1}
C[m,m-1) m-1
_

a+ Fm--1 m-1 m-1

(16)
Finally, for the case b = m, there is only one equation, namely: a
a
m

(17)
whence Fm 0, provided (-a+a

Once them ^{2}+2m+l = (m+l ^{2} functions F^b have been obtained, where 0 < a '317s, m, and Ibl ~ a, the associated representation of N (x^-, E) according to the general pattern (Z4) of Sec. 6.3

M a rr(X,Q = E
a=0 b = -a
F^b W (D^b M a a
(18)
Equation (18) is the requisite mth order spherical harmonic approximation to the radiance function N on a stratified plane-parallel source-free optical medium in the steady state.
Vector Form of the Truncated Solution

It is of interest to place the truncated system (14) to (17) into the compact form of (13);.
Thus let us write
"F(b,m)" for (F ^{b} b
F^{b} + , . . . , F^{b}) (19)
b l m
F(b,m) is a function which assigns to each depth z in the plane-parallel medium the (m+l) -component vector F(b,m;a), i.e.,

Fm(z)
By studying the general = fors of (14) to (17) , we see that
the truncated associate of (2D in (5) is the (m-b+l) X (m-b+1) matrix:

0	C(b+l,b)	0	...	0	0
C(b+1 ,b)	0	C(b+2 ,b)	...	0	0
0	C (b+2 ,b)	0	, . .	0	0
0	0	C (b+3 ,b)	...	0	0
0	0	0	...	0	0
. . .	0 . .	. . .	0 . 0	...	...
...	...	...	...	0 . .	r . 0
. . .	. . .	. a .	. . .	. . .	. .
0	0	0	...	C(m-l,b)	0
0	0	0	. , .	0	C(m,b)
0	0	0	...	C(m,b)	0

which we shall denote by "C(b,m)". This matrix is invertible whenever (m-b) is odd as we shall see below. Furthermore, if we write
-
then the general representative of the systems of equations (14) to (17) is of the form:
- ^{d} F(b,m) C(b,m) F(b,m) l.(m) (22) UT
Finally, if we write
11$_{(g(b}$, m) ^{11} for c,(m) ['$^{-1}$(b , m) (23)
we have .

- z F(b,m) _ F(b,m) B(b,m)
0 < b < m; m-b odd

(24)
which $_h$is the desired vector form of the system (14) to (17) of m order spherical harmonic equations. We have now reached the stage where the system (1) is in a form amenable

to solution by any of several well-known theoretical or numerical techniques in the theory of ordinary differential equations (see, e.g., [23] or [47]), of course (1) itself can always be programmed directly for solution.

There is one instance of (24) whose solution can be written down immediately in "closed form," namely the case where a and v are independent of depth; in other words, for the case of an homogeneous medium x. Then, if we write
exp ~6(b,m)~
for

(25)

where 40i (b,m) is the j th power of the matrix 6 (b ,m) ,
note the value of F (b , m) at z by "F (b , m ; z) " then
and de -
F

F(b,m;z) = F(b,m;U) exp {-P (b,m)z
0 < b < m; m-b = 1, 3, 5, ...

(26)
Using the theories of [37], (26) may on the one hand be put into closed algebraic terms using the Jordan canonical forms o f matrices; and on the other, (2 6) may be programmed for direct evaluation on general-purpose electronic computers using the techniques, for example, in [23].

To facilitate computations of P (b, m) using (26), we may arrange matters so that the inverse oaf ~(b,m) can be written down by inspection whenever it exists.. This may be done as
follows. First we verify the fact already noted, that (b,m) has an inverse whenever m-b is an odd integer. For example,
when m-b = 1, and b > 0
0 C (b+1 ,b)
C (b, m)
C(b+l 3 , b)
0
then
det C(b,m) = - C^2 (b+1 ,b)

where "det A" denotes the determinant of a matrix A. Hence d'(b,m) has an inverse.
Again, when m-b = 2, and b ? 0

r - The pattern forming should now be clear. By induction we have, for integers b > 0, p > 0 such that m - b = 2p + 1.

P

det C (b , m) _ (-1)p+l 7T Cz(b+(27+1),b) f 0 (27) J'=0

We next introduce the permutation matrix P which permutes the m-b+1 rows of L °(b,m) , where -b = 2p + 1, p > 0,

in such a way as to near- d i a f (o n a l i x e Cb.m) i n t h e f o l T o w i n sense. Return to d(b,m) above where m - b = 1 and note that we can diagonalixe it by interchanging its two rows. Similarly, by interchanging the rows of C(b,m) where m : b = 3 in pairs, starting with the first two rows, theft going on t o interchange the second pair of rows (i.e., row three and four) we obtain:

PC(b,m) _

r

C(b+1 tb) 0 E C(b+2,b) C 0 0 C(b+l,b) 0 0

f

0 0 t C(b+3,b) 0

0 C(b+2,0) 0 C(b+3,b)

where

The general structure of P for the case of an arbitrary m - b ~= 2p +1) should now be evident: P is a 2 (p+l) x 2 (p+l) m-a - trix obtained from the identity matrix I of the same order by interchanging the rows of I in pairs;, as illustrated by the special case just considered. The utility of the permutation P rests in the fact that the inverse) of P C[b, m] where m - b = 2p +1, is readily written down by 'inspection. To see how the inversion proceeds, consider once again the case of m - b

3 = 2p + 1 (so that p = 1). To simplify the illustration, we shall write "C -" for C (b+ j ,b) , with "b" understood. Inspection of Pe(b,l), with m-b = 3, shows that its inverse must have the same overall structure: as Pof(b,m) itself and whose main diagonal consists simply of elements of the form 1/Cj. With this in mind, we may write:

[PC(b,m)1 [PC[b,mf 1

C_1	0 i i C_2	0 1	1	0 , x 1 i	0
0	i C_1 i 0	0		i	
	i 0 i C_3	0	0	0 1	0
	C2 0 i	C3	0	r x i 2 0 i	1 C_3

1	i 0 , 0	0	
0	1 i 0	0	
0	0 ' 1	0	I
0	i 0 , 0	1	

As yet the entries xi, x2 of the matrix are not known. However, it is clear that x1, xz satisfy the conditions

0
x C + C2 = 0 2 3 C1
whence

C_2
$^{U}I^{L;}3$
C_2

As another example, let m - b = 5 = 2p + 1 (so ~that p = 2). Once again the overall structure of [PC b mis the same

as P e(b , m) ; i.e., near diagonal t, where P is now the requisite 6 x 6 row permutation matrix. To find [P C(b ,m)] i , we write

PL' (b

F	I	F	I
I 0 0	I O C_5 0	0 0 ! 0	0 1 F
F		F	I
I 0 F O	I C_4 I 0 C_S	I 0 o I o	I x_4 1 o
I	I	I	

The remaining entries xi, ... x4 are now readily determined as in the case o f p = 1. By direct inspection

x_3
$^{C}4$ C~5

These two examples for the cases p = 1,2 clearly indicate the nature of [PC(b,m)] with m-b = 2p +1-for general integers

p > 0. The general rule may be phrased as follows: the main diagonal of [P C(b,m)] " i consists of elements of the form 1/C(b+(2j+1) , b) arranged successively in pairs r j $=_w$0, 1,
*00_P p. The nonzero off-diagonal elements in [P (b,m)] occur in exactly the same places as in PC(b,m), and each may be obtained by dividing the corresponding entry Cj of PC(b,m)

by (-l) times C C~ , where Ck and C_Z are, respectively, the elements o f P C(b , m) in the same row 'and column as C1-. The reader should now construct the [P C(b , m)] $^-$1 for p =3 'to test this rule, What is the rule's general form?

Finally, we can rearrange (26) so as to specifically include within the formalism the preceding simple inversion procedure. Returning to (22) , we can write
d Uz
we have

G(b,m) = G(b,m) 49(b,m)
4 < b < m; m-b = 1, 3, 5, ...
(30)

as the present counterpart to' (24) . The inverse [P42(b,m)] -' is the one whose simple rule of formation we have generated in the preceding discussion. Then, Corresponding to (26), we have

G(b,m;z) = G(b,m;0) exp.{'#3(b,m)z)
0 E b < m; m-b = 1_9.3_9 5, ...
(31)

Because of the autonomy of these equations with respect to b, we can vary the truncation parameter. m for each given b, so as always to have m-b odd, and therefore, to always have the algorithm J31) at hand. Suppose, f a r example, we wish to find all F_a with a < 4, as indi sated; by the diagram in Fig, 6.2, and so as to have the representation of N (x, E) in (18) for the case m = 4.Thus we are to find $4+1)^2$ = 2 5 functions in all. In solving for the $_f$amily {Fj) we accordingly may truncate at FT (rather than F4) ;and solve for F^O, a = 0,% 1, 2, 3, 4, 5 using (31), taking advantage of the oddness of m-b = 5-0 = 5 . In solving for the family { Fa) , we use(31) directly since now m-b = 4-1 = 3. A similar tactic i s employed for extending by one additional member the family {Fad, as in the case of Fa , and so on, do the end of the computation procedure.

Equations (26) and (31) are thje final forms of the mth order spherical harmonic equations w'e shall study in this work.Having deduced (26) , (or its variant (31)) we reach

 the threshold of the invariant imbedding domain of radiative transfer theory. Thus the equation (26), say, may be viewed on the one hand, as the logical culmination of the train of deductions begun in Sec. 6.2 in the development of the classical spherical harmonic method; and on the other hand (26) forms a bridge between the classical method of solution of the equation of transfer and the invariant imbedding techniques for the solution of the equation of transfer. These latter techniques will be considered in Sec. 7.10.

Summary

In the preceding four Secs. 6.1 to 6.4 the spherical harmonic method is developed and applied after an appropriate motivation of the method in Sec. 6.1. The main purpose of the discussions is to make clear the fundamental ideas on which the method rests, in particular the general role of the orthonormal family of functions used to represent the radiance function as a sum of products of purely spatial and directional terms. This was done in Secs. 6.2 and 6.3. To show the applicability 'of the method to the case of plane-parallel media, the setting of greatest utility in the study of hydrologic and meteorologic optics, the discussion of the present section is added to the general remarks. In particular, equation sets (14) to (17) above explicitly exhibit the truncated forms of the spherical harmonic equations, where the truncation arbitrarily sets to zero all functions F_a with indices a > m. The resultant system (24) cp be used to solve for the unknown complex valued functions FO, 0 < a, < m, JbI

a. To solve (24) directly we must know A (in (31T or (32) of Sec. 6.3) from experiments.. If No is to be found theoretically, we may use invariant imbedding methods which will give the aerosol's or hydrosol's reflectance to incident light (Volume IV, et seq.).