6.7 Solutions of the Exact Diffusion In Equations

n n -r n

The exact diffusion equation on which we base the discussion of the present section is (57) of Sec. 6.5. In full notation, this equation is of the form:

h(X) (h h) n

1 (h (x') + s (xl)- h(x') K (x' ,x) dV(x') n

4 7r

a

X

T

1 h (x') + s (x' h (x' r-r dV(x') (-1)

n 1x_x112

X

The current settings in which this integral equation is to describe the scalar irradiance field hare infinite and semi-infinite homogeneous media with arbitrary sources described by h_n within X. Once a solution h is found for a space X, the associated radiance distribution throughout X is obtained by means of (60) of Sec. 6.5. The first of our two main goals in this section is to solve (1) for a point source in an infinite medium and arrange the solution in such a manner as to be directly applicable to problems of finding radiance distributions associated with general source conditions in X. It will be seen that by judiciously tabulating the point source solution of (1), all solutions of (1) corresponding to the possible source conditions within X, are obtainable in principle by relatively straightforward numerical procedures based on the tabulated solution. The second main goal is to discuss the solutions of (1) for semi-infinite media (infinitely deep, plane-parallel media) with arbitrary internal sources.

Infinite Medium with Point Source

we begin with (1) for the case of an infinite homogeneous medium x with a point source at the origin. The homogeneity assumption frees o (x) and s (x) of dependence on x throughout x and lets us write:

T_{r-r}1 (x' r0

where, as usual "x" denotes a point in x, and where Ix-x' lis the distance between points x and x'. The point source condition is represented by

h_n (x') " P_n 6(x^1) (3)

where P i s the quantity o f radiant flux emitted steadily in time any uniformly in all directions by the point source at the origin. We may leave the nature of this source quite arbitrary throughout the discussion. As a result, we shall be able to adapt various solutions of (1) for the point source case, by means of integration, and in such a manner that the actual nature of the source may vary from true emission processes, through transpectral scattering processes, on through elastic scattering processes. This will be illustrated later in the discussion. For the present we go on to investigate the case of (1) with a single point source. The requisite form of (1) is:

P 6(x') + sh(x') e

n , dV(x' (4)

I x-x

The theory of the solution of (4) is thoroughly understood; a representative detailed development of the solution of (4) may be found, e.g., in [40]. Therefore, beyond the general observations leading from (39) to (59) of Sec. 6.5, we shall not need to discuss the details of the solution procedure of (4) in the present work. However, we wish to display the solution of (4) in such a manner that the results of [4©] may be readily adapted to the radiative transfer context. Such an adaptation requires the preliminary transition to a certain class of dimensionless geometric parameters, which we now define,

Throughout this section we shall write:

It T (x,x') for

a(x") dr" (5)

where a is the volume attenuation function for the medium. The integral is a line integral along a path 9.(x , E) with initial point x and terminal point x^1. Since he medium X is isotropic and homogeneous, paths are straight-line segments and

T (x,x') = aI x-x' I (6)

When no confusion will result, we will simply write

o f T o f

for T (x, x') ,

with x, and x' thereby being understood.

The quantity T assigned to the- distance Ix-xtl between x and x' is dimensionless, and by virtue of (6) may be viewed as the number of attenuation lengths L_a between x and x^1,

Next, for every subset Y of X we write

1 t v $_a$(Y) it

for I $a^3(x^t)$ dV(x')

Y

The quantity V (Y) is dimensionless. Throughout this section, both T {x,x' ~ and 'V (Y) may be thought of and referred to as optical lengths and optical volumes, respectively, with

out fear of confusion with the classical notions of the same names,

With definitions (5) and (7) in mind, (4) may be re

written as

P -r h(x) = 1 n S(x') + p h(x') e dVa[x')

a T (x 's x')

X

where p is the scattering-attenuation ratio s/a. Equation (8) is the required dimensionless version of (4) ; and for purposes of a solution tabulation, we now impose the unit source condition in the context of (8):

P n~ a

provided that the Dirac-delta function d with dimensions L^{-3} (to go with the volume measure V) is retained. Otherwise, if a dimensionless Dirac-delta function 6 Ito go with the optical V$_a$)   is adopted, in (3) we write h,, d (x') and the unit source condition is

The scalar irradiance field h governed by (8) is clearly spherically symmetric about the point source so that h depends only on radial distance r or (now that the transition to dimensionless parameters has been made) on T. Let us denote the solution of (8), under the unit source condition 09a), by "K ". Then it can be shown (cf. [40]) that the scalar irra5iance at optical distance T from the origin is KE(T), where

K_E ('r) = A(p, T) K_a(T) + $ (P,T) K_{lc} ('r) (1

and where, in turn we have written:

RA(P T)" for 1
E (P, T) (1

and

3k2

"$_B$ (p. T)" for D_o (1
P

to point up the fact that K,:(T) is simply a linear combination of the dimensionless diffusion kernel KK (T) (cf. (52) of Sec. 6.6) where now we write:
'K VT
11KK ($_T$) to for en T v
(13)
and the dimensionless beam transmittance kernel Ka(T) (cf. (43) of Sec. 6.5) where now we write
11K (.0 it

(14)
It remains to specify the terms E (p , T) , K RV aki / ap, and D_o. The latter term is simply aD, where D is t~e diffusion constant (cf. (27) of Sec. 6.5) for the classical diffusion theory. The remaining three terms form the heart of the exact solution and are tabulated in Tables 1 and 2 below for various values o f p and T.
Thus from (10), we have

K (T) ` EC T) B TE 4 Tr r^2
ak^2
+ ~ 1 $_B$_ICOT ap-7r –T
(15)

TABLE 1
The function e(P,rt)

T	P	P =0.1	P=0.2	P=0.3	P=0.4	P=0.5
0,0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.1	1.0000	1.0210	1.0418	1.0542	1,0526	1.0420
0.2	1.0000	1.0382	1.0773	1.1000	1,.0962	1.0756
0.3	1.0000	1.OS32	1.1088	1.1409	1.1346	1.1046
0.4	1.0000	1,0667	1.1375	1.1781	1.1692	1.1301
0,5	1.0000	1.0790	1,1640	1.2126	1.2008	1.1529
0.6	1.0000	1.0904	1.1888	1.2448	1.2300	1.1736
0.7	1.0000	1.1010	1.2121	1.2752	1.2571	1.1926
0.8	1,0000	1.1109	1.2342	1.3038	1.2826	1,2100
0.9	1.0000	1.1202	1.2552	1.3311	1.3066	1.2262
1.0	1.0000	1.1291	1.2753	1.3571	1.3293	1,2412
1.5	1.0000	1.1674	1,3644	1.4724	1.4273	1.3034
2.0	1.0000	1.1990	1.4402	1.5699	1.5068	1,3504
2.5	1.0000	1.2258	1.5068	1,6551	1.5738	1.3874
3.0	1.0000	1.2494	1.5667	1.7311	1.5314	1.4171
3.5	1.0000	1.2704	1.6213	1.8000	1.6818	1.4415
4.0	1.0000	1.2895	1.6718	1.8630	1.7265	1.4617
4.5	1.0000	1.3070	1.7188	1.9214	1.7665	1.4786
5.0	1.0000	1.3231	1.7630	1.9757	1.8026	1.4928
6.0	1.0000	1.3521	1.8443	2.0745	1.8654	1.5147
7.0	1.0000	1.3779	1,9182	2.1630	1.9182	1.5304
8.0	1.0000	1.4010	1.9863	2.2432	1.9634	1.5412
9.0	1.0000	1.4222	2.0497	2,3169	2.0024	1.5486
10.0	1.0000	1.4417	2.1094	2.3851	2.0366	1.5531
11.0	1.0000	1.4599	2.1659	2.4499	2.0667	1.5554
12.0	1.0000	1.4770	2.2196	2.5086	2,0933	1.5559
13.0	1.0000	1.4931	2.2710	2.5652	2.1172	1.5550

14,0	1.0000	1.5084	2.3204	2,6188	2.1385	1.5529
15.0	1.0000	1.5230	2.3682	2,6700	2,1578	1.5498
16.0	1.0000	1.5370	2.4141	2.7190	2.1752	1.5459
17.0	1.0000	1.5503	2.4586	2.7658	2.1910	1.5413
18.0	1.0000	1.5632	2.5019	2.8109	2.2055	1.5361
1910	1.0000	1.5757	2.5439	2.8543	2.2186	1.5304
20.0	1.0000	1.5877	2.5849	2.8963	2.2307	1,5243

Now that it is clear how K, (ˑr) depends on the diffusion kernel K_K ((52) of Sec. 6.6) and the attenuation kernel K_a ((43) of Sec. 6.S) we write (10) in its explicit form;

TABLE 1--Concluded The function E(p,T).

	p=0.6	p=0.7	p=0.8	p =0.9	p=1.0
0.0	1.0000	1.0000	1.0000	1.0000	1.0000
0.1	1.0269	1.0099	0.9921	0,9745	0.9554
0.2	1.0474	1:0162	0.9843	0.9528	0.9222
0.3	1.0643	.0206	0.9757	0.9341	0.8934
0.4	1.0786	1.0236	0.9593	0.9173	0.8683
0.5	1.0909	1.0257	0.9621	0.9019	0.8460
0,6	1.1017	1.0271	0.9551	0.8878	0.8260
0,7	1.1113ˑ	1.0279	1,9483	0.8747	0.8077
0.8	1,1198	1.0282	0.9417	0.8625	0.7910
0.9	1.1275	1.0282	0,9353	0.8510	0.7755
1.0	1.1343	1,0278	1.9290	0.8402	0.7612
1.5	1.1601	1.0229	0.9002	0.7936	0.7019
2.0	1.1763	1.0149	0,8748	0.7562	0.6568
2,5	1.1866	0.0054	0ˉ.8519	0.7250	0.6207
3.0	1.1929	0.9952	0.8313	0.6982	0.5908
3,5	1.1963	0.9847	0.8124	0.6749	0.5655
4.0	1.1978	0.9742	0.7951	0.6543	0.5437
4.5	1.1976	0.9637	0.7791	0.6358	O.S246

5.0	1.1963	0.9534	0.7643	0.6191	0.5076
6.0	1.1912	0.9334	0.7374	0.5901	0.4788
7.0	1.1838	0.9144	0.7137	0.5654	0.4550
8.0	1.1749	0.8964	0,6926	0.5440	0.4349
9.0	1,1551	1.8793	0.6734	0.5253	0.4175
10.0	1.1547	0.8631	0.6560	0.5086	0.4024
1110	1.1438	0.8477	0.6400	0,4936	0.3890
12.0	1.1327	0.8330	0.6252	0.4800	0.3759
13.0	1.1215	0.8190	0.6114	0,4676	0.3661
14.0	1.1102	0.8055	0.5985	0.4562	0.3562
15.0	1.0989	0.7926	0.5864	0.4456	0.3471
15.0	1.0876	0.7802	0.5750	0.4357	0.3387
17.0	1.0764	0.7683	0.5643	0.4265	0.3310
18.0	1.0653	0,7568	0,5540	0.4178	0.3238
1910	1,0542	0.7457	0.5443	0.4096	0.3170
20,0	1.0433	0.7349	0.5349	0.4019	0.3107

TABLE 2

The p	functions K 0 is 0	and 0 dk2/dp 0 dk2/dp
0,0	1.000000	0.000000
0,1	1.000000	0.164892(-S)*
0.2	0.999909	0.009094
0,3	0.997414	0,116201
0,4	0,985624	0.373272
0.5	0.957504	0,731896

0,6	0,907332	1,145954
0.7	0.828635	1.590033
0,8	0.710412	2.051119
0.9	4,525430	2,522370
0.92	0,474002	2.617473
0.94	0.413976	2.712805
0.96	0.340829	2.808348
0.98	0.242983	2.904085
0.99	0.172511	2,952020
1,00	0.000000	3.000000

*Note: "(-5)" means "multiply by 10`."

In this way we .can see that, for computation purposes, the scalar irradiance K_e (T) at optical distance T from the origin consists of two terms, one which may be attributed to residual flux (the first term) and the other which may be attributed to scattered flux. This type of partitioning of the exact representation of h (x) into a residual part (ho) and a scattered part (h*) was already encountered in the classical diffusion theory, e.g., in (7) of Sec. 1.5, in (57) of Sec. 6.6, and more generally in (79) of Sec. 6.6. Also, in the time-dependent case, this partition was encountered in (90) of Sec. 6.60

A tabulation of 4n T 2g (T) is given in Table 3 for two cases o f p and for a range of T from 0 to 10 units. These choices of p are representative orders of magnitude for p in the case of the ocean (p = 0,3) and the atmosphere (p a 0.9) for wavelengths around 500 mp, for the middle of the visible spectrum. For the determination of IC_E(T) for values of p other than p = 0.3, 0.9, Tables 1 and 2 may be used. It must be kept in mind that these tabulations are for the unit source condition (9a) .

TABLE 3

The function 47r T^{i*} IC_E (T)

T	P=0.3	p=0.9
0.0	1.0000	1.0000
0.1	0.9644	1.1211
0,2	0.9196	1.2343
0,3	0,8710	1.3384
0:4	0.8209	1.4326

0.5	0,7708	I.S168
0.6	0,7215	1.5914
0.7	0.6737	1.5567
0.8	0,6277	1.7130
0.9	0¯.5838	1,7607
1.0	0,5421	1.8006
1.5	0.3675	1.8974
2.0	0.2441	1.8660
2,5	0.1599	1.7547
3,0	0.1037	1.5992
3.5	0.0568	1.4239
4.0	0.0427	1.2454
4.5	0.0272	1.0742
5,0	0.0173	0.9158
6,0	0.0069	0.6483
7.0	0,0028	0.4467
8.0	0.0011	0.3018
9.0	0.0004	0.2007
1010	0.0002	0.1318

Infinite Medium with Arbitrary Sources

We now develop a procedure whereby Table 3, and more generally (15), may be used to compute scalar irradiance fields generated by arbitrary sources. Suppose the source term h_n (x) its given throughout an infinite medium X; h, (x) may be associated with plane sources, finite volume sources of flux, etc., and may be of quite arbitrary spatial dependence throughout y. I t is clear either intuitively or formally (from the interaction principle using the theorems of Sec. 3.16) that the scalar irradiance h(x) associated with h, (x) is given by

h (x) 1 h (x') K (x' , x) dV (x') X

(16)

where we have written:

'1X c (x' ,x)'+ for IC_E (T (x,x')) (17)

The reason for the presence of "a" in (16) may be found by tracing back through the unit source condition (9a) and ultimately to (3) and (4). I f h is given in watts per cubic meter, and a in per meter, tRen h is given in units of watts per square meter.

A practical computation scheme for h(x) may be based on the following procedure: given h, (x) throughout a subset X_n of X, divide X_n into n small cubes C (xi) (or any other conveniently shaped regions) over each of which both T (x,x') and h_n (x) vary only slightly. Thus each cube C (xi) i s representative of the radiometric properties of X around xi, where x • is the cube's center point. Then (16) may be replaced by tLe approximating finite sum:

n
h (x) 1 1 h (xi) X (xi !x) V (C(xi))
a iml nE a
(18)

The evaluation of h(x) using (18) is facilitated by using Table 3 for optical distances T (x,x') up to 10. More generally, (15) would be used with Tables 1 and 2.

As a specific example of a setting in which (18) may be applied, consider the problem of determining the irradiance field generated in an infinite homogeneous medium by a beam-type source, such as that associated with powerful search lights or laser beams. The geometrical relations of the present example are summarized in Fig. 6.5. The source may be represented as a small sphere of radius r with surface radiance N_o and which is allowed to emit uniformly over a conical set "H_o of directions with central direction t_o. Thus o may be all directions & such that & *E_9 > cos 8o where 9_0 is the half angle opening of $H_0$0 By varying 0_0, the cone can represent everything from narrow beams (small 60) to uniform point sources (e_o = 7r) .

With these geometrical preliminaries fixed, we now return to the discussion in Sec. 6.6 which developed the theory of primary scattered flux as source flux and which culminated in the formulas (67) through (69) of Sec, 6.6. We can

FIG. 6.5 Geometry for a nonisotropic point source of radiant flux in diffusion theory,

immediately adopt for our present purposes the formula (67) of Sec. 6.6 which describes the primary scalar irradiante h*(x') in terms of the inherent radiance N_o, the total scattering coefficient s_f the beam transmittance e-ar', and the solid angle P(r') subtended by the point source at point x^1.

(See Fig. 6.5.) Now h* (x') replaces h, (x') in (16) or $h_{,l}$ (xi) in (18). Thus (16) becomes

e^{-a} I x' ^{l} Q (I x' 1) X (x' , x) dV f i x') (19) a

and (18) becomes

h (X) ' PN_o I e^{-alXll} n (I X i l) K_e(X i ' X) V_a(C(Xi)) I a I

In (19) the integration may be limited to the subset X_o of X defined by the cone

? of directions. Thus point x' is in
if and only if $x^1/1$ x' is in o. In (20) the sum is over all

cells C(xi) which partition X.. Because of the exponentials and the solid angles SI (I xi 1) in (20), the sums (for a given N.) need not be extended over very many attenuation lengths within X_o before good estimates of h(x) can be made.

Semi-Infinite Medium with
Boundary Point Source

The exact diffusion solution (15) holds for media which extend indefinitely far in all directions about the point source. Such a situation will hold more or less in natural waters when the source and observer are at relatively great depths (several attenuation lengths, say). However, if the source is relatively near the surface, the reflectance properties of the remaining thin layer, of medium above the source would differ noticeably from that of an infinitely deep layer above the source, so that the scalar irradiance h(T) at shallow depths in a light field induced by a point source near the boundary would differ markedly from that predicted by (1G). Similar observations may be made for fogs and cloud banks in the atmosphere. In the present example, we summarize some results of exact diffusion theory which can predict h(T) for relatively shallow depths in natural waters (or for points near flat cloud or fog boundaries) when the point source is on the boundary. The reflection effects of the air-water surface are not included in the present analysis and must be accounted for separately. In the the second example below the results will be extended to the case of internal point sources. Both examples are based on the results by Elliott given in Ref. [88]. A generalization of the equations developed below and their appropriate place in the general theory of radiative transfer in media with internal sources, will be given in Sec. 7.13.

The starting point for the present discussion is equation (8) in which the medium X is now an infinitely deep homogeneous plane-parallel medium exhibiting isotropic scatter ing and with a point source of small positive radius r_o at depth x = c y 0, We shall use the terrestrially based reference system for natural hydrosols (cf. Sec. 2.4). Furthermore we use the unit source condition (9a) in (8).

Thus we start with (8), now in the form:

_T (x, X')
h (x) 4n (a(xI-x_o) + ph(x')) e dV_a(x') (21)
T (X, X')

where X_+ is the set of all x (= (x1 ,x2 ,x3)) in the terrestrial coordinate frame such that x_3 = z > 0. The Dirac-delta function & in (21) is dimensionless, and is centered on the point x_o (M(^{0}1000) f c z 0, Furthermore, it is to be explicitly noted that for the remainder of this section all coordinates x 1 , X2* x 3 (hence all distances, are as, and volumes) are to be measured in units of optical length (cf. (5). (7)).

Now the procedure in Ref. [88] is to take the Fourier transform of (21) with respect to 'the variables x, and xa over an arbitrary horizontal plane at depth x_3

(=z). Thus let w, and w2 be the spatial frequencies along the x, and xz directions and let us write:

If f_o (z ! w19'2) to for h (x) e 1 x1w1 + X
z
x2w2) dA(x)
(22)

where X_z is the horizontal plane at depth z, and A is the area measure over X_z, Thus f is the Fourier transform of h over X_z, and f_o has the same dimensions as h. Therefore, applying the operator:

$_e$i (xlwl + x2w2) dA(x)
X_z

to each side of (21), we obtain:

0
[1] 6 (z' ⁻a) + f (z' M I (z-z' E ~w) dz' P o
(23)

where we have written

tt i (I z _ $_z$ t I $_{t\ w}$) it for

2-Z~~ A^Z-1 IdL
(24)

*In the present exposition, we retain the Fourier transform conventions used in [881 in order to facilitate the study of the results therein.

where J_o is a zero-order Bessel function, and where, for brevity, we have written;

fff_o(z ;w)" for f_o(z ,WI_,w2) (25)

The next step in the solution procedure is the observation that (23) can be solved using the Wiener-Hopf technique provided that c = 0 , i.e., that the source is at the boundary.

This solution procedure is quite intricate and beyond the immediate interests of the present work; therefore the interested reader is referred to Ref. [88] for details and further references. The main results of the present example may be understood without recourse to the solution details, We need only observe that the required scalar irradiance is obtained from the solution f_o (• ;w) of (23) by means of the following integration which is the inverse Fourier transformation to that in (22):

(27)

$_r$2 $_{a\ x}$2 + x2 (28). 1 2

Since h (x) depends only on depth z and the radial distance r, we agree to write "h(z,r)" for h(x)

(29)

Figure 6.6 depicts the geometrical details of the case where the point source is at the boundary. Observe that the medium is divided into region A (shaded) and conical region B (unshaded). It is found that h(z,r) for points x w- (x_1, x2 , Z) in region A i s approximated by the relation

h(z,r) V3 h_n ~I(Z) e"Kor(I + Kor) =
2nar
(Valid in region A, Fig. 6:6.)
(3v)
where in turn *1(z) is evaluated in [172] and is tabulated in Table 4 , and K_o is given in Table 2. -Table 4 may be extended,
r y
and are tabulated. The farther the point x (w (XI ,x_x ,x3) in region A is from the dashed dividing lines between regions A and B, the better the approximation (30).

TABLE 4 Evaluation of *I (z)

	z + ^{t}o	1(^{z})
0	0,7104	0.5773
0.01	0.7204	0.5982
0,02	0.7304	0.6154
0.03	0.7404	0.6312
0.05	0.7604	0.6607
0.1	0.8104	0,7279
0.2	0,9104	0,8495
0,3	1.0104	0.9633
0.4	1.1104	1.0731
0,5	1.2104	1,1803
0.6	1.3104	1.2858
0.7	1.4104	1.3901
0.8	1.5104	1.4935
0.9	1.6104	1.5963
1.0	1.7104	1.6985
1.2	1,9104	119019
1,5	2,2104	2.2051
2.0	2,7104	2.7079
2.5	3.2104	3.2092

3.0	3.7104	3,7098-
3,5	4.2104	4.2101
4.0	4.7104	4.7102

The error of the approximation by (30) is of the order of magnitude of j z^1/r^s 1 and (30) is applicable when p is 0,6 or_ more,
Furthermore, i t i s found that h (z , r) for points x
(= (x 1 , xz , z)) in region B is approximated by the relation

d3 h_n cos e Kod
h(z,r) e^W (1 + Kod) 2-ff ad
(Valid in region B, Fig. 6.6.)
(31)
where we have written
"d" for ~r + (z+zo) (32)

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and where

t an e r

z + z
0

and:

$z_0 = 0.7104$

This approximation improves with the distance of x (=x $_1$, x_x , z)) in region B from the dashed dividing line's between regions A and B. The error of approximation by (31) is of the order of magnitude of 1 $1/d^5$ l and (31) is applicable when p i s 0.6 o r more*
A study of (30) and (31) readily shows the effect on h(x) of the presence of the boundary at depth z = 0. Suppose for the moment that $K_o = 0$ (no absorption case). Then in
region A of Fig. 6.6, and for fixed z, the scalar irradiance falls off as the inverse cube of the distance r from the symmetry axis of the field, whereas in region B, which is relatively farther removed from the boundary than region A, the scalar irradiance falls off only as the inverse square of the distance d. The fixed number z_o (known as the

"extrapolation length") in (34) arises in the correct adjustment of boundary conditions of the present problem.

Semi-Infinite Medium with Internal Point Source

The results of the preceding example will now be extended to the case of a semi-infinite homogeneous medium with a point source at x_4 = (0 , 0 , c) , c _> 0 , i. e. , with a point source in the interior of the medium rather than on the boundary. Let us denote the solution of (21) for this case by " f_c (z ; w) " . Hence, when c = 0 we are to have f_o (z ; w) of (23) back once again, and f_c is to be a proper generalization of f Now assume a general point source condition h_n/a (cf. (%*)). Then the functional relations connecting f_c and derived by Elliott [88] are of the form:

z
fc(z,w) = f0 (1 z-cl ~W) +h
t_o
f_o(t_VW) f_0 (t+c-z,w) dt, , z < c
f_c(z sw) = f_o(f z-cl ,w) + s n
f_a(t,w) f_o(t-c+z,w) dt z > c
0

FIG. 6.7 Domains of validity of approximate solutions (38) through (40).

Once f_c(z,w) is obtained using (35) or (36), h(z,r) can be obtained by means of the inversion formula:

00
h(z,r) = 1
a
f $_c$ (Z-, w) w J_o (wr) dw (37)

which is simply (26) now with f $_c$ in place of f 9, A few observations on these functional relations will e made below, but for the present we go on to their immediate consequences. Figure 6.7 depicts the semi-infinite medium with point source at (o,0,c). The medium is divided into two regions with the shaded region A and the conical region B, exactly analogously to the partition depicted in Fig. 6.6. Corresponding to (30) we now have the approximate solution:

FIG. 6.8 Relative placement of source (c) and observation (z) levels in (35) and (36).

z
~`3h
h(z,r) _ - (c$^-$ ~$_1$(c- z) + , (*,(t)+*,(t+c-z) dt 2nar
o

for z E c
41
c
V3h$_n$ --'Kar + 2 war
J
0
(Valid in region A, Fig. 6.7.)
for z > c
(38)
(39)

All the terms occurring in (38) and (39) were defined in (30). The ranges of integration may be visualized with the help of Fig. 6.8. Observe how (39) reduces to (30) when
c = 0. The errors of approximation are on the order of 1c^3/r^{5}j for (38) and 1z'/r'j for (39). The approximations (38), (39) are applicable far media with p = 0.6 or more.

Corresponding to (31) we now have
r

h ~h $_n$ ~ z , r) _ -----~ (1 + c v'3$^-$) cos 8 e -Kod (1 + Kod) 2wad
(Valid in region B, Fig. 6.7.)
(40)

Observe in this instance, also, how (40) reduces to its limiting case (31) for c = 0, where now in (40) we have written:

far r^2 + (z+ $_z$ -c) z 0
(41)

and also where
r
tan 9 = z + z - 0
(42)

The approximation (40) holds for large i z-c I and has an error on the order of magnitude of c/d' , for media with p = 0. b or more .

Observations on the Functional Relations for f_c and f_o

The various solutions displayed above for h(z,r) in a semi-infinite medium are of great interest for two reasons, The first reason is clear enough: They supply additional information on the behavior of h(x) in deep plane-parallel media in which there are point sources near the boundaries. The second reason for interest in these solutions does not exist so much on a practical level as on a theoretical or conceptual level. This interest centers on the form of the functional Z relations (35) and (36) which seem to hold considerable importance for radiative transfer theory. These two remarkable relations show how to connect the point source solution for the case c = 0 with that for the case c > 0. The general form of the functional relations (35) and (36) are those of the relations usually found by the techniques of invariant imbedding, the techniques growing out of the classical invariance principles of Chandrasekhar. It will be shown in Sec. 7.13 how the general counterparts of (35) and (36) for radiance fields may be deduced from the invariant imbedding relations (cf. also examples 2, 3, 5 of Sec. 3.9). As a result of the derivations in Sec. 7.13, there will be a

unified set of analytical techniques for solving internal source problems in general optical media.