7.2 Differential Equations Governing the Time Dependent R and T 0perators

We now extend the formulations of the preceding section to the time dependent case. The geometric setting and optical properties of the medium are unchanged except that now all functions in addition vary with time. The first step in such an extension is the derivation of the time dependent version of the local forms of the principles of invariance for a plane-parallel medium,

Time Dependent Local Forms of the Principles of invariance

We begin with the time dependent equation of transfer (4) of Sec. 3.15. For every y, such that a y b and in H , and time t in an interval E _ (to,ti):

1 DN(Y99,t) ¯a(Y.9t)N(Yv~jt) + N(Y,C vt)a(Y;t';&;t) dQ(&') v̄ Dt

., f

where we have written:

I'D it

Dt

for

ay at

Hence (1) above differs from (1) of Sec. 7.1 in only one essential respect: the presence of the. time derivative term. Therefore the transition to the time dependent versions of (5), (6), and (9) of Sec. 7.1 should be a straightforward matter. Thus, let N_+ (Y, t) , and N_ (y, t) be the upward and downward radiance distributions restricted to , E_, respectively, at level y in X(a,b) and at time t in the time interval E. E may be finite or infinite and is generally of the form (t_o, t i) , where t_o :st I . Furthermore we write

lip (Y,t)^{11} for (3) J~ekj

in which E isin E _ , and

11T(Y,t)11 for 1-0-~f C)a(Y;~';E;t) dQ(C')- - ^{1} a (Y9, t)

JE͵okj E + j E k j (4)

in which ~ is in w_+,

and in both of which a :5y _-sb , and t is in E. The requisite pair of equations now follows directly from (1)

MOR

D_N_+ (Y, t)

N_+(Yvt)T(Ypt) + N_(Yjt)p(Yft) (5)

DY

D_+N_(Y;͵ t)

N_(Ytt)T(Ypt) + N+(Yvt)p(Yft)(6)

Dy

where we have written:

"D_+^{11} for +a
+ 1 .a (7)

Dy ay I&·klv at

Equations (5), (b) are the required time dependent local forma of the principle a of invariance. As in the steady state case, each is obtained from the other by an interchange of + and - subscript signs throughout; the only salient difference between the time dependent and steady state sets is the time derivative term, as can be seen from (7). If we write
and:

¯T(Ypt) P(Yvt)

IT for

-p(YPt) T(Yft)

'IN (Y "
v t) for(N+(y,t),N_(y,t))

(9)

(12)

In applying D_o/Dy to N(y,t), all operations proceed as usual except that in the case of the time derivative term the derivative operator 8/at acts (say) first and then C acts on the resultant derivative. Equation (10) is the time dependent version of (9), For brevity of notation we will subsequently write:

II_u r t
for I t•kI

Time Dependent Invariant Imbedding Relation

The next step in the present-discussion can be made on any one of several levels of generality. Since our present goal is a set of time dependent versions of (18). $(27)_v$ (28), (29),, of Sec. 7,1,. the most immediate route is the development of the time dependent principles of invariance, along with the R and T operators they govern. We could develop the latter principles very much after the pattern set in Examples 2 and 3 of Sec. 3.7. However, with only slightly more effort we could outline the development of the time dependent version of the more general invariant imbedding relation-, following the pattern of miscellaneous Example(iv)of Sec. 3.17. This we now do, as it affords some further illustrations of the interaction method of Chapter 3 as we make our way towards the present goal. It will also allow us to illustrate once again how the principles of invariance (now in time dependent form) are derivable from the more basic invariant imbedding relation.

Following the three main stages of Sec. 3.18, for Stage I, let X(x,z) be the isolated subset of the optical medium X(a,b). Let the current set of radiometric functions be radiance distributions defined over the plane surfaces X_y of X(x, z) , x :~y ::sz and over a time interval E. Thus N+ (y,E) and N-(y,E) are upward and downward radiance functions defined on the general plane XY over the time interval E.

The sets of incident radiometric functions are enumerated as:

A1

all incident radiance functions like $N_+(z,E)$

A2 all incident radiance functions like N_(x,E)

The sets of response functions of interest are (x :~ry :sz)

Bi : all response radiance functions like $N_+(y,E)$

B2 : all response radiance functions like N_(y,E)

The¯interaction principle then asserts the existence of four interaction operators sij:

s $_{1l}$ -- 7 (z,y,x,E) s12 -- a(z,y,x,E) s_{zl} -- jo(x,Y,z,E) s2$_2$ -- Cr(x,Y, z,E) The corresponding interaction equations thus are:

.N_+(y*E) $ N_+(z,E) $v^{0'''r}$(z,Y,x,E) + N_ (x,E) a(x,Y,z,E) (13)

N_ (YpE) = N_ (x,E) Z"(x,Y,,z,E) + N_+(z,E) a(Z,Y,x,E) • (14) The requisite invariant imbedding relation then is:

(N+(y,E),N.(y,,E)) _ (N+(z,E),N_(x,E))~(x,y,zjE)

(15)

where we have written:

I'M (x.Y. z E)" for

~(z*Yfx.E) 4?(ZPYPx,E)

(1G)

.a(x,Y, z,E)~r(x.Y. z,E)

Integral Representation of. Time Dependent and ,+*¯r operators

we currently need to go further than the preceding operator statement of the invariant imbedding relation. In particular we wish to obtain specific representations of the

and 9' operators as integral operators over the time domain E. Thus we must enter Stage II of the interaction method, To set Stage II in motion, we choose an arbitrary time t in E and hold it fixed until further notice. Next, consider operator ~r(x,y, z,E) , Choose and fix a point p in X_y and fix a direction 4 in E., . Then, by (13), for every N- (x,E) in A_z

N~(p,E,t) a CN.(x,E)7(xyyjzqE)3(pV'E0t) (17)

is a non negative number--indeed, it is the radiance N_(p,&,t) induced by N _ (x, E) . Thus in the present setting with fixed

p, &, t, the operator 0'`(x, Y, z , E) is a positive linear functional on Ai. By Theorem A of Sec. 3.16 there is a measure jj(x,y,z,•,p,~,t) on the set E such that:

.7(x,y,z,E)
dv(x,y,z,&,p,~,t)
E

(where p,t,t are implicit in the notation on the left) so that (17) may be represented as:
N_ (P f ~ p t) * I N(x, t') du (x,y, z, t' ,P, ~, t) . (19)
E
where N (x, t I) is the value of N_ (x, E) for the variable t' in E. The next step is to observe that the measure v(x,Y,z,',P,t,t) is absolutely continuous with respect to the time measure on E., This simply amounts to -the physically based assertion that: for every subinterval F of E, if F is of zero duration, then: v(x,Y,t,F,p,t,t) a 0. In other words X(x,y,z,F) will not transmit any finite incident radiance N,. (x, F) where F is of zero duration. Thus the measure has the AC property and Theorem B of Sec. 3.16 asserts the existence of a kernel function 7^{-}(x,y, z, t' ,p, t, t) such that
u (x,y, z, G,,p, E, t) = I X (x , y , , z , t' ,p, t, t) dt' (20)
C
for every subset C of E. Theorem C of Sec, 3.16 now lets us write (IS) as
1 [ ] 3 (x , y , z,t' ,t) dt' (21)
E
where we have suppressed the p and ~ in going from (20) to (21), since they were arbitrary, and we now wish to work on the function level. In this way we arrive at the following integral representation of (17). For each t in E

N_ (x,E) 7^{*}(X,Ypz$_{t}$]E) - N_ (x, t') 7(X,Y,Z,t^{l},t) dt' , (22)
E
It should be noted that .7(x, y, z, t' , t) just found is the time dependent version of the complete transmittance operator defined in Sec. 3.7 and is itself an operator which, under suitable regularity conditions, can be represented as an integral operator over E= _ and over the upper boundary X_x of X(x,z), and which acts on downward incident radiance distributions over X_x. Since this particular type of integral representation has been used in the steady state studies throughout this work, there are ample examples of such operators which the reader may turn to, so that we may go on with the main line of discussion:
In a similar way we define the remaining three complete and *7r' operator kernels and derive the remaining three integral 'farms of the operations in (13) and (14)

N_{+}(z,E) X(z,Y,x,E) = N+(z,t v) X(z,Y,x,t.* ,t) dt' (23)
E

N_ (X,E) a (x,Y, z,E) =N_ (x, t') a(x,Y, z, t' t) dt' (24)

E

N_+(Z,E) A(z,Y,x,E) '~ N+(Z,t I) .gR(z,Y,x,t' ,t) dt' .(2S)

E

This^{-}type of integral of N_+ with the j&R and)' operators occurs so often, let us agree to write:

"f (t)g(t)" for

' f(t)g(t) dt (26) E

where f and g are any functions or operators such that for every t the "product" f(t)g(t) is defined over E, The "product" could be the customary multiplicative numerical type, or matrix type, or general operator type. With this convention we may write the invariant imbedding relations (13) and (14) as:

N+(Y,t) " N_+(zit') (ZRY#x,t^{v},t) + N_(x,t')(~?(x,Y,zJ',t) (27) N.. (y,t) '~ N+ (xit')7(x,Y, z t) + N_+ (z,t') 6t (z, Y,x,t' ,t) (and so (27), (28) can be written succinctly as:

(N+(y#t)*N.(yvt)) = (N+(Zp"f')*N_(Xpf,))I(xsyvzo-f,pt)

(29)

Time Dependent Principles of Invariance

From the time dependent invariant imbedding equations (27), (28), we can deduce the four time dependent principles of invariance for plane-parallel media. First, in analogy to

(44)-(47) of Sec. 3.7, we write:

"T (x, z., t I *^{1}

$_{o}$ t) for t7'(x, z , z $_{9}$ t' , t) (30)

"R (x, z,t' ,t) " for 6?(XVxV z,-t' ,t) (31)

and require:

(32)

(x, z, z, t' =

, t) 0 (33)

Definitions (30) and (31) define the time dependent versions of the standard reflectance and transmittance operators for X(x,z). If we had derived these standard operators directly from the interaction principles (after the manner of Ex. 3, Sec. 3.7), then (30) and (31) would have become derived equality statements (as in the case of (44), (45) of Sec. 3.7),

In the time dependent setting we impose two further conditions on R and T which are useful in numerical work, as well as theoretical manipulations, namely:

and

R(x, z, t' , t) a 0fort - t' < 0 (34)

T (x, z , t' , t)0 for t - t' < . I z-x I /v . (35)

Conditions (34) and (35) are causality conditions, whose physical significance is readily seen. For concreteness in the present formulations, we will specify the time interval E of the general derivations above, as the interval (-*°,t), where t is an arbitrary fixed time throughout a given discussion.

With these conventions and observations, we can write down the four time dependent principles of invariance for X(x, z) and X(a,b), a ~5x:sy s-.z mob, after the manner of Ex. 3 and Ex. 4, Sec.¯3.7, as follows:

Letting x = y in (27) and using (30), (31)

$N_+(y,t) = N_+(z,t')$ T(Z.,yet',t) + N_ (yft')R(y,z,t' ,t)

Setting

in (28) and using (30). (31)

II. N_ (y,t) N_ (x,t')T(x,y,t; ,t) + N+(yft')R(yfx:t',t)

Using I twice, first let y = a, z = b; then y = a, with z arbitrary:

III. $N_+(a,t)$ = N+(b,t')T(b,a,t',t) + N_ (a,t') R(a,b,t' ,t) = N+(z,t')T(z,a,t',t) + N.(a,t')R(a,z,t',t)

Using II twice: first let y = b, x = a; then let y = b with x arbitrary

IV. N_ (b, t) = N_ (a, t') T (a, b, t' , t) + N_+ (b, t') R(b, a, t' , t)

N_ (x,t')T(x,b,t' ,t) + N+(b.,t')R(b,xj'.,t)

Differential Equations for the Time Dependent R and T Operators

The differential equations for'the time dependent R and T operators may be derived by imagining a powerful short pulse of light pumped into X(a,b) at its upper boundary X_a. The directional structure of this incident radiance distribution may be arbitrary as also its dependence with location on X_a. We shall assume that this is the only source of flux in X(a,b) and that N_ (a, t) is such that N., (a, t) = N_ (a, t_o) 6 (t-t_o) t_o ~ t, where t_o is the time at which the pulse is incident on 'Xa.

The subsequent operations with Dirac-delta functions are governed by the usual conventions which may, e.g., be found in [95]0

We begin by applying the operator D,./Dy (for time variable t) to principle I in which we have set z = b, and have used the fact that N_+ (b , t') = Q for every t' c t

D_N_+(Y0t) D_N_(Yvt') _ D„R(Yi bft'9t)

Dy DY Dy
(36)
Next the operation lim $_{ia}$, is applied to each side of (36). The left side of (36) yields, by means of (5)
N_+ (a, t) T (a, t) + N_ (a, t) p (a, t)
= N_ (a, t_o) [R(a,b, t_o, t) T (a, t) + 6 (t-t_o) p (a, t)]
The second equality is derived from principle III and the adopted form of N_ (a, •) . By (5) and (6) we have, on applying
the operator D-/Dy (for time variable t) to N_ (y,t')

D _ N- (a, $_t$ 1 6, (t, $_{-t}$ } - a (t, $_{-t}$) T (a, $_{t,)}$ _
y-joa DY o uv o 0
_ R(a2bvto*t')P(a,t'3
where 6'(t'-t_o) is the symbolic derivative of the Dirac delta function with respect to t'. Finally:
D_R(y,b, t' $_t$ t) $_s$ _ 3R(a,,b, t' ,~,.t) 1 aR a b t' t)
1 im + -.. .
Y$^-$'$ Dy a a UV a t
Assembling al-1 these results and using them in (36), rearranging (35), and cancelling the arbitrary function N-(a,•), the resultant operator equation is obtained:
3R(a,b,t_o,t)
.+

3R(a,b, t_o, t) aR(a,b, t$_©$., t)

-- 6 (t-t_o) p(a,t) + -r (a,t_o)R(a,b,t_o,t) + R(a,b,t_o,t)T (a,t) +
+ R(a,b, t_o, t') P (a, t') R(a,b, t' $_t$ t)
Now starting with principle II and applying in turn the operation D_+/Dy and the limit operation lim $_+$b, and then making use of (5), (6) and principle Iv, we have
y in a similar way:

II^I aT(a,b,t_o,t) . $_1$ DT(a,b,t_o,t)
ab uv atT(a,b,t_o:t)T(bpt) +, T(afbvtoft')P(b,t')R(b,a,tt,t)

The third differential relation follows from principle III by applying the same general procedure used to establish I' above:
IIP aR(a,b,t_o,t)
ab
T(asb,t_o,t')P(b,t')T(b,a,t^t,t)
The fourth differential relation follows from principle IV by applying the same general procedure used to establish IIt above

IV
a T (a,b, t_o, t) $_1$ aT (a,b, t_ot)
-

as uv at$_o$

r(a,t_o)T(ajbvt_o*t) + R(a,bptopt')P(-a,t')T(a,b,t',t)

Discussions of the Differential Equations

The set of equations I'-IV' above is the desired set of differential equations for the time dependent R and T operators for plane-parallel media. These operators are homogeneous with respect to time--i.e., they depend on the difference jz-xj of depth parameters--only if the medium is separable, where "separability" by definition means that v/a is independent of depth (so that a may be separated into two factors: one spatial, the other directional). When the medium is homogeneous--or more generally, separable--then there are precisely two R and T operators associated with X(a,b). However, when X(a,b) is not separable, then there generally are four R and T operators for X(a,b)$^-$: a reflectance-transmittance pair for flux incident on X $_g$, and a pair for Xb. Thus in nonseparable media, R and T ex ibit polarity, i.e., we have

R(a,b,tt ,t) 0 R(b,a,t' ,t) or T(a,b,t t ,t) 0 T(b,'a,t',t) for some t',t in E. The general order of solution of the preceding equations is the same as the steady state case; thus one may solve the above system in either the order I',IV;II',III' or I',IV^1,III^1,II'. Furthermore, the pair I^1,IV' is the autonomous pair of the set of four equations in the sense that they determine R(a,b,t',t) and T(a,b,t',t) for X(a,b); and by interchanging "a" and "bt" throughout and wherever necessary, the algebraic signs of the spatial derivatives, they also determine R(b, a, t' , t) and T (b, a, t t , t) . Hence I' and IV' may be used for the determination of all four R and T operators for X(a,b). Numerical procedures for the solutions of I'-IV' may be constructed for the set in either undecomposed form or in decomposed form (cf. Sec. 7.1) of the T operators.