7.3 Algebraic and Analytic Properties of the R and T Operators

Consider a thin layer of scattering material in a plane-parallel medium. Suppose that this layer and another layer twice as thick are irradiated with radiant flux in the same manner. Is the reflectance of the latter layer twice that of the first layer? Intuition seems to say yes. Another question we may ask concerns the transmittance of the doubly thick layer relative to that of the layer of half its thickness. Intuition says the transmittance is simply the square of the single thin layer. In certain special cases both these intuitive guesses are essentially correct. But what of the general relation between the reflectances and transmittances of a medium of arbitrary thickness with those of its parts, arising under a general partitioning of the medium? Is there a general formula which relates the reflectances and transmittances of the 'sum' of two parts with those of each 'summand'? In this section we answer such questions--for the case of an arbitrary, stratified plane-parallel medium, The various formulas we shall find are characteristic of the general case, i.e., they are essentially unchanged if one makes the transition to more general geometries and asks the same questions there. Hence the derivations which take place below are algebraically representative of the derivations in the more general settings, but the details have the advantage of being intuitively and analytically simpler than the general case.

Partition Relations for R and T Operators

The setting for the present derivation is depicted in part (a) of Fig, 7.2 in which a plane-parallel optical medium X(a,c) has been conceptually partitioned into two parts X(a,b) and X(b,c), we ask: what is the connection between the operators R(a,c), T(a,c) of X(a,c) and the reflectance and transmittance properties of its parts X(a,b) and X(b,c)? Another way of looking at essentially the same problem is to imagine that a given medium X(a,b) is imbedded in a larger medium,

X (a, c) by the adjunction of the given layer X (b, c) , and it is required to find the properties of X (a, c) in terms of the imbedded medium X(a,b) and the added medium X(b,c). See (b) of Fig. 7.2. Throughout the present discussion in our quest for the answer to the preceding question, we will draw freely on the concepts and relations developed in Secs. 3.6 and 3.7, especially Examples 2, 3 of Sec. 3.7-.

We begin by assuming that X(a,c) is irradiated by an arbitrary N- (a) and with $N_+(c)$ 0. Using principle of invariance III of Ex. 3 in Sec. 3.7 applied to X(a,b), we have:

$N_+(a) = N_-(a)R(afb) + N+(b)T(b,a)$ (1)

Principle III is again applied, now to X(a,c), to yield:

. $N_+(a,) = N_-.(a)R(a,c)$ (2)

FIG. 7.2 Deriving the Partition Relations for R and T Operators, by using the condition $N_+(c) = 0$.

Next, (25) of Sec. 3.7 is adapted to X(a,c) by replacing each "y" by "b" and each "b" by "c" in that equation. The result is:

$N_+(b) = N.(a)T(a,b)R(b,c)\ [I - R(b,a)R(b,c)]'=$ (3)

in which we have again used the condition $N_+(c) = 4$. Using (2) and (3) in (1), and noting that N- (a) is arbitrary, we obtain the first of the desired partition relations:

$R(a, c) = R(a,b) + T (a,b) R(b, c) [I- R(b, a) R(b, c)]^{-1} T(b, a)$

Before giving the physical interpretation of (4), we go on to find. its companion formula for T(a, c) . Using the same incident lighting conditions as before, we appeal to principle Iv of Ex. 3 in Sec. 3,7 applied to X(b,c):

N_ (C) = N_ (b) T (b , c) (5)

in which $N_+(c) = 0$ was used,

Principle IV is again applied, now to X(a,c):

N_ (c) = N_ (a) T(a, c) (6)

using once again the condition $N_+ (c) = 0$. Next we use (2¯6) of Sec, 3,7 adapted to the present case by replacing each "y" by "b" and each "b" by "c" in that equation. The result is

N_ (b) - = N_ (a) T (a, b) [I - R(b, c) R(b, a)] ` (7)

using the condition N+(c) = 0.

Using (7) in (5) and using (6) to represent the left side of (5), and also recalling that N_(a) is arbitrary, we obtain:

T(a,c) = T(a,b) [I - R(b,c)R(b,a)] -'T(b,c)

which is the second of the two desired partition relations.

We now-discuss some of the properties of (4) and (8). First of all, we see that our simple intuitive guesses about R(a, c) and T (a, c) givers at the outset of this section are hardly correct for general media. The presence of the term [I -R(b, c) R(b,a)] ~' in each equation (recall the observation on (28.) of Sec, 3,7) represents the complex activity of interreflections between X (a,b) and X (b, c) . However, to see that our intuitions are not wholly misleading, suppose this interreflection factor were absent from (4) and (8) or practically equal to I, This occurs when, e.g., the media X(a,b) and X(b, c) are optically thin so that R(b, c) and R(b, a) are very small. For example, to within first order of infinitesimals we have from (19) of Sec. 7,1:

R(a,b) = R(b, a) = p (b) I b-aI (s)

R(b, c) = R(c, b) = p (b) Ic¯b I

(10)

where the numerical differences 1-c-bI and Ib-aI of layer depths are small compared to the attenuation length $L_a = 1/a$. (This is what it means for X(a,b) and X(b,c) to be "optically thin".) Similarly, from (27) of Sec, 7.1 we have, retaining only the first order of infinitesimals:

T(a,b) = T(b,a) = I + T(b) Ib-ai (11)

T (b: c) = T (c,b) = I + 'r (b) jc-bj (12) where I is the identity operator and once again the absolute values Ic^{-}bl and Ib-al are small compared to L_a. It may be noted that the argument "b" of the p and T operators may be replaced by "a" without changing the validity of the approximations.

With (9)-(12^{-}) in force, (4) becomes

R(a,c)= R(a,b) + R(b,c)+ o(I c^{-}b I) (13)

and (8) becomes:

T(a,c) a T(a,b)T(b,c) + o(I c-b I) (14)

where "od c-bl)" denotes a quantity which goes to zero faster than the difference Ic-bI, that is lim o(x)/x = U. Thus for practical work with optically thin
x^{+}~ media X(a,b),X(b,c),
one may drop "o(I b-c I)" from (13) and (14) ; the result is a pair of equations which bears out the intuitive guesses stated in the introductory remarks above. In brief, with respect to composite properties of optically thin media, their reflectances add and their transmittances multiply.

We next cast (4) and (8) into alternative forms using the complete reflectance and transmittance operators associated with X(a,c). The advantages accrued from such a reformulation are both formal and intuitive. Thus from (40) of Sec. 3.7 now adapted to X(a,c) by replacing each "b" by "c" and each "y" by "b", we have as an alternate to (4):

R(a,c)= R(a,b) + fz(a,b, c) T(b, a)
(15)

and (8) becomes:

T(a,c) = T(a,b,c)T(b,c)
(16)

using 42) of Sec. 3.7 suitably adapted to X(a,c). The relation o (Ib), to (51) of Sec.'3.7 should not escape notice. We see also that (15) is an important addition to the family of functional relations studied in Chapter 3, of the semi-group type for interaction operators over X(a,c). We shall see repeatedly below and in Sec. 7.4 the important uses to which (15) and (16) and their generalizations may be put. The relations (15) and (16) characterize the alternate point of view suggested in the introductory remarks; namely, that X(a,b) may be considered as imbedded in a larger medium X(a,c). The operators .Gt'(a, b , c) and 0" (a, b , c) point up this alternate view, being the AR and X operators of the invariant imbedding relation.

Similar formulas hold for R(c, a) and T(c, a) associated with X(a,c). For purposes of reference these are given below:

R(c, a) = R(c,b) + .42(c,b, a) T (b, c)
- (17)

T(c,,a) _ J(c,b,a)T(b,a) '(18)

Partition relations (15)-(18) will be generalized to the case of the R and Z' operators in Sec. 7.4 (see (7G) of Sec,-7,4),

Alternate Derivations of the Differential
Equations for R and T Operators

In Sec. 7.1 we derived the differential equations I'-IV' for the four R and T operators associated with X($_a$,b).-- One of the main ingredients of the derivations was the local forms of the principles of invariance. (S), (s) of Sec. 7.1, i.e.,,the equation of transfer in operator form. The purpose of the present discussion is to show how one may derive the differential equations for the R and T operators without direct recourse to the equation of transfer. The knowledge we gain, from such a tactic is of great theoretical importance: by deriving the differential equations for R and T directly from (15) and (16) above, we show that the theory of radiative transfer can be made to rest on the principles of invariance,. the equation of transfer then being a law derived incidentally from the principles. This point of view of radiative transfer was explored in detail in Ref, [251]. We now present a simple exposition of this matter in the setting of plane-parallel media. We begin with a derivation of the simplest of the four differential equations from (4), namely, (28) of Sec. 7.1. Rearrange (4) as follows:

R(a,c) -
RLa'b) T(a,b) R b c
- (19)
(c-b) (c-b) [I R(b,,a)R(b,,c)]-IT(b,a)

we next assert that:

-
1im Rac Rab 3Rab (2v)

cab (c-b) ab

1im R (b c
= (21)
p (b)

cab (c-b)

1im T(b,c) - I= z (b) (22)

c-I-b (c -b)

for every

a,b,c: acbsc (23)

Observe that in (20)-(22) we are boldly asserting the existence of these limits, and are not deducing them from such established results as, e.g., (9)-(12). In effect we are defining the members on the right sides of (20)-(22). This is in accordance with our present aim of starting with the principles of invariance as initial points of the derivation of the theory. In the present approach statements (20)-(22) are then called regularity properties of the R and T operators occurring in the statements of the principles of invariance. Any physical theory using the calculus and the concept of limit in particular must, somewhere along the path of its construction, in effect assume regularity properties of the principal functions of the theory under study. In the case of radiative transfer theory, a detailed and systematic enumeration of such properties was made in Ref. [216]. In the present work, commensurate with its different goals, these properties were for the most part implicitly assumed as each was needed (see, e.g., Secs. 3.10-3.15).

With the preceding assumptions in force, we let c approach b in (19). Physically, this amounts to letting the slab x(b,c) in Fig. 7.2 approach zero thickness. Mathematically, this results in the statement: .

R L a $_{j}$, b) = 7(a,b)P(b)T(b,a) (24) 3b

which is (28) of Sec. 7.1.

We next derive from (4) the most complicated of the four differential equations, namely (18) of Sec. 7.1. Subtracting R(b,c) from each side of (4) and using (28) of Sec. 3.7 to establish the fact that

(I - R(b, a) R(b, c) I . 1 = I + R(b, a) R(b, c) + o (b-a !) (25)

(which follows from (21)) we can rewrite (4) as

R a$_{t}$c _ - RC~,C2 a R a b + Tab R b c T b a) - R b c)

b-a b-a b-a

T(a,b)R(b,,c)R(b,a)R bac)T. (b . y a I + o f b¯a)

b-a b-a (26)

Applying (22) to the operator T(a,b) and T(b,a) we have: T(a,b) = I + T (a) r b¯a I + of (I b¯a I)

T(b,a) = I + T (a) (b-al + a2 (Ib-al)

where oi(Ib-al) and ox(Ib-al) are analogous to o(Ib-al) defined above. Hence:

T(a,b) R(b,c)_T(b,a) -R(,b,c) = R(b,c) T (a) + $_{T}$ (a) R(b$_{,c}$) + of b-a)

b-a b¯a

Postulating that:

we then see that equation (26), under the application of the limit operator lim , becomes:
b+a
8R a ca p (a) as
+ T (a) R(a, c)+ R(a, c) T (a) + R(a, c) p (a) R(a, c)
which is (18) of Sec. 7.1 in the setting of X (a, c) . We may go on to deduce (27) and (29) of Sec. 7.1 from (8) in a similar manner, However, the point of the present derivation now seems well wade and we leave such details to the interested reader and pass on to the next matter of the present section.

Asymptotic Properties of R and T Operators

How do the reflectance and transmittances of optically thick media behave with depth of the media? As in the case of optically thin media (cf. (13), (14)) our intuitions supply some rough answers to this question. In the case of reflectance, imagine an observer over a horizontally extensive homogeneous fog bank illuminated by the sun. the fog bank is optically very thick and is virtually blinding to the observer. Suppose now that, as the fog is under observation, it is noticeably decreasing its depth. However, the brilliantly reflected light does not seem to lose any of the intense magnitude until the final stages of dissipation. From common occurrences such as this we form the opinion that as the optical depth of a very deep homogeneous optical medium increases, there is eventually no appreciable change in its reflectance properties so that an upper limiting value of R(a,b) is expected as lb-al increases without bound. On the other hand, to an observer on the ground below the great fog bank every bit of decrease in the thickness of the layer is noticeable as a corresponding increase in the general radiance distribution transmitted down to the observer. From recollections such as this, we form the expectation that the transmittance T(a,b) should decrease rapidly to zero as lb-al increases without bound. We now show how these empirical facts are borne out by means of simple arguments using the differential equations for the R and T operators. A more detailed and rigorous analysis of the present ideas will be made in Chapter 10, For the present we simply pursue these ideas on a heuristic level. That is, we shall attempt to discover the requisite properties of R(a,b) and T(a,b) by treating them as if they were numerical magnitudes and the equations they satisfy ~as ordinary algebraic or differential equations of numerical valued functions.

Suppose a slab X(a,b) is imbedded in an infinitely deep homogeneous plane-parallel medium X(a,co). The homogeneity of X(a,oo) requires p(y) and T(y) to be independent of y for all
y such that a : y. It follows from (18) of Sec. 7.1 that the

differential equation for R(a,b) is:
-aRab a p + TR(a,b) + R(a,b) T + R(a,b) pR(a.b) • (27) as
From this we infer at once that R(-a,b) = R(b,a) and the common value depends only on the difference lb-al. Equation (28) ~of Sec. 7.1 implies that

ab

since T(a,b), T(b,a) and p are analogous to positive valued functions. Thus. we recover within the theory the empirical fact that for fixed a, R(a,b) b, and. hence lb-al increases: -It is a bit more difficult to establish:

aT a b~

ab

i.e., 'the fact that for fixed a, T(a,b) decreases as lb-al increases. This can be made plausible by noting that-the term T+ R(b,a)', in (27) of Sec. 7.1 is negative when there is absorption but no scattering in the medium, i.e., when a = 0 and a >0.' Since T(a,b) is positive, (27) of Sec. 7.1 then implies (29) above. However, by slowly increasing a from [3- to small positive values, the inequality (29) clearly persists, for a while ;and indeed, in all natural optical media, (29) can be shown to hold with only mild regularity properties imposed.

. From (29) and (28) of Sec., 7.1 we now can see that:

b-o-w ab

0

so that, by (27). above (i.e., since R(a,b) = R(b,a))

p + TR(a,°°) + R(a,w) T + R(a,°*) pR(a,c*). = 0 (30)

where

we have written:

"R(a,w)" for lim R(a,b) (31) b+~

Equation (30) shows that R(a,co) is independent of a since p and T are. This property was formally used by Ambarzumian in (1] to derive some of the earliest forms of the integral equations indigenous to the invariant imbedding point of view of transfer phenomena. When certain reciprocity conditions hold for the medium, we have:

R(a,co)T = TR(a,co)

and

R(a,(*) p = pR(a,°°)

i.e., we have commutativity of the T, R, T and p operators,

Under such conditions (which hold, e.g., when scattering is isotropic) (3©.) becomes

p + 2TR(a,m) + pR^2 (a,°°) a 0 (32)

The solutions of (30) or the special case (32) yield the form for R(a,°°). A numerical procedure leading to R(a,b) for a range of finite b (from which R(a,W) is estimable will be given in Sec. 7.6 for spaces x(a,b) in which scattering is isotropic. We shall return to these heuristic operations with the R and T operators in Sec. .8.7. The operations just performed can be redone in the irradiance context and can be made fully rigorous without the need for advanced mathematical techniques. See (35)-(38) and (39)-(42) of Sec. 8.7.