7.4 Algebraic Properties of the Invariant Imbedding operators The various invariant imbedding operators introduced in examples 4-7 of Sec. 3.7 will now be studied in greater detail. Our main purpose in the present section will be to demonstrate the fact that the collection r2(a,b) of operators of the form 7f (x,y) , which we found in Example 4 of Sec. 3.7-to constitute a partial group, may be used as basic building blocks to systematically construct, via simple algebraic procedures, all other operators of the collections r3(a,b) and r4(a,b), and hence all R and T operators and their simple combinations. The net result of these possible constructions will be novel procedures for solving transfer problems in plane-parallel and, indeed, general optical media. In other words, we shall demonstrate that the operators M(x,y) can serve as the computational work horses on both theoretical and practical levels in the theory of radiative transfer and thereby have them earn their right to reside among the giants, the elements of r4(a,b), which in turn serve to unify the theory and to link the theory with the interaction principles. Throughout this section, unless stated otherwise, we shall work with an arbitrarily source-free plane-parallel medium X(a,b), a s b, with arbitrary incident radiance distributions $N_-(a)$ and $N_+(b)$ over the upper boundary X_a and lower boundary Xb respectively. Generalizations of the indicated results to general one-parameter media are immediate; generalizations to arbitrary media can be patterned after the discussions of Sec. 25, Ref. [251]. Throughout the discussion all reference to various regularity properties required for inverse operations, differentiations, integrations, etc., has been avoided so as to bring out the highly intuitive flavor of the operator algebra.

The Operator M(x,z)

The simplest interaction operator associated with a general plane-parallel medium X(a,b), a :sb is that which maps (or transforms) the pair (N+ (b) ,N_ (a)] of incident radiance-distributions on X(a,b) into the pair (N_+ (a) ,N_ (b)] of response radiance distributions for X(a,b). It is a simple exercise in the use of the principles of invariance for X(a,b) to determine this operator. Thus, from principle III., in Example 3 of Sec. 3.7, we have:

N_+(a) = N+(b)T(b,a) + N-(a)R(a,b) and from principle IV we have

N_ (b) = N_ (a) T (a,b) + N_+ (b) R(b, a) The matricial form of this system of equations is:

(N+(a),N.(b)) (N+(b),N_(a)) T(b,a) R(b,a) R(a,b) T(a,b)

The displayed matrix of R and T operators is the requisite interaction operator. More generally, let X(x,z) be an arbitrary plane-parallel subset of X(a,b , a < x s z :sb, and suppose (N_+(z), N_(x)) and (N_+(x), N_(z) are, respectively, the incident and response radiance distributions on X(x,z) as they exist in the medium X(a,b) which is irradiated by an arbitrary set N_+ (b) , N_ (a) of radiance distributions on its lower and upper boundaries, respectively. (See Fig. 7.1.) Then principle I in Example 3 of Sec. 3.7 yields for the case X = Y'

N_+(x) = N+(z)T(z,x) + N_ (X)R(x, z)

Similarly, principle II yields: for the case y = z:

N_(z) = N_(X)T(x,Z) + N_+(z)R(z,x)

The matricial form of this system of equations is:

(N (x),N.(z)) (N+(z),N_(x)) T(z,x) R(z,x)

R(x, z) T (x, z) (1)
Let us write:
$^{l'}$M (x . z) ^{11} for T(z,X) R (Z , X) (Z)
R(x, z) T (x, z)

where a s x s z s b . Thus M(x, z) is a 2x2 operator matrix which is defined for depth variables x, z such that the preceding equalities hold. Some experimentation with (1) will show why this restriction (namely x :sz) is necessary if we are to retain the useful convention of always writing radiance distribution pairs with the upward (+) distributions as the first member of the pair. Another advantage in preserving the fixed order of variables x,z in M(x,z) shows up in the detailed computations below wherein it will always be clear whether an operator on an upward or downward flow in X(a,b) is being represented. Thus in all that follows, M(x,z) with x:5 z is a useful conceptual anchor whose components have simple physical

significance. Let us denote by '^{1}G_2 (a,b) f f the set of all operators M(x, z) , a 5 x :s z s b.

The Connections Between M(x, z) , M(x, z) , and n (z , x)

We now establish the connections between the operator M(x, z) and the operators -T(x, z) , 7 1 f (z , x) in the setting of an arbitrary sub-medium X(x, z) in X(a,b). (Recall (78) of Sec. 3.7.) Once this connection is established, we will have an effective means of computing M(x, z) and ?M(z,x) in terms of the standard R and T operators for X(x,z): and conversely, the operator M(x,z) will be directly representable in terms of the operators M(x,z), M(z,x). This latter representation will be a prototype of more general representations of the members of r3(a,b) and r4(a,b) to be derived subsequently, and will be instrumental in developing novel methods of solution of light fields in X(a,b), later in this chapter.

To establish the requisite connections we require the partition of the identity operator I on rz(a,b):

In C_+ and C_, I_+ is the identity operator$^-$on the set of all* upward radiance distributions and I- is the identity operator on the set of all downward radiance distributions associated with X(a,b). No confusion will result if in the subsequent discussions we drop the signed subscripts from the identity operators (their positions in the matrices provide adequate identification). The general working properties of C_+ and C_ are obtained by direct computation:

C+ = C_+ (6)
C^2 = C

C_C

a b a 0

(IC - 0)

\c d/

b
(a d) 0 b
C (0

\c d

We drop +,- on I_+ when direction is clear.

Ic

a] to o,

al Ic al

(a,b)C_+ = (a.0)
(a,b) C_ = (©,b)

Hence, via, suitable pre- and post-multiplications by C_+ or C_, various elements, of .a matrix of operators or of a vector can be isolated as needed.

Now, equation (1) holds for all incident radiances (N+(z),N- (x)) on ·x(x, z) * From the definition of the operators ¯Mx, z) and· Mt z,x3 and the partition operators of I, we have:

(^{N}+(x)· 0) = (N+(Z).oN_(Z))~'(Z,X)C+
(0 N. (z)) (N_+ (z) N_ (z)) C_

Adding, we have:

(N'$_+$ (x) N._ (z)) _ (N_+ Cz3 N_ (z)) [~(zx) C+ + C..] (7)

Further:

(N_+ (z) 0) (N_+ (z), . N. (Z) C_+ (O.N.(x)) (N+(z),N.(z));~(z,,X)C.

Adding, we-have:

(N+(z),N.(x)) _ (N+ (z),N_ (z)) C_+ + ??j(z,x) C_ (8)

Combining (1), (7) and (8).

N_+~z) ~ ·N.¯(z)) [C_+^+ (zix) C-7M(x=z)$^=$ (N+(z) N_ (z)) [*Z,X)C++C_]

This holds for every incident light field on x(x,z). Hence:

"((Zfx)CAM(xfz) = C"{ (ZVX)C+ + C-1

whence

On the other hand, solving (9) for ?7(z,x) , we have:

2r(Z fx) [C-M(X*Z)
[C' - C+M(x, z~~

whence:
On.taking inverses of each side of (11):
'7(x, z) = Ec $_{-}$M(x f Z)
7 Ec _
C+M(x $_{o\ z)}$]- i

This may be solved for M(x, z) to yield a companion formula to (10):
M(x,z) [N.(X.'z)C+ + Q $^{-1}$ [^{C}+ + M(x'z)C

Equations (10)-(13) are the desired connections between the 'operators M(x, z) , ?7(x, z) , and ?~(z,x) for levels x, z in X(a,b) with x :sz.
Invertibility of Operators
The inverse operators in the preceding representations can be examined in detail so as to allow us to establish some conditions sufficient to insure their existence. The inverses generally encountered in computations with (10) -(12) are of the form:
1C$_{+}$ + AC$_{-}$ I - [AC+ + Q _ i [C. - C+Aj _. Ic--A C+1
where

bl dJ
"A" denotes either the 7j or M matrices so that a, b, c, and d are generally operators on radiance distributions. To evaluate these inverses consider for example the first; we require a 2x2 matrix with elements a, 0, Y, 6 such that:

I
0 1 -
From this are obtained the four equations: $_{a}$+ by = I
0 + bd = 0 d = 0 dd = I which in turn determine the elements of the inverse: a = I
-bd^{-} '
y = 0
$_{d}$-1
a ~ i o a ~
IC$_{+}$ + AC-1
v R fI r~ v fi
(14)
The remaining three inverses are obtained similarly:
-a-1 -a-lb
r
0 I
0 L-d_^{1}c d^{-}'
(17)
From an inspection of this collection of inverses i.t is clear that their existences depend in turn on the existences of the inverses of the component operators a and d in A. When A is M(x,z), this requires the transmittance operators T(x,z) and T(z,x) to have inverses. In most natural optical media (oceans, atmosphere), the volume scattering function a and volume absorption function a are positive throughout the media. This property of a and a generally insures the norm contraction property of I-T(x,z) or I-T(z,x)

so that under these conditions the inverses of T(x, z) and T(z,x) exist. Of course in any specific instance, it is good practice to have

the invertibility of the transmittance operators verified in detail. In the present discussions our interest is solely in the algebraic structure of and interconnections between the various interaction operators, and the discussion proceeds on the assumption that all required regularity properties are in force*

Representations for the Components of M(x, z) , ?~(z,x)

By means of the functional equations (11), (12) for M(z,x) and M(x', z) we can find explicit formulas for the components of these operators in terms of the four standard R
and T operators for X(x,z). Thus let us write

74_{++} (x, Z) 711_{+-} (x, z)

$\__{+}$(x,z) (x P Z)

for Or(x, z) (18)

thereby defining, in context, four operator components of M(x, z) . A similar definition is made for ~7(z,x) . Next we observe that the two factors comprising 'j(z,x) in (11) may be written :T(z,x) -R(z,x)

and, by (17)

[C _ M (x 9 1 Z) - C+1_1 =

$_{T}$_ 1 (x , z) R(x, z) $_{T}$_ 1 (x , z)

With these specific representations of the factors in (11), we have-:

T(z,x) -R(z,x) $T^{-'}$ (x, z) R(x, z)-R(z,x) T^{-1} (x, z)

[$T^{-'}$ (x, z) R (x I Z) T - 1 (x , z)

whence

7n++(z'x) = T(z,x) ^{i}

- R(z,x) T_ (x, z) R(x, z) (19)

7 n + . . (z , x) = _ R ($_{z,}$ $_{x}$) $_{T}$_ x (2 0)

(x , Z)

=

+ (z,x) $T^{-'}$ (x, z) R(x, z) (21)

771 = T - I

_ (z,x) (x,z) (22)

Next, we use (12) to find the component operators of ^{5}~ j (x , z) . The first factor in (12) is

-i 0

[C $_{-}$M(X, Z) _ C+

R(x, z)   T (x, z)
The inverse operator is evaluated by means of (16)
~T_I (z'x)   _T-I (z,x)R(z,x) 0 . I
Then (12) becomes:
CC_ - C_+M(x' Z)
M (x '   z)   =
whence

T- x (z ,x) $T^{- I}$
(z,x) R(z,x)
-R(x, z)$T^{- i}$ R(z,x)(23)
(z,x) T(x, z) -R(x, z)T^{-i} (z,x)
?1++(x'z) = T_i (Z'x)

(x, z) = $T^{-'}$ (z, x) R(z,x)   (24)

,1_+(x, Z) _ -R(x, z) $T^{-'}$ (z,x)   (25)

-M_ = - T_ i
_ (X, z) T(x, z) R(x' z) (z,x) R(z,x)$^{\cdot}$ (26)

The components of M(x,z) may be represented in two equivalent ways, depending on whether (10) or (13) is used. Using (10), the factors are, explicitly

[)~(Z,X)C_+ + C_-]   =   . 17++ (Z,X) 0

Fj++ (z ,x) -;_+(Zvx);1+.. (Z.Ox)$^{;}$~-1 (Z ox)   ' +-(z,x)7?
1(Z,x)I_+(z,x)   `-~ (z 'x)
r
From this:
T (z ' x)   =   / 1~1++(Z.'x) -   7'_+(z,x) ?~^{1}+_ (z,X)^{9}I - i (Z,x) (2 7)
R (z , x)   =   _   - (z , x) 5 r " (z , x)

R(x, z)=   _.~ (z,x) j-+(z,x)
T(x, z) a M-' (z,x) Alternatively, the factors in (13) are: I 7 ' +_ (x, z)
(x z)
r   D
7)7(x,z)C+ + C_ -' a
- ?-+ (x, z) M++ (x, z)   I
Then

J++(x’z) ^{2}7++(xPz)1+-(x9z)
M(x.$_{p}$Z)m
-1-+(xjzA++(x,*z) -_tx~z)_ _+Cx~z)~I++fix=Z) '+-tx~z)
From this:
T(z,x) = ^{7}I++ (x.z) (31)
R(z,x) = A1++(xP z) 7+- (x, z) (32)
R(x, z)$^{-7}$~-+(x’ ^{z})7~$_{+}$+(x, z) (33)
T (x, z) =, 27-.. (x, z) - ..+ (x’ z)``[++ (x, z) ;~+- (x, z) • (34)

The connections between the two sets of representations (27)-(34) of M(x, z) rest on the fact that I(x, z) and I(z,x) are mutual inverses. The four component equations harbored by:

7?j[x,z)7rf~z,x) ~ I

provide the necessary explicit link between the two preceding sets of representations.

It is interesting to observe that one may go from one set of representations to another by simultaneously interchanging the arguments "x" and "z" along with the subscripts "+" and This interchange rule also works for the sets (19) - (22) 'and (23)-(26), and also^{-}for the functional equations (1D)-(13) (leaving M(x,z) inviolate). The physical basis of this rule is that such interchanges applied to the radiance vector [N$_{+}$ (z) , N _ (x)) and the matrices M(x, z) , M(z,x), effectively reverse the incident and response radiances and the operators applied to them.

The Isomorphism ~ Between r2(a,b) and G2(a,b)

The algebraic links just established between the operators M(x, z) and M(x, z) suggest a close overall structural resemblance between the members of the set G$_{2}$(a,b) (i.e., all M(x, z) , .with a A x S z _-5b) and the members of the partial group rz(a,b) (i.e., all 'Or(x,z), a sxs b, a :~-z =b) . We can use this strong tie between the two sets to induce a means for multiplying together members of G2(a,b) in a way that faithfully mirrors the natural multiplication of elements of r$_{z}$ (a,b) . The practical utility of the newly formed multiplication process will become clear as this discussion nears '.’its Close.

Let us denote by "0 (M (x, z)) " the operator M(x, z) ,found from M(x, z) using (3I)-(34). and let "0~$^{;}$ (M(x, z))" denote the operator ~J(x, z)obtained from M(x, z) using (23)-(2.6)..

FIGS. 7.3 ,7.4 The meaning of the isomorphism between r2(a,b) and G2(a,b).

where, a s x :5z t^{-}-b . In this manner we define in context a function ~ on part of r2(a,b) (call it the upper triangle of r2 (a,b) onto G2 (a,b) . (That is, we do not define $ for all pairs x,z, but only those such that x -cz.) This function is one to one in the^{-}sense that to each M(x, z) in G$_{2}$ (a,b) there is assigned at most ^{7}~' (x, z) in the upper triangle of r2 (a,b) for any choice of levels x,z in x(a,b), where x sz. The term "upper triangle" of r2(a,b) is suggested by the fact that in a Cartesian coordinate plot of the pairs depths (x,z), those pairs such that x s z, lie above the diagonal line. (See shaded region in Fig,. 7.3.) An alternate one to one mapping * from the lower triangle of_r2(a,b) onto G2(a,b) is possible using the systems (19)-(22) and (17)-(20). Either mapping 0 or will suffice for our present purposes. We choose to work with as far as possible. With this choice of (12), (13) may be rewritten as:

M(x,z) = W) N X . V z)) _ [17 (^{x}9,z)C+ + C] -, [C+

(x, Z) _ $^{-1}$ (M (XIZ)}.. [C -^{M}($^{x' Z}$) G+ [$C_{.}$ - C_{+}M (x,z)~ -~

The induction of the multiplication process on C2(a,b) is now carried out as follows. Let M(x,y) and M(y,z) be any two elements of G2(a,b)-, provided that they have a depth level in common (e.g., y, as shown). It seems natural to require that their "product" be such that the usual matrix product of the corresponding operators 0 (M(x,y)) and ~- (0T (y, z)) in r2(a,b) maps back, under , to the required "product". (See Fig. 7.4), Thus we agree to write:

"M(x,Y) *M(Y, z)" for ~ [0- 1 (M(x,Y)) O^{-1} (M(Y, z))] . (35)

By definition of 0-1 and the one to one properties of 0:

and:

-)~O((xj,y) _ ~-1 (M(x,Y)) 7(YPz) _ \$$^{-1}$ (M(YPZ))

Hence:

O (m i l (x ,9Y)) = M (x j y) O P V Y , z)) = M(Yjz) Therefore an alternate way of expressing (35) is:

0PY(x.%y)^{7}t(ysz)) = O(n(x2y))*0(71(Y2,z))

(3b)

This alternate form of describing the star product of elements of G2 (a,b) defined in (35) shows how the structure of multiplication in G2 (a,b) mirrors that of r2 (a,b) . In modern algebra the function ~ which induces operations such as the operation * is called an isomorphism, the etymology of the word in this physical case being most appropriate (iso = same;

morph = form). Under the introduction of the-star product, G2(a,b) becomes a partial semi-group, with an identity operator of the form M(x,x), and with the associativity property and inverse properties holding.

The Physical Interpretation of the Star. Product

The star product on Gi(a,b) introduced above has a most interesting physical interpretation. It is worthwhile to pursue this interpretation as it will permit us to tie together the territory covered so far in this section with that of section 7.3. Since M(x,y) describes the reflectance and transmittance properties of X(x,y), and M(y, z) describes those of X(y, z) , we ask: What physical description, relative to X(x, z), does the star product M(x,y) *M(y, z) represent? The clue to this description is given by examining (35). The ri ht side of the definition is the image, under ~, of (x,z). simply Hence we see that:

M(x, z) = M(xpy) *M(Y p z)

(37)

Therefore the star product of M(x, z) and M(y, z) is the operator M(x, z) associated with the union (the sum) of the two^{-}contiguous. slabs X(x,y) and X(y,z) (as depicted e.g., in (b) of Fig. 7.2).

Let us find the components of the star product
M(x, z) *M(y, z) directly in terms of the components of the factors M(x,z) and M(y,z). We begin the derivation with (35). Thus, ~ by (23) - (26)
$T^{-'}$ (Y,x) T-' (Y,x) R(y,X)
~ -' (M (x, Y)) m
-R(x,Y) T-' (Y,x) T(x,Y) -R(x,Y) T-' (Y,,x) R(Y px)
Similarly:
$T^{-'}$ (z,Y) $T^{-'}$ (z,Y) R(z,Y)
C^{1} (M(Yfz))$^{=}$
L-R(y_{j}z)$T^{-'}$ (z py) T (Y, z) $^{-}$R(Y, z) $T^{-'}$ (z,,Y) R(z py) The product of these matrices is:
C' (M(xVy)W' (M(Y!z)) = AI(xvz)
and where:
$_{++}$(x,z) 'T ' (Y,,x)$T^{-'}$ (z,Y) - T-' (Yfx)R(Y~x)_R(Yz)T-' (zy)

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I _ C x , z) T^{-i} (y,x)T^{-l} (z,y) R(z,y) +
+T- 3 (y , x) R(y,x) [T (y, z) $^{-}$R(Y, z) $_{T\_}$ ' ($^{-}$z,y) R(z ,y)]
~'_+ (x, z) =- R(x, y) T_ ^{1}(y, x) T_ ^{i} (z,y) -
- [T(x,y) -R(x,y) T~ ' (Y,x) R(y,x)] [R(y, z) $T^{-'}$ (z,y)) f _ (x , z) a _R(x,y)$\cdot$$T^{-i}$ (y,x) T_' (z,y) R(z,Y) +
+ [T(x,y)-R(X,y)T~' (y,x)R(z,y)] [Toy,x)- R(y, z).T_'(z,y)R(z,y) I

Each of these may be reduced considerably if we use algebraic formulas developed earlier. For example:
$^{7-0}$7$_{++}$ (x, z) = T^{-1}(y,x) [I -- R(y,x) R(y,'z) $^{-}$] T-1 (z,y)

when the last inequality is based on (18) of Sec. .7..3. (See also (8) of Sec, 7,,3*) In a similar (but-slightly more arduous) manner the remaining components may be reduced so that they may be used in (31) -.(34) . The net result of the mapping back to M(x, z) .from M(x, z) is:
M (x, z) M$\cdot$(x py) *M (y O Z)

r' (z,y,x)T(y~x) R(z,y)+ GR(z,y,x)T(y, z)
.R (x,y) + $_{j}$6~ (X,y, Z) T (y,x) ~7(xf y, z) T(y, z)

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In this way the representation of they star product is rendered into a mathematically self-contained form by means of the partition^{-}relations developed in 7.3. The representation is made particularly meaningful physically by using the complete reflectance and^{-}transmittance operator -for X(x,z), so that each component of the product can be read directly in terms of reflectances and transmittances. We summarize (38) by saying that: the star product of M (x,y) and M (y, z) is the mathematical form of the partition relations (25)-(28) of Sec* 7-,J for the medium X (x, z) , and therefore contains a t a the information for determining the standard reflectance and transmittance operators o f the union X(x,y) U X(y, z) o f two contiguous media, knowing the respective operators o f each component o f the union.

The Link Between $^{;11}$7(a,x,b) and $^{;}$Na,y,b)

Two invariant imbedding operators for X(a,b)', such as ^{2}f (a,x,b) and M(a,y,b), may be linked by the operator ^{7}&(x,y) as follows. The definition of the invariant imbedding operator yields the equations:

(N+(y)tN_(.y)) = (N+(b).,N_(a))M(a,yb) ,

(N+ (x) ,N_ (x)) = (N+ (b) ,N_ (a)) 7z7(a,x,b)

Since

(N+(y)pN_(y)) = (N+ (x) ,N_ (x));1(x,y) , it follows at once from these three equations that:

$(N_+ (b) ,N\_ (a)) (- M (a , x , b) r n (x , y)) = (N_+ (b) ,N\_ (a))$ 7T(a,y,b) The incident radiance distributions being arbitrary, we have:

^{75}I(a,y,b) = 3I(a,x,b) ^{211}1(x,y)

(39)

for every x, y in (a,b). If the inverse of $^{;}$~7(a,x,b) exists, we find:

J(x,y) = ^{21}7' ^{1} (a,x,b) 7-,I(a,y,b)

(4D)

which shows how $^{;O}$f(x,y) is represented in terms of the third order invariant imbedding operators.

It is interesting to view (39) not as representing a static link between members of.rs(a,b) but as depicting the transformation of the interval ja,b] into the set r 3 (a,b) . This new view is obtained by first fixing level x, a s x :s b. Then for each choice of y in the interval [a,b] equation (39) assigns to y the operator M(a,y,b) in, r 3 (a,b) . In this way ?W(x,•) serves as a mapping or transformation from [a,b] to r a (a,b).

Building on the preceding viewpoint, equation (39) may be envisioned as stating four "principles of invariance" for the complete G' and C operators. Thus, unfolding (39) component by component:

~^{7}(b,y,a) = 7(b,x,a)?PI++(x,y) + G?(b,x,a) 7

7 $\_ _+$(x , y)

4 ? (b , y , a) = 9'(b,x, a) $91_+\_$ (x,y) + rat(b,x, a) ;~V_ _ (x,y)

eat (a,y,b) = Q(a,x,b) 71++(x,y)- + 7(a,x,b) 711 $_+$(x,y)

~(a,y,b) = 6? (a,x,b) $7\text{-}17_+\_$ (x, y) + T(a,x,b) 7 ' _ _ (x , y)

In the present point of view the R and 2 operators act the role the radiances did in the final statements (e.g., Ex. 3, Sec. 3.7) and the components of ZoY~(x,y) act like transmittance and reflectance operators: those with like signs are transmittance operators, those with unlike signs are reflectance operators. This analogy is exact in the sense that an operatorial theory for the 4R and .^{7}' operators can be developed

which is essentially parallel to the radiometric theory for Nf (Y) . This and still other analogies (-some of which are brought to light below), open up vistas in algebraic radiative transfer theory which are beyond the scope of this work but which are potential areas of basic research in the theory. See Problem X, Sec. 141, Ref. [251].

Representations of ^{7,}f(x,Y, z) by Elements of r_2 (a,b)

In view of the success in representing the basic operators M(x,y) by means of the imbedding operator ^;W(x,y) (See (14)-(13)) we are led to seek still further representations of interaction operators by members of the partial group r2 (a,b) . We shall find that the set r2 (a,b) is an extremely powerful set of operators in the sense that virtually all operators in modern radiative transfer theory are representable by suitable algebraic combinations of members of r_z (a,b) . In the next few paragraphs we shall assemble some evidence in this direction. The formulas so gathered will be employed in Sec. 7,5 to find various differential equations governing the interaction operators, equations which should suggest novel solution procedures in radiative, neutron, and generally linear transport theory.

On the one hand the light field at level y in X(a,b) is obtained from arbitrary incident light fields at levels x and z, x s y s z , by the relation:

(N+(y)vN_(y)) = (N+(z),N.(x))/0W(x,,Y,,Z)

(41)

On the other hand those on levels x and z are related by that on level y by using-the following operators:

$(N_+(z),O)$ a (N+(y)jN_(y))7)t(yjZ)C+

$(4_j, N_.$ (x)). = $(^N$+ (Y) vN_ (Y)) 7,^{1}/'(y, X) C_

Adding these equations:

$(N_+$ (z),N_(x))= $(N_+$ (Y) 9N_(Y)) ' (Y0 Z) C+ + 0I(Yfx)C_

and using (41)

(N+(z),N.(x)) _ (N+(z),,N_(x))M(x,y,,z)F~(yfz)C+ +M(Ytx)C_

which, in view of the arbitrary nature of the incident distributions, yields the desired representation:

7/7(x2y2Z) = [71 (Y.9 Z) C_+ + '~47(YYX)C- -i

. (42)

It is interesting to speculate what would happen if we allowed the variables x,y, z in (42) to take on any three values in the depth interval [a,b]. The derivation of (42) by

convention (but not essentially) is performed only for the depths x, y, z, in the usual order x .:sy s z within x (a,b) . - But since the operators '01(y, z) and 'W(y,x) are defined for all pairs of depths, and since the inverse of-the-indicated linear combination of these 'operators should exist just as often as .those in mare orthodox settings, there now is a way, as indicated by (42), of .formally extending the domain of definition of the invariant imbedding operators.

A Constructive Extension of the Domain of ^{0,}7(x,y,z)

The preceding observations of the potential extensibility of the domain of definition of the invariant imbedding operator 7P('(x,y, z) is reinforced by recalling equation, (39) , in particular the interpretation of the equation as implicitly defining a mapping which, in effect assigned to each y in the interval [a',b] an operator ;1~(a,y,b) , as explained above. Suppose. then we write, ad' hoc:

,, M($_{x,u,}$ z),,, for . ^{7,}f(x,y, z) M (y,,u) 10 (43)

I t follows that, as long as. we have x :5u Z r f t z , the operator

3# (x,u, z) is, by (39). simply O'(x,u, z) . But the product of the operators in (43) is certainly compatible for any u, given
each factor associated with that ,u. In this way, then, we can formally extend the domain of M(x,y,z) so that the parameters
.may fall outside of the subinterval [x, z] in [a,b]. Once the extension is fully and unambiguously made, the bar above-I'M" in (43) may be.' dropped in practice.
. The extension just made is a constructive extension of (x,y, z) in the sense that, given '7'7(x,y, z) and ;;)^{l}(y,u) there is a definite construction procedure that may be followed in this case, a simple matrix product effecting the extension,
It should be recalled, of course, that ^{7}·P ' J (x, y, z) , is in already extended" form as it is cut directly from the more comprehensive mold of the generalized invariance imbedding relation. (See the discussion of (7b) of Sec. 3.7.) Thus we may simply write:
"MX,y. Z)" for ~)l(x.y; z,y) (44)
where x,y,z are any three levels in x(z,b), and study M(x,y,z), so formed, as a special instance of the generalized invariant imbedding operation. Thus (43) without the bar over
is in the last analysis simply a consequence of the semi-group property (84) of Sec. 3.7, Further, if one returns to the derivation of (42) or repeats its derivation, now using
the definition (44) for ;~Y/(x,y, z) , the same functional relation (42) would be obtained, and the speculations on the extension of (42) to general parameters x,y,z, now have a solid affirmative basis.

Representation of ^{7}Av, z;u,y) by Elements of rz (a,b) and rs(a,b).
We begin the derivations by representing M(v,z;u,y) as a product of two simpler operators by means of the semigroup relation (84) of Sec. 3-, 7.
M(vv z;u,Y) M(vjx;uPxPJ(x, z fx,9y) (45)
in which we have set x = w, with this simple identification of x and w we have managed to represent M(v,z;u,y) as a product-of two operators of the extended type M(x,y,z). Thus, the first factor M(v,x;u,x) in (45) is simply an extended invariant imbedding operator 71~(v,x, u) as defined in (44). The other factor appears to be the inverse of such an extended operator. Indeed ,.using the semi-group relation (84) of Sec. 3.7 once again; it is clear that:-
77(x, z;x:Y) I(Zvx;ypx)
(46)
Hence:
;Pf(xtz;xvy) = M~' (zox;YIX)
_m ~1
z $_{1}$x fY)
(47)
It remains only to return to (42) and make the appropriate substitution of variables to obtain the desired representation of ;~I(v, z;u,y) . Thus from (42)
1(v,x;u,x) _ ^{7}I(v,x,u) ' AT L (XIU)C+ + I(x~v)C.. _~ . (483 Once again from
(42):

'j(zix;Y,x) _ ;1'$j(z,x,Y) _ $^{j;}$1(x#Y) C+ +
M(xoz) C _ " _ . (49) In view of '.(47), equation
(45) therefore becomes
1'(v, z ; u, Y)
(x, u) C + + M(x,v) C ., -^{i}7 - ^{0}l (x s Y) c_{+}+ ?#(x v z) C . .
(50)

which is the requisite representation of an arbitrary member of r4(a,b) by members of rz(a,b), and which holds for every u,v,x,y,z in [a,b], provided, of course, that the inverse operator in (5p) exists in a given setting. An alternate form of (50), using the generalized invariant imbedding operator, is:
(51)

In equations (5a) and (51) the depth variable x is free to be chosen anywhere in [a,b] . Observe in (51) how the first factor, as a generalized invariant imbedding operator, maps

(N+ (u) , N _ (v)) into (N+ (x). ,N _ (x)) , and then how the inverse factor maps the latter radiance distribution into
(N+ (y) ,N_ (z)) -, The composite mapping of these functions is precisely that performed by 0f(v,z;u,y). Thus one could almost write down (51) by sight if the various ranges and domains of the operator are kept in mind. It is of interest to compare (51) with (40) which yields a representation of
7' (x, y) in a similar vein to that of1j1(v, z ; u, y) above in (51).

The representation (50) may be used to yield at once, under suitable confluence of the variables u,v,y,z, the entire family of interaction operators considered so far in this section. This is left as an exercise for the interested reader. The derivation of the- following alternate representation of
7j (v, z;u,y) 'by extended members of T_3 (a,b) is also left to the reader:
;'(vf z;uvy) = 1(v:y'u) C_{+} + 0'(v' z,u) C_
. (52)

The Connection Between T(x,y) and $^{-}$?~1(s,y)

The interaction operators for media with internally distributed sources of radiant flux differ fundamentally from those designed to describe radiative transfer in source--free media. The origin of this difference was pinpointed in the equation (31) of Sec. 3.9 for the operator $T_{++}(s,y)$; and the subsequent discussion of this operator showed that it was discontinuous at the point (s,s) of its domain, a property not possessed by operators of the source-free kind. The operator T(s,y) introduced in Example 3 of Sec. 3.9 (of which $f_{++}(s,s)$ is one of four components) is specifically designed to describe light fields in a media which have internally distributed sources. Since we have apparently reached in this section a culmination point in the discussion of source-free media, it would be of interest to relate the operator T(s,y) to the basic operator ?~(x,y) for source-free media.

We now momentarily abrogate the standing condition about source-free media y(a,b) made at the outset of this section, We postulate instead a source of flux arbitrarily distributed over level s in x(a,b), a s s sb. The source is represented as an arbitrary radiance distribution $N^{O}W$, where NO(s) is conceptually partitioned into the pair (N°(s),N_(s)) of upward (+) and downward (-) radiance distributions. Then the radiance distribution N(y) (= (N+(y),N-(y)) at any level y in x(a,b) is given, according to (15) of Sec. 3.9, by:

(N+ (y) p N _ (y)) = (No (s) j N_ (s)) T (s 1 y)

What we must do next is to use the operator . '~7(s,y) , which is designed for use in source-free contexts, to relate the radiance distribution at level s to that at level y. It is important, therefore, to renew acquaintance with the manner in which the source radiance function NO(s) is viewed in radiative transfer theory. A re-reading of the opening paragraph of Example 3, Sec. 3.9 will serve this purpose. We see that the source is pictured very much like a thin transparent layer of pure light sandwiched between the media X(a,s) and X (s,b) . For true internal sources, we require a < s <b, and this is the condition used throughout the earlier and ,the present discussion. Furthermore, the presence of the source is detectable in practice, by an effective discontinuity of the radiance readings of a radiance meter as the meter passes through the layer containing the sources. However, in levels y of X (a, b) distinct from level. s , the general properties of the light field are identical with those ,of any source-free medium. Therefore, to relate N(y) to the light field at level s # y we may use ;Pf (s,y) provided we feed into 7'(s,y) the total incident light field as it is measured at level s.

This means that the input radiance distribution , for ^{0}f(s,y~ is to be N°(s) + N(s)', where N(s) is the resultant light field at level s generated throughout X(a,b) by the source NO(s). Therefore

$(^{N}$+ (y) p N_ (y)) _ $(N^{+}$ (s)

N_{+} (S) ,N0 (s) + N- (S)) ^{7}~'(s,y) (53)

and this equation holds only for y # s. By setting y = s in. (53) we obtain a contradiction. Herein, then, lies the-salient difference between 07(s,y) and T(s,y) in general media: M(s,s) = I, but T(s,s) # I; thus T(s,s) is that irreducible core of T(s,y) whose task it is to take specific cognizance of the presence of the obtrusive layer of light at level s.

With the metaphysics over, we can now proceed to the final steps that relate 1'(s,y) to IF(s,y). Equation (53) may be written:

$(N_{+}$ (y) ,N: (y) = C(N 0 (s) $_{O}$N 0(s)) + (N+ (s) ,N_ (s))1 ^{7}~(s,y)

(54)

Setting s = y in T(s,y)

(N+(s)pN_(s)) -_ (NO(s),No(s) (s,$_{9}$ 5) (55) which, when used in (54) in conjunction with the equations (15) and (35) of Sec. 3.9, yields ($C_\pm$ far [y-s] Z 0)

(N^o(s) sNo(s))f(s,y) = (No (s) sN~(s)~ ~Ct^+ 'y(sts)] M(s,y)

Since NO(s) is arbitrary, we obtain the desired connection in the form

T (sly) = [C±+ ^{T} (^{s} t s)] M(s f y)

(5b)

where we use C+ for ' s < y and C, for s > y in accordance with the jump property o f IF (s , y) a t s = y. $C_\pm$ are defined in (4), (5) .

A Star Product for the Operators /^{3}'(X,y, z)

We end the present section on three ascending general notes, of which the present discussion sounds the first: we wish to extend the concept of the star product of the operators M(x, z) , as developed in (35), to the invariant imbedding operators '·(x,y, z) . This product, which we found to be the algebraic essence of the partition relations (15)-(18) of Sec. 7.3., serves to show how to combine the interaction properties of two contiguous media X (a,b) and X (b , c) to find the corresponding interaction properties of their union X(a,c). We now attempt to do the same for the complete reflectance and transmittance operators of any two adjacent media.

Figure 7:5 depicts the present setting. We imagine .a plane-~parallel optical medium X(a,b) to be the union of two arbitrarily overlapping sub-media: X(a,z) and X(x,b). Let y be any level in X(a,b) such that x s y -sz.The problem before us is: to represent 4R(a,y,b), J-(a,y,b), and M(a,y,b) in terms of a suitable algebraic combination of the complete and w ' ¯operators associated with X (a, z) and X (x,b) . The present problem is geometrically slightly more general than its counterpart posed in Sec. 7.3 for the R and -T operators in the sense that we require not contiguity of X(a, z) and X(x,b) (.so that necessarily x z), but merely intersection of the media (so that x s z) .

The incident radiance distributions N_(a) and N_+(b) on X(a,b) generate .a light field at general levels x,y,z in . X(a,b) which may be computed several ways depending on which medium one envisions the levels to be in, i.e., as light fields in X (a,b) , or in X (a, z) or in X (x,b) . Thus N_f (y) , as

r

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FIG. 7.5 The setting for the star product in $r_3(a,b)$.

radiance distributions in X (a, b) are given-by:

N+(y) a N_ ($) .6?(a,y,b) + N_+ (b) ’(b,y, a) (57)

N., (y) a- N. (a) r (a, y, b) + N+ (b) 6? (b,y#a) (58)

which follows at once from the invariant imbedding relation which X(a,b). On the other hand, the distribution N+(y) 'considered as being in X(a, z) , is given by:

N_+(y) N_ (a)f(a,y, z) + N_+(z) (z,y.,a)

and N. (y) -, considered as being in X (x, b) is given by

N_ (y) '~ N_ (x) ..T(x,y,b) + N_+ (b) 4R(b,y,x) - (60)

which are the results of applying the invariant imbedding operators of X(a,z) and X:(x,b) , respectively. Now'-_.t .e _4~ distributions .N_+ (z) and N,. (x) -~ appearing on (5.9) a 'id. .(= n be found by solving the system:

N_+ (z) = N_ (x)AR (x, z, b) + N_+ (b) T(b, z x) (61)

N (x) N_ (a),7,,ˉ (a,x, z) + N_+ (z) z~x~a : (62)

which is derived similarly to (59), (60) by considering level x as occurring in X(a,z) and level z as occurring-in X(x,b).-

The solutions are:

N.A. (z)

(a)7(afx., z)6~(x, z,b)+N+(b)l(b., Zix)] [I AR(z,,x,,a)4?(x,, z" b)] ~ ~ (63)

m [N.. (a)y(a,x,,z) +N_+ (b)X(b, z,x)g(z,x, a)] [I-A(.X._._,z,,.b)jR(z,x:,a)~ -'

These equations should be compared with (9), (10)$_t$ (25) and (26) of Sec. 3,7 and (49) , (5n) of Sec. 3.9 for structural similarities.

Next consider the two alternative ways. of describing N+(y) in (57) and (59). If these two expressions for N+(y) are equated and if N_+ (z) as given in (6.3) is' used.ˉ, : then since

N_+ (b) and N_ (a) are arbitrary, we derive the following two operator equations as a result:

A(a,yp.b)

^{m} a(aty, Z')$^+$Y(a,xtz)4R(xo z,b) [I_R(z,x,a)Axgz,b)] "- (Z,y,a)

These two equations constitute a rather interesting generalization of (15) and (18) of Sec. 7.3. For by letting y = z in (65) and (66) we have:

and

.R(a, z,b) = J(a,x, z) A(x, z,b) [I- o0(z,x,a),6'~(x, z,b)] - i (67)

Jlo'(b, z,a) = 9-(b, z,x) [I - a(z,x,a) 6?(x, z,b)] - } (68)

and these representations may be plowed back into (65) and (66) to yield the following compact forms of (65),(66):

6?(a,y,b) = 0(a,y, z) + a(a,z,b)j*'(z,y,a) (69)

%.;'(b,y,a) = 5'"(b, z,a) 7'(z,y,a) (7©)

The latter equation is simply the semi-group property'(52) of Sec. 3.7 for the "operator. However, (69) is a relatively novel equation, much in the way (15) of Sec. 7.3 was a newcomer to the semi-group scene in that setting. Equation (69) will be used at crucial points of the investigation in 7.13. The analogy between the present derivation and those leading to (15) and (16) of Sec. 7.3 appears to be a through going one, on the strength of which we can write down the remaining two correspondents of (16) and (17) of Sec. 7.3:

%)"-(a,y,b) _ j_(a,x,b)Zr(x,y,b) (71)

a (b , y , a) _ .9C(b,y,x) + ,&T(b,x,a) 7(x,y,b) (72)

where

J-(a,x,b) = O'(a,x,z) [I-.A(x,z,b)Ag(z,x,a)] $^{-i}$ (73)

a (b,x,a) = 0^1~'(b, z,x) R(z,x,a) [I-._R(x, z,b),R(z,x,a)]-' (74)

The requisite star product for the invariant imbedding operator ^{0}I (a,y, z) and ~'(x,y,b) associated with the submedia x(a, z) and X(x,b) may then be defined as follows.-

We write:

ft/~7 (a,y., z) *;1(x,y,b) •1 for ^{7}~^{1}(a,y,b) (^{7}5)

where $^-$)n(a,y,b) in (7S) is constructed from the operators of M(a,y, z) and M (x,y,b) using (69)-(72) in which .~R(a, z,b) , X(b, z,a) , 3'(a,x,b) , and AR(b,x,a) are as given in (67),(68)$_t$ (73) and (74), respectively. Thus:

7j(a,y,b) = Vj(a,y, z) * .?f(x,y,b)

_ [T(b' z. a) 7(z,y, a) (b,y,x) +.g(b,x, a),'(x,y,b)

d?(a,y, z)$^{+,}$R(a, z,b),T(z,y, a) 9-(a,x,b)e`r(x,y,b)

(76)

This star product will be used subsequently in the study of irradiance fields in interacting media (cf. (91) of Sec. 8,7).

Fig 7.6 The star product of invariant imbedding operators can be defined for arbitrary media.

Recall that the depth variables x and z in (75) are arbitrary, subject only to the condition that the media x and x overlap., and that y be chosen in the intersection of these media. Equation (76) is to be compared with (39).

The power of the present algebraic approach to radiative transfer theory can be appreciated I some detail if we now turn to the general invariant imbedding relation (51) of sec 3.9 and observe that all activity we have gone through to reach)(76) can be repeated for the general medium x of examples 4 and 5 of sec 3.9. Thus if we have a medium x with two overlapping submedia a and b of a one-parameter medium as in Fig 7.6, and more generally, if we have two media x and y which intersect in a region z as in Fig 7.7, then aw can form a star product of the invariant imbedding operators of x and y to obtain the invariant imbedding operator of their union in exact analogy to (76).

In view of these observations, the possibilities for further exploration of the algebraic theory of radiative transfer are clearly mounting in number and in depth. The possibilities branch off into topological and algebraic directions which, if kept bound tgether by suitably defined concepts, will raise the theory of radiative transfer the remaining distance to its logical haven: a possible general theory of linear transport processes. Such a pursuit is unfortunately beyond the scope of the present work , and we rest the matter here.

FIG. 7.7 A general setting. for the star product of interaction operators in general optical media,

Possibilities Beyond M(v,x;u,w)

In this the penultimate note of the present section,. the possibility of operators more comprehensive than those in r4(a,b) will be considered. We ,shall show that such possibilities of arbitrarily great comprehensiveness are easily constructed. However, in a sense, such generality is no longer needed now that operators like ;Pl(x,y) harnessed in parallel have been shown to have sufficient computational power (cf.

(50), (51) and (52)) to do everything M(v,x;u,w) can do.

For simplicity, we shall remain in the setting of one-parameter media during the present discussion.

To see what direction we may take in generalizing 7,'!(v,x;u,w), let us return to its definition in (56) of Sec. 3.7. Recall that the primary motivation for 7xz(v,x;u,w) was

the need for an operator which would take as input the pair (N+(u),N_ (v)) of radiance distributions 'on arbitrary levels

of u and v in X (a, b) and yield as output the pair (N+ (w),N _ (x)) on still two more arbitrary levels w, x in X(a,b). In this way we achieved a comprehensive, symmetric setting for all classical operators. In particular, these choices of input and response distributions constitute the natural generalization of the classical type of inputs and responses of M (x, y) (cf. (2) above) and the general invariant imbedding operator

N(x,y,z). Having thus extended the input and output types to ,a reasonably general kind (there is still room beyond here too--consider, e.g:, partition¯ing .+ and '.-H. into many and sundry pieces) we turn to consider the effect of an increase in the number of levels. Thus, suppose we ask for an operator which takes as input the 2m component radiance vector:

$(N_+ (u1) \sim N\_ (vl)\ {}^{.N}+ (u2)\ tN\_ (v2)$;

... ; N+(urn),N.(vrn))

(77)

and yields as output the 2n component radiance vector:

$(N_+ (wl)\ 9N\_ ({}^{x}1)$;N+ (w_2) 9N_ (x2) ; . . .

$N_+ (w_n)\ pN\_ (x_{\ n})$) (78)

where ul,...,u_m and v ,...,v_m are 2m arbitrary levels in X(a,b) and wl,...,wn Ind x1 ,...,x_n are 2n arbitrary levels in

x(a,b). N_+(ui) is as usual the upward radiance distribution over level ui.. Similarly with the other radiances, Then the interaction principle supplies an operator

71($(v_m,x_m$;

vl,xl;un,wn; ... ; ulwl)

(79)

which is a 2m x 2n matrix of operators of which.2mn are -like and 2mn are Q-like (which we need not display here) and which clearly reduces to 7.*?(v,x;u,w) by setting m = n = 1. We shall now show that the operator (79) can be represented as a linear combination of generalized invariant imbedding operators of the form ?y (vi,xi;u- ,w-) . Then, in view of (5D), the algebraic representation of t79~ in terms of members of the partial group I'_2(a,b) will stand established.

The key to the desired representation of (79) rests in the following two partitions of identity operators:

m

I_m = Z Ci [trn Cil (89)

i=1

n

I = D . [trn D j] (81)

where Ci is a 2m x 2 matrix and D- is a 2n x 2 matrix of the form:

(82)

and where "0" denotes the 2 x 2 zero matrix and "I" the 2 x 2 identity matrix considered earlier (e.g., in (3)). The notation "trn Ci" denotes the transpose of Ci, i.e., 2 x 2m ˉmatrix obtained by turning Ci on its side so that the identity operator I_m in Ci is the ith .matrix counting from the left as usual, That (80) and (81) represent, respectively, the 2m x 2m and the 2n x 2n identity matrices is readily established, and is left as an exercise to the reader. Observe also that [trn Ci]Ci is the 2 x 2 identity matrix I, for every i ,

The operators Ci have the useful properties that:

(^{N}+ (ui) , N. (vi) a C_i

(N+(wj),N_(xj) b Dj

(83)

for 1 s i : s m , 1 s j :s n, and where "a" and "b" denote (77) and (78), respectively. We shall continue to use these abbreviations "a" and'"b" in the remainder of this discussion.

Now, we know how to relate (N+(ui), N_ vi)) and (N+ (wj) , N _ (x j)) , Such relating is the specific task of M(vi,xj;ui,wj). Thus

(N+ (w_j) . N_ (xj)(^{N}+ (u_i) , N_ (vi)) ;~r(vi ,xj ~ u_i,w_j) • (84)

In other words (84) states that:

b D. = a C 7..

1 i]

where we have written, ad hoc:

(8Sˉ)

"A3.j" for '(V_i ,x_j ;u_i,w_j) (86)

Equation (85) therefore suggests that we start with:

where "M " at present denotes (79), and insert the 2m x 2m identity operator I_m, in the form (80), between a and A'in
(87) to obtain:

m

b = a C_i [trn C_i]   (88) i--.1

Once this is done we operate on each side of (88) with Dj to obtain:

b D_j = a

C_i{ [trn Cj 7lJ D_j)   (89)

It is clear from (89) and (85) and the fact that these equations hold for every incident radiance vector a, that.'

j _ [trn C_i] n D li   i

(90)

In this way we see the 2x2 operator matrix via is a special case of 01.   Going on with the present analysis of (87), we operate on each side of (89) with trn Dj and sum over all j. Thus

n

b = b E D~ [trn D~] j=1

a

Ci 'b7ij [trn D $_j$]   (91)

i¯1

Again, since a is arbitrary, we have, from (87) and (91).

2)

By means of this equation, we see that '~Iis representable as an mxn block matrix with Orij as the element in the ith row: and j th column.

Equation (92) is the desired representation of ''~'in terms of ¯74i ,   i.e.,   in terms of the members of r_4 (a,b) .

Thus we see hat '~(may be represented by a suitable algebraic combination of elements of r_z (a,b) , using (50).

Possibilities Beyond r_2(a,b)

We conclude this section with some observations on the possible direction in which the notion of the partial group rz(a,b) for a .plane-parallel medium X(a,b) may be extended. An immediate extension of rz(a,b) may be made to a one-parameter three-dimensional optical medium in which "a" and "b" are indices of the two-parameter surfaces bounding the general curvilinear medium X(a,b). The resultant algebraic structures are isomorphic, (i.e., algebraically identical) to that of the plane-parallel case and so will not be explicitly considered.

An extension of r2(a,b) beyond one-parameter media would be to an arbitrary connected medium X in which "x" and "y" in ^{7}/7(x,y) now denote two arbitrary points' of X or possibly small subsets of X. We shall call x a point in either case in what follows. This extension is of great physical interest and we pause to examine it using formal operations in just enough detail to see how the generalization may go.

Let X be an optical medium in three-dimensional Euclidean space, i.e., the space which represents an ordinary everyday world. Within X we can simulate portions of the earth's atmosphere, or its seas and lakes. Let N° be the incident radiance function on X and N the associated response radiance function on X. Then the interaction principle supplies an interaction operator 7.fl(x) which maps N° into N:

N = N°;~,/(X)

(93)

The reader will recall that N is a function which assigns to each x in X and E in F the radiance N(X,E) at x in the direction E. Thus, from N we can obtain the radiance distribution N(x) at point x. Let E(x) be the operator which assigns to point x in X the radiance distribution N(x) at point x as induced by the radiance function N. Thus:

N (x) = NE (x) (94)

In other words, E(x) is a continuous (or generalized) version of Ci or D- introduced in (82). Conversely, from knowledge of N(x) It each point x of X, we can reconstruct N. Let "trn E(x)" be the operator such that:

I = f [ ]E(x) [trn E(x)] dV(x) (95)

X

where I is the identity operator (transformation) on the vector space V(X) of all radiance functions defined on X. (The use of vector space concepts was introduced in an earlier discussion; see Example 15 of Sec. 2.11.) We shall not go into the details of construction of the operator trn E(x). It will suffice to note that it is intended to be analogous to the transpose operators discussed 'in (80) and (81) and may be constructed using theorems A,B,C of the interaction method in Sec, 3.16. Using this partition of I in (93) we have:

$N = N^0I$ 7f(X) = f $N^0E(x)$ [trn E (x)] ' ;~j (x) dV(x) . (96)

X

Applying the operator E(y) to each side of (96) we have:

N (Y) = NE (Y) = N^0E (x) [trn E (x) I ( ) E (Y) dV (x)

• (9 7) f

X

Let us write:

ff Mo (X; x, Y) " or

ff w ^{0} (x Y) "

for

[ ] [trn E (x)] 7?j(x) E (Y) dV(x) (^{9}\$)

X

Then (97) can be written as:

N (Y) = No (x) T1?^{o} (x P Y)

(99)

where

No ($_x$) = N^0E ($_x$) (100)

We shall now assume that the operator 71~o(x,y) is one to one for every pair (x,y) of points in X, in the sense that two distinct incident radiance distributions NIO(x), N2 (x) always are mapped into distinct corresponding response radiance distribution functions NI (y) , N2 (Y) using (93). By distinct radiance distribution Ni, NZ it shall be understood that for

some set $\sim_o$ of directions in E, f wo [N $_i$ (x, C) -Nx (x, C) j df2 (C) >O . Matters can usually be arranged so that an optical medium X can be$_0$partitioned into pieces Xi over each of which the operator M (Xi;x,y) is one to one. Hence no essential loss in$_0$generality will be engendered in what follows if we assume

(X;x,y) is one to one over an arbitrary optical medium X. The one to one property of

^{o}~x,y) is used to insure that

the inverse (71 (x , y) 3 $^{-1}$ of(x,y) exists. For, once this inverse is available, we can directly relate any two radiance distributions in X. Thus, from (99) used twice:

N (Y) a No (x) $^{7`}$ 1 ^{o} (x f Y)

whence:

N(z) = N^o (x) Z1^o (x, z)

N (Y) (~I^0 (X.Y)) 'i

N(z) (Mo(x:z)) -'

whence again:

N (z) = N (Y) [j ^{o}(x,Y)] $^{-1}$ ~^{o}(x.z) which holds for every x in X, so that if we write: ~ s

(Y , Z) " for [MO(X,Y)I _ , m^o (X I Z) : (101)

we have:

N (z) = N (Y) -M(Y, z)

(102)

for every pair y,z of points in X. In this way we generalize the invariant imbedding operator 717 (u,x) of Sec. 3.7 to a wider geometric setting, i.e., to one in which x and y are not surfaces, but possibly points or subsets of X. We retain the notation '^{1}77/(x,y)"without fear of confusion with the simpler concept in the present discussion. Recall that x and y are now points or subsets of X rather than depth parameters for surfaces. We shall denote the set of all ^{7}~J(x,y) with x and y in X,

by ^{11}r2 (X) * to

I t follows at once from (101) that the operators 291(x,y) in r$_2$ (X) form a partial group in the sense explained in the discussion around equation (79) in Sec. 3.7. Hence

r$_2$ (X) is a proper generalization of r$_z$ (a,b) .

Several directions of further development of (102) are possible at this exploratory stage of the analysis. For example, using Stage II of the interaction method we can represent ?V(y,z) as an integral operator over W. Alternatively, we could partition ?1j(y,z) into a 2X2 matrix analogously to the partition in (18) for the plane-parallel case, and develop a theory for 07(y, z) analogous in every detail to that between (18) and (92) above, but now for the general medium X. Since this is representative of a nontrivial extension of the invariant imbedding group r$_z$(a,b) to more general settings, we shall now explore the initial details of such an extension.

In the plane-parallel case we had the terrestrially-based coordinate system as a frame of directional reference for the partition of M (x, z) as shown in (18). In the present case there is no preferred or pre-existing coordinate frame from which to launch the construction of the present counterparts to ^{7}~^{f}++ (x, z) ,))~+- (x, z) , A/_ + (x, z) ,

and 2~_ - (x, z) Therefore for the first stage of the present extension we simply assign to each point x in X a partition x (x), .2(X) of into two parts. This partition can follow any rule, so that Ei(x) need not be a hemisphere. Once the partition is specified at each x in X, the radiance distribution N(x) is restricted to 71 (x) and 2 (x) resulting in N3 (x) and lye (x) , respectively--in complete analogy to the N_+ (y) ,and N. (y) of the plane-parallel case. This partitioning of N(x) at each x in X into the pair (NI (x) ,N2 (x)) in turn induces a cleavage of M (y , z) into a 2x2 operator matrix such that:

(Y,Z)

(103)

$^)$~21 (Y, Z) ' h 1 2 2 (Y, Z)

The details of this partitioning of ^{7}I(y, z) are very much like those used to establish ')~° (y, z) from 7~(x) above or El_i from XI in (90), except now (95) is replaced by a formula like (80)

2

I = C_i[trn C_il (104)

where Ci, i = 1,2, is the operator which assigns N^-(y) to N(y), so that the Ci are like C+ and C_- in (4), (5~. Indeed, the partition-(103) is obtained in precise analogy to (92) for the case m = n = 2. Hence we may refer the reader to equations (80)-(92) for the general outline of the details.

With this decomposition (103) of $^{7..}$'/(y, z) , equation (102) may be written:

^{177}(}1(Y,Z) ^{1}~?12(Y~Z)

(NI (Z) N2 (Z)) (Ni (Y) ,N2 (Y)

7Yj21 (Y, Z) V722(Y,Z)

(105)

As a specific instance of (10S), let ~(yz) , be a smooth directed ath in X connecting point y to point z (in that order).

Once , z) is specified then any point x along - it is located (Y by a single parameter--the distance of x from y along the. curve, and the tangent to the curve is given the usual sense at X. See Fig. 7.8. At each point x of &P(y,z), let E(x) be the tangent to the curve. Then let wi(x) and '2(x) of the general discussion above be E+(E(x)), E _((x)), respectively, where ~+(~(x)) , it will be recalled (Sec. 2.4). is the hemisphere of E consisting of all directions ' such that ~' • C (x) ? 0, and = _ (E (x)) consists of all E' such that ' -4(x) s 0. With these assignations of -I(x), E2 (x), the formula (105) takes the form:

FIG. 7.8 Extending the $^;$r(x,y) operators to general geometries,

? I + + (Y 0 z) + _ (Y 2 z)

(N + (Z) O N _ (Z)) ' ~ (N+(Y)pN_(y)) , (106)

m i l _ + (Y I z) 9 ~ _ _ (Y P Z)

where N_+ (x) and N_ (x) are now the restrictions of N (x) to +(~(x)) and respectively. Thus (106) is formally indistinguishable from its plane-parallel counterpart; furthermore the algebraic properties of $^;$~$^;$I(y, z) in (106) are identical to those of its algebraic counterpart and (lab) reduces to the stratified plane-parallel case when F~' (y, z) is the straight path from level y to level z and such that &,~P(y,z) is perpendicular to the parallel planes of X(y, z) in x(a,b). As we shall see in Sec. 7.11, (106) reduces the type of solution procedures used for light fields in a general medium X to those used in plane-parallel media, with arbitrary lighting and optical conditions In the preceding explorations of the possibilities beyond rz(a,b) there is a general pattern forming for one such family of extensions. We conclude these explorations with a summary and review of the incipient pattern for the case of an arbitrary subset S of a medium X. The formation of the extensions begins with an invocation of the First Stage of the interaction method. This yields the generic equation:

where S may be all of X- or a proper part of X. Furthermore N may now be 'radiance' functions, for .polarized light, and may depend explicitly on scattering with change in wavelength, - etc, Hence X may be-more-than three-dimensional. 'Let us assume X is n-dimensional.. (See opening remarks, Sec. .99.0f

Ref, [.251].) Using the technique of - decomposing -the identity

operator, as in (80), (8 1) .,(95) ,. or (104)., . the basic, equation (107) can be systematically taken apart leaving an operator which -forms a member of, a new partial semi-group r z (S) .

The ways in which (107) .may , be so analyzed are .manifold. The . examples cited above show that the partition of' the identity.. operator may ~be "over -spatial variables (as in .the case of

(,80), ('81), and (95)) or over directional variables (as, in

the case of (104)),' The work of Ref i- [251] shows how the . partition of the identity. operator may in' other contexts be over the location space of a 'di8cre'te optical medium .(Sec. 90,

Ref* [251]).with the resultant generation .of the local operator *o analogous to ~l^{o} in (98). In-addition, the technique of partitioning the identity operator As applicable to the polarized radiance context (Sec. 114, Ref: [251]) and also the heterochromatic radiative transfer and even the general Markov-process context-of general radiative transfer of equa-,

ti-on VII, (Sec,-119 of Ref. [251]) . With these examples in mind let us assume a quite general partitioning of the identity operator I on the vector space of radiance functions on S, thus:

[ ]C(x) [trn C (x)] dV(x) , (108)

f

S

where now x is a point of the-subset S-of the n-dimensional space X, and V is the volume measure on X. (The various dimensions of X may arise from the various parameters needed to describe N--location variable's,-direction variables, polarization parameters, wavelength parameters, etc..)'. The operators C (x) are analogous to E (x) in (95). Therefore we write

'IN (x) ~~ for ~~C (x) (109)

in complete analogy to the earlier special .cases of N (x) .

Next we insert I, in the form given by (108), between No and 7.~7(S)' in (107) to obtain:

N = N ° I /?1(X.)

J N°C(x)[trn C(x)] 1~/(S) dV(x) . (110) S

By (109) we have

a

No (x) = No C (x) [111) and in analogy to (98) we write:

" ^{0}(S;x$_9$Y)~~ $_{or}$; 11Mo(x,Y)11 for.

J] [trn C(x)]~((S)C(y) dV(x) . (112) $_X$[

Therefore, upon operating on each side of (110) with C(y) we have:

N(y) = N^o (x) Mo (x , Y)

Assuming the integral operators W(^{o}(x,y) to be one to one for every x and y in S we de f ine

";7((y, $_z$) " for [Or^o(x.9Y)1$^{-'7}$~^{o}(x' z) (114)

in analogy to (101) , The collection r2(S) of a.11 operators OY(y,z), with y,z in S is seen to be a partial group as in the earlier instances; so that for every y,z in S:

N(Y)7,)7(Y.Z)

which holds for an arbitrary radiance field N in S.

Further partitions of the identity can now be made on the vector spaces of radiance distributions with elements N(x), x fixed in S. For example, if x is simply the spatial variable then further partitioning of the direction space or wavelength space can be made if desired. Thus in general, let D_a(x) and D_0 (x) be operators such that,:

I fix) fA D_a (x) [trn D_a (x) du (a) (116)

is the partition of the identity operator on the vector space of functions N (x) at x in x.

The space A is the space (either discrete or continuous) which is being partitioned and u is the measure, and could be direction space or wavelength space, etc. Let us write:

"N_a (x) " for N (x) D_a (x) (117)

and

S

i R ~ a , (5 ; s , Y) " or "71a6 (x,Y) i t for [trn D_a (x) 7 (x,Y) D~ (Y)

The functions N_a(x) with a in A; and ~(x,y) are generalizations of Ni (x) , and M i - (x, y) in (I0 5 , where now the space A is quite arbitrary. Se^{r}e also the discrete example (85) of [118]. Then to see how far these generalizations can go, we return to (115) and observe that we may write:

N(z) = N (Y) I (Y) *$^;$~I(Y ., z)

= f $_n$N (Y) D_a (Y) [trn Da (Y)] ~'(Y z) du(a)

which is one of the possible generalizations of the type to which (92) belongs. This concludes the summary and overview of a possible general method of constructing partial groups r2(S) of operators on the subset S of the optical medium X. The problem of generalizing r_z (a,b) to r_2 (S) will be considered once again in Sec. 7.11. _