7.7 General Solution Procedures for R a b and T ab in Plane-Parallel Media

We return to the general integrodifferential equations for R(a,b; t' ; t) and T*(a,b; t' ; Q as given in (12) and (29) of Sec. 7.6, and develop a general numerical procedure for their solution without the benefit of homogeneity and isotropic scattering within the medium X(a,b). The approach we shall follow is quite direct, one which requires a minimum of numerical preliminaries, thereby leaving such matters for choice in the individual programming procedure for the numerical solution. For example, with only minor changes, the following analysis may be repeated using Gaussian quadrature procedures. For the purposes of the present exposition, the determination problem for the operators R(a,b) and T(a,b) is considered solved when their correct functional equations have been found and suitably reduced to an initial value problem for some set of approximating (or occasionally exact) differential equations. Perhaps the greatest value of the following discussion is to allow students of the subject to come to grips with the inner workings of the integral operators R(a,b), and T(a,b). Once this is done, perhaps some efficient solution procedures will eventually come to mind.

We begin with a partition of ?_ and + into m and n sets of directions Ai and Bi, respectively (see Fig. 7.15); that is, we assume:

m
w
= u A_i ; A_i n A_j w,.
n
"+ _ B_i Bi n B_j = (Z) i=1

These partitions of E_ and +, if sufficiently fine, let the integrals over them be reduced to simple numerical sums, as follows.

Consider, for example, the integral term occurring in (12) of Sec. 7.6:

FIG. 7.15 General partitions of each half of the sphere of unit directions,

f a (a 3 E°; E') R(a, b; ~' i ~) dSt

which with the partition of _ in force, may be written equivalently as the sum:

m
fa(a;to~ da)
i=1 A u

Now, if the partition of W .. is sufficiently fine and a and fit, as functions of V are continuous (a condition always available in geophysical settings) the integral over Ai may be represented arbitrarily closely by a term of the form:

o (a; E°; g') R(a,b;·&·! ; E) A.
1 1 t

where El is a direction in Ai and vi is JE!ekl-,

Now V and E as they occur in the reduced term, happen also (for some j and k) to be in partition members Aj and Bk respectively, so that we can completely replace the functions

v and R, for the present purposes, by sets of numerical quantities, as follows. We write:

"R($_a$,b; i, j) " for R(a,b; ~i; c_J) (3)
where tl and Ej are selected fixed directions in Ai and Bj, respectively. Hence the function R(a,b;•;•) is replaced by the mn numbers R(a,b;1,j). These direction selections are fixed for the remainder of this discussion. Further., we write:
f f p (a ; i !)) 1 ' for (4)
whenever ~i is in Ai and ti- is in Bi:
tlp+(a; $_{i, J}$)'~ for a(a; Ei; tj) (5) whenever Ei is in Bi and E] is in A_j:
whenever E_i is in Ai and 9J is in Aj
1 ' $_{T+}$ ($_a$, i, j) • 1 for a (a; i ; $_J$) (7) whenever E_i is in Bi and &j. is in Bj.
The values in each of the four cases just defined are readily determinable from the given volume scattering function a. Observe that we are not assuming that the medium is
isotropic, so that no use will be made of reciprocity principles [of which r(a;u',v) = r(a;v,u') was an instance in Sec. 7.6). With these four definitions we may disassemble (12) of Sec. 7.6 into the following system of differential equations which can approximate (12) of 7.6 arbitrarily closely:
3R(a b; i, j 2y + y ( ) 1 -L I (J)
+ nnm~ rm~i~rr r`r

The unknowns in this system are the mn functions R(a,b;i,j), i = 1,..., m; j = 1,..., n. Holding b fixed, a is allowed to vary from b up to any arbitrary distance above b. Hence we have an initial value problem, which is got underway using the initial condition:
R b,b;i,j) = 0
for every choice of the mn directional pairs. The (m+n)2 dimensionless numbers such as:
and
p (a; i ' --~---
.I\ j
a (a)

(A k) 1 i

Tls - 9 k

k $_{a (a)}$

n(B)

n n
can be assembled neatly into four matrices, denoted, say, by: "r_ (a) ", "t_ (a) ", "t_+ (a) ", "r+(a)II, respectively. (For an alternative procedure, see (17)-(21) below) Then writing "R(a,b)" for the mxn matrix formed of the numbers R(a,b; i, j) , (8) can be written succinctly as:

1 aR a, a + [DR(a,b) + R(a,b)E] s
oc (a) a

r_ (a) + t_ (a)R(a,b) + R(a,b)t+(a) + R(a,b)r+(a)R(a,b)
lr_(a) + R(a,b) t_+ (a)]
[t_ (a)+ R(a,b)r+(a)] R(a,b)
where we have written:
"$_E$lf
f or

A check of the linear dimensions of the matrices shows that they are all properly commensurate. Thus, r_(a) is mxn, t_(a) is mxm, t_+(a) is nxn and r+(a) is nxm. The choice of
the partitions of 't,. and w_+ governs the dimensions of these matrices. In some settings it ^ is quite possible and, indeed,
desirable to partition :_ and ~+ in essentially the same manner so that m a n and the vi and vi are equal for each
i = 1,..., n. Hence we would have D = E and all matrices would be nxn, and there would be n^2 equations in n2 unknowns implicit in (9). As in the earlier reductions in Sec. 7.6 a transition to dimensionless depth parameters would be numerically convenient. Hence the change of variables given there should be adopted for numerical work. Equation (9) can be used to find both R(a,b; ~' ; C) and R(b, a; C"; C') in the case of non-separable media, simply by integrating from level Xb to Xain the first case and from Xa to I in the second case (cf. Fig. 7.10 in which the integration rom Xb to Xa is depicted).
It remains to reduce Equation (29) of Sec. 7.6 to its approximating matricial counterpart. We-retain the general partition of E used for R(a,b) and write
"T*(a,b;i,])" for T*(a,b;Eitc i)
whenever 4i is-in Ai and ~j is in Aj. In this way we generate the mxm matrix T (a,b).

The system of differential equations approximate to (29) of Sec. 7.6 may then be written:

n Q(B .k)
(a,b) I T-(a;i, j) + 1 R(a,b;i,k)p+(a;k,J3 (12)
k=1

where "Try (a,b) " denotes the beam transmittance Tr- (a, Cj) ane r3 is ,now the distance r, as defined in (27) of Sec. 7.6 associated with E-. Using the battery of matrices defined above, this can bi cast into- the compact matricial form:

*
- 1 ~ aT a b + DT *(a,b) a (a) as
t-(a)·T (a,b) + R(a,b)r+(a)T (a,b) +
+ [t- (a) + R(a, b) r+ (a)] T^o (a, b) (13)
where we have written:
"T^0(a,b)" for
Trz (a,b) .
(14)
The differential equation (13) may be written, compactly as:

,~ ~1 aT a, , + DT *(a,b)= [t-(a)+R(a,b)r+(a)]T(a,b) a (a) as
(15)

where we have written:

"T (a,b) " for $T^{\circ}(a,b)$ + $T^{*}(a,b)$. (16)

The resemblance of the right side of (15) with that of (29) of Sec. 7.1 is striking. Similarly with (9) above and (18) of Sec. 7.1. The solution procedures for (9) and (15) can run along parallel to each other, for as (9) is solved for R(a,b), these values could be fed into (15) a fraction of a second later to help construct T*(a,b). Indeed, a study of (9) and (15) shows that whole groups of terms of matrices are shared by both equations and that their simultaneous computation would help produce efficient computation programs.

A concluding word about the choice of definitions of the various matrices, such as r_(a) and R(a,b) made above, is in order. These definitions are not unique and may be replaced by variants which, in the press of numerical work, may be found more amenable to the computational procedures than those exhibited above. For example, instead of R(a,b;i,j), we could use as unknowns the terms:

$R(a,b;i,j)/v_j$

(17)

and then R(a,b) would be made up anew of such entries: Once this is done, it automatically dictates the following recombinations of terms which are guided by an examination of (8) and the list of terms following it. Thus, instead of p-(a;i,j)/a(a) we would have: resulting equations would have the same gestalt as (9) and (15) so-that the same general numerical procedure would be applicable to either system.