7.9 The Method of Semi-Groups for Deep Homogeneous Media

The results of the preceding section, in the form of the module method of solution of radiative transfer problems in the sea and the air, were so simple and direct that we are encouraged to explore the method in more detail, with an eye toward obtaining a general method applicable to all media. Thus our purpose in this section is to begin with the basis for the module equations, namely the system (13) of Sec. 7.8, and study the effect on the module equations when the module . thickness is allowed to go to zero but with the depth z (=jd) held fixed. The resultant equations will reveal a general pattern which suggests the requisite generalization, namely the method of semigroups.

The Semigroup Equations for 0 ' (z)

Consider (14) of Sec. 7.8, which describes the downward radiance distribution at depth jd, j a 0. and where d is the thickness of the module X(0,d) for the homogeneous infinitely deep medium X(O,co). If we halve the depth of the modu1e ; then we must double the powers to be used to find N,: (j d), i.e., we are observing that:

~`"^{2j} (d/2) = ~'^{i} (d) 9 (1)

which follows from (12) of Sec. 7.8. More generally, for every positive integer n:

17- nj (d/n) J-J (d) (2)

which follows by induction on n, starting with (1). Setting in (2) we see that the resulting equation, namely

17 ^{n}(d/n) " g(d)

demonstrates quite graphically that the module transmittance is the product of an arbitrarily large number of transmittances of 'sub-modules' of thickness d/n. Now, (12) of Sec. 7.8 shows that 9(0) = I, the identity operator on the set _ of all downward radiance distributions, M - is defined in the invariant imbedding statement Sec. 3.9-]. The continuity of X(s), which holds for the most part in all natural optical media, then implies that the transmittance operators for the submodules X(O,d/n) approach the identity operator I. That is:

lim X(s) = I s'0

All this is quite clear when one reflects on the definition of 7 (s), being an instance of a complete transmittance operator. But now, in the light of the present approach, wherein the analogy of r(s) and beam transmittance T_s just waiting to appear explicitly upon the scene, in this light we are moved to consider next the limit:

s-'0

which is motivated by the defining equation for volume attenuation function a (Sec. 3.11)

1 -
1 im
_P1 r s 0

The limit involving 7"(s) will, of course be an operator of some kind rather than a number, as is a; however, the analogy now a-building seems so suggestive that we are next moved to write:

"All for

 Hence, for any depth differences s , we .have directly from (3)

A + c(s) = I - s (4)

where 'fe(e)" denotes an operator (actually defined implicitly by (4)). which goes to zero as its argument goes to zero. Solving (4) for 7(s)

7(s) " (I -- sA) + o (s)

where "o(•)" denotes e(•)s. The closer s is to zero, the closer o(s) is to the zero operator. This equation is analogous to

(1 ~ so) + o (s)

for beam transmittance (where "o(•)" in the latter equation is of course distinct from that in (5)).

The momentum of these definitions, and discoveries of analogy carry us on to consider the present analogous structure to the differential equation for beam transmittance: (cf. (2) of Sec, 3.11). Thus, we are led to form the difference quotient:

and obtain its limit as s goes to zero. In preparation for this, we write:

7 ~r+ +5I_~~rI w ,. .~

(A + e (s)) T(r)

which follows on use of (12) of Sec: 7.8 and (4) above. Therefore we have:

d 7(r)
dr

wherein we have written

"d ~7r i'

for 1 im r,7-(r+sL - Z(r)

m m n n n r n . m r ~ r ~ i i i Z (r)

dr s+U s

We pause now to restate the purpose of the present discussion, we wish to find the continuous counterparts to the module equations (14) of Sec. 7.8. The first step, just completed, makes clear the structure of 7(s) when s is allowed to approach zero: This structure is shown in (5) and (6).. The next step find the continuous counterpart to 7-J (d)

as d goes to zero but such that j d is some fixed depth z1, In view of the analogy between 9_r and Tr which has guided the developments so far, it is clear that this next step should be equivalent to finding the operator version of:

Tr = exp (-ar-)

This observation requires us to find the operator analog of exp {-ar}. At this point we recall that the Maclaurin series development of exp {-ar) shows promise of being extend

able to the operator context, especially since we have the basic derivative formula (6)- to work from. . Therefore by means of (6), taking all the integral derivatives of 7(r) in dr 3 dr dr and in general:

z d.,7"'-(r)

A 3 7(r)

dJ T~ - j

-.---.r- (1) A g-(r) (7) dri

3"(0)

Using the identity property of T(r), namely that I, (7) yields:

R d)T(r)

Following through on the Maclaurin series analogy we then write:

"exp { -Ar } •"

for

This definition makes sense from a strictly operational point of view. For we can perform, at, least in principle, the iterations of the operator A- to find A) for every integer. j .

Furthermore we can multiply AJ by the real number (-1) J r~ /j!

in each case; and we can add together any finite number n of such combinations to end up with

as a well-defined operator. Granted all this, it appears that, at least on a numerical or empirical level, the exponential characterization of T(r) is settled. ~ The mathematical reader, however, will wish to dwell on the convergence problem entailed in the definition (8). Such considerations are quite simple and are readily characterized in terms of the radiometric norm (Ex; 15 of Sec. 2.11) which supplies the necessary machinery in algebraic radiative transfer theory to handle problems of convergence of operator sequences. Such a digression is not pertinent in the present discussion, and we can safely pass it by without serious effect on the remainder of our study,. Interested readers wishing to study. such matters in more detail are referred to Re-f. (110], a book devoted almost exclusively to the extension of the ideas, inherent in Car, to their most general settings.

With the definition (8) `and .on the basis of the analogy of (8) with the scalar (numerical) context, we see that:

(r) = exp (-Ar)

With this representation,, 9"(r) exhibits directly and succinctly all its important` properties ((12) of Sec. 7.8, and (3), (5), (6) above). We now may write (13) of Sec. 7.8 as:

N_+ (Y) = N_(O) exp {-Ay}-R,,, N_ (Y) = N_ (O') exp { -Ay

(1D)

Equations (9), (1©) are the requisite semigroup equations for O~(r) as they are applied to the determination of N, (y) in an infinitely deep homogeneous plane-parallel medium. The operator A is called the infinitesimal generator of the semi-group formed by the transmittance operators T(r). (The semigroup structure stems primarily from. the property (12) of Sec. 7 . 8 .) Readers acquainted with the theory of stochastic processes (in continuous time, say) will observe via (6)-, or (9) that a radiative transfer process in a deep homogeneous optical medium may be viewed as a Markov process which evolves continuously with depth in that medium.

The Infinitesimal Generator A

One final point remains in the preparation of the system (lo) for actual numerical application, or in further theoretical work, and that is in. determining the explicit

dependence of A on the inherent optical properties of the medium. Thus we, have the task of finding how A depends on the volume attenuation and scattering functions a and u.

The key to the required answer rests in the equation (37) of Sec. 7.5. For, setting a = o and b ~, in that equation, we have:

a (4 a) _ j~r0?(0 jYj,°°) P (Y) + 7(0

VY 9) (Y)

ay

in the contracted notation presently in use, this is:

d.: R(0VY1,Q*)Pr

dy

J(y)T

I(Y) [Roop * T]

Comparison of (11) with (6) leads us to the required representation:

(12)

Equation (9) can. now be written as.:

(r) exp (T + R.p) r. . (13)

Thus, the infinitesimal generator A of the semigroup { 7(r)} of complete transmittance operators is characterizable as the sum, of two operators: T the local transmittance operator, and R,.p, the product of the (global) reflectance operator R.,, of X (D,°°) and the local reflectance operator p. observe that, by the homogeneity of X(o,co), all three operators comprising A are independent of depth in,X(o,c*), so that A has the same property, Further, observe that:

T(r)A = A 7(r) ,

by virtue of the discussion leading to (6), and in particular the semigroup relation (12) of Sec, 7.8. The approximate matricial form of A is readily forthcoming from those of T, P and R,, as given by the discussions in Sec. 7.7 and 7.8. In particular, we use p+ and T_ in place of p and T.

The role of the infinitesimal generator A as compared to that of the volume attenuation function a, as these roles are viewed from the theory of radiative transfer as a whole, is characterizable succinctly as follows: A is to N as a is to No. That is, A is the logarithmic derivative of downwelling observable radiance distributions in a plane-parallel medium, while-a is the logarithmic derivative of directly transmitted (i.e., residual) radiance along a path. Putting it still

 another way, A is the 'volume attenuation function ',f-or the natural (undecomposed) light field in deep homogeneous media such as the seas, lakes, and optically dense atmospheric med-. la, This may be seen by returning to (13) of Sec. %8 and taking the derivative of N_ (y) with respect, tv y: , Thus

dN y (y) d - .

a N_(D) . .

dy dy.

N_ (O) (-A T(y)) N. (o)(y)(-A) -N _ (y) A

When we apply the derivative operator d/dy to the upwelling radiance distributions as given in (13) of Sec. .7.8, we have: dN_+ (y) a T

N. (0) R o N_ (0) (-A'7-(y) R~ dydy

-N_(y)AR,,

This representation of the depth rate of change of N+(y), while not as direct as that for N_ (y) , still shows that the logarithmic depth rate of change of N+(y) is essentially

A. Since commutativity of A and R. need not generally hold, we cannot generally place "R,," next to "N_(y)" in the preceding equation to get N+(y) as a result. This asymmetry in the local behaviour behavior of N+(y) and N_(y) is a slight and inessential notational irregularity in the otherwise conceptually pleasing and powerful formulations of the method of modules and the method of semigroups. A search for a more symmetric treatment of the depth rates of change of N_+ (y) and N_(y) leads to the method of groups to be considered in the following section.