7.10 The Method of Groups for Deep Homogeneous Media

Once the flush of discovery of the semigroup equations (10) of See. 7.9 has passed and the critical eye runs over, their asymmetric forms, one is moved to search for a new set of equations which incorporates both the conceptual and computational power of that set with a more pleasing symmetry of form. In this section we embark on' such a search and are rewarded with a set of equations which fulfills all these requirements and more. The additional dividend is a novel perspective of Chandrasekhar 's classical method of solution of the transfer equation in plane-parallel homogeneous media [43] from the heights of group theory and the modern theory of differential equations. As a result, we can view Chandrasekhar's classical method as but one of a large family of possible solution procedures unified from the viewpoint of invariant imbedding theory. This insight then unites with that encountered in Secs. 6.1-6.4, in which novel views of the spherical .

harmonic method were developed, to give an overview of all the classical solution techniques .'in radiative and neutron transport theory, and, indeed, all linear transport theories..

The setting for the present section is once again (as in Sections 7.7-7.9) an infinitely deep homogeneous source-free plane-,parallel optical medium X(a b) with a-= O_P b = ~, X 0!
.is irradiated at each point of its upper boundary by a given arbitrary incident radiance distribution N_(0), and has an arbitrary volume scattering function a, and scattering-attenuation ratio s/a.

The Return of the Group rz(0,00)

The natural candidate for the task of symmetrizing the semi-group. -relations (10) of Sec. 7.9 is the group r2(^{0},^{00}) introduced in its general form in Sec. 3.7 (see, in particular
.(79)-02) of that section) and studied at some length in Secs. 7.4 and-7.5. Toward this end, we direct some attention to the specific form of rz (o, co) . Now that we have a particularly simple physical setting, the structure of r2(o,co) takes on some rather interesting properties. For example, the homogeneity of X(o,=) makes each
member ;7#T(x, z) of r2'(0,°*) depend only on the difference z -x, where x and z are any two depths in X(0,co) . This fact may readily be seen by an inspection of equations (19)-(2b) of Sec. 7.4. Consider, for example (19) of Sec.7.4. Since X(0,w) is homogeneous and isotropic our findings (of Sec. 7.7, e.g.) show that T(x,z) and R(x,z) depend only on the absolute difference Ix-zl. Further, from (23) of Sec. 7.4 we see that 71~++(x,z) is not generally the same as ?~l++(z,x), but still 71++(x, z) depends only on the magnitude of the difference z-x. Hence the operator matrix 1-7(x,z) depends only on z-x for which we shall write "s" for brevity, so that Ir(x, z) is written as " M(s-)" whenever s = z-x. The general group closure property of r_Z (0,°°) , namely:

17(x, y) 7.IT (y, z) = 27(x z)

now takes the form

;7I(r+s) _ n(r)M(S)

This should be compared with (12) of Sec. 7.8. We see that there is an important difference in the range of parameters s in 7'/(s) and those of '(s) . Whereas s 't 0 in (12) of Sec. 7.8, we have -w ---- s :S w for s in (1) .We may summarize these differences as follows: The set r2 (0,^{c*}) forms a group which is isomorphic to the-additive group of real numbers. Thus, to each pair of real numbers r,s there correspond operators,
(r) , ~ (s) of r $_2$ (0 ,~) , and to the sum r+s corresponds te operator '*'7 (r+s) , such that (1) holds. On the other hand the set { X (r)) o f complete transmit lance operators discussed in See. 7.9 forms a semigroup which is isomorphic to tixe additive
semi group of non-negative real numbers. Thus to each pair r,s

of non negative real numbers, there ,correspond*operators T(r), 7-(s) of :"(r) I and to the sum r+s corresponds T(r+s) such that (12) of Sec. 7.9 holds. in this way ,I by means o f (1), we can view the theory of radiative transfer in homogeneous in-finite plane-parallel media as an instance o f the theory o f continuous groups on the real line. (That .all- of radiative transfer theory is essentially attainable .via r2 (a,b) was. demonstrated in Sec. 7:4. See also the remarks leading to (19) of Sec. 7.3.)

The Infinitesimal Generator of r2(^{0},^{0*})

The concluding insight arrived at in the paragraph just above can be put into quite concrete terms. One way of putting it is to say that ,'conceptually, the theory of determining the radiance distribution N(y) at depth y in X(9,oo) is as simple as determining the reduced radiance NO of a beam a distance r from the source, for both quantities are governed by the exponential law. To see this in the case of N(y), recall the operator forms of the equation of transfer. (9) of Sec. 7.1:
dN(-_ N(Y)X(Y)
dy
where A (y) is defined in (7) denotes (N+ (y), N- (Y))': Next governing 7yt(x,y)- as given in
of Sec. 7.1, 'and "N (y) as usual recall the functional equation (1) of Sec. 7.5:
d 77f (x
dY
7"7 (x 1~ Y) X (Y) (3)
where '?C(Y) is the same operator as in (2). In view of the homogeneity properties of X(O,w) we can write (3) simply as:
dam(= W(Y) fC
dy
with the initial condition:
I^{o}w (0) = I
and where:
Here o and i are the local reflectance and transmittance op

e ra t or S for X (©, oo). They are independent of depth y in X (O , co) . K is the z: infinitesimal generator of the group r2(0,[00])-

The Exponential Representation of M(y) and N (y)

By following the same motivations as those leading to (8) of Sec. 7.9, we write:

t'exp (KyItt for

in which -a[6] :S_y :5c*. This operator is a function of y and satisfies the same differential equation as 721(y) in (4). Further, exp {1C o } 1 = '-T(0) . Hence:

.

7 (y) exp (X y }

(8) .

The 'exponential representation of N(y) - a.t any depth y in x(o,o*) follows immediately from (2) using the same reasoning which yielded (8); or one may use the fact that: .

N(y) = N(O) W(y)

which with (8) implies:

N(y) = N(O) exp {fit y)

From this, we also have:

. N(z) = N(y) exp {k (z-y) } (10)

for every pair z,y of depth in X(O,co). Equation (9) is the requisite symmetric rendition of (13) of Sec. 7.8.

The Exponential Representation of Q(y)

In Sec. 7.5 it was noted how close the connection was between the .operators [7.J]I(x,y) and the pair of complete operators .fit (a,x,b) , X(a,x,b). The basis for this connection is summarized in (38) of Sec. 7.5. We pause to explore this similarity in the light-of the present developments.

The similarity between equation (38) of Sec. 7.5 and (3) above shows that the operator 6Z(y) has the same depth behavior as ~(y), though their initial values differ. Thus, (38) of

Sec. 7,.5, adapted to X(O,w), becomes:

in which:

d_(2 ~ = Q (Y) X (Y) dy

a c v _ (. G ? (y) , x (y))

where we have written: 114R (Y) [11]

for .R (0 jy,w)

and

"7(Y)[t'] for 7(01Y,C*)

Hence, while 42(y) and 20r(y) satisfy the same differential equation, the initial condition for 0(y) is:

Q (0) a 69 (0) , 910)

which follows from (40) of Sec. 7.5. Therefore, analogously to (8) we have:

a(Y) ' C1(0) exP {Ky}

It is interesting to note the effect of the presence of "(2-0)" in (11) on the multiplication law of the operator a(Y). It turns out that the set (&(y)) does not form a semi group under ordinary operator composition. Indeed, from (11), used three times as follows: we

Q (r) = GZ (0) exp {J~ r} 67(s) = 42(0) exp. {?(s)

Q (r+s) = Q (0) exp {?C (r+s) } find that, at least formally:

eZ(r+s) _ C2(r)4Z-1(0)a(s)

(12)

This shows that we have gained the symmetry of (9) at the expense of the simple semigroup property for the set {4Z(y)j. However, the loss is ~not essential. For by defining the following star product of members of { Cl (y) % we establish a group structure for {2(y)). In view of the semi-group properties (52) and _(53) of See. 3.7, let us write:

12(s) ,t for (9-(r) A(s) (r) J-(s)) • (13)

Then it follows immediately that:

6? (r) * 6Z (s) = a(r+s)

(14)

The set { CL (y) } , with the preceding star product defined for its, elements in the manner shown in (13), becomes a group isomorphic to r2 (0,°°) , once the definitions of 7(r) and R (r) are extended to negative values of r. This can be done directly through (11) by simply computing exp (?Cyl for negative values of y.. Further, the semigroup relation

a (r+5) = 7 (r) 4 R (s) (15)

which is a special case of (5 3) of Sec. 3 . 7 , . is formally extended, for this purpose, to the domain of negative arguments. This extension can be rigorously included in the theory deducible from the interaction principle by adapting the extension of the group rs(a,b) (now for the special case a = 0, b = } suggested in (44) of Sec. 7.4.

Numerical Procedures for N(y): The Exponential Technique

Equation (9) for the radiance field N(y), as already noted, is the primary goal for the present section. In its symmetric form rests the solution of the problem of the penetration of light into the sea, atmosphere, and other plane-parallel media. There are several ways of coaxing numbers and general information from its terse mathematical form, and we shall study such ways in this and the following paragraphs. Each technique to be considered is based on a preliminary. reduction of (9) to an approximating matrix statement. This reduction is quite analogous to those developed in Sec. 7.7 for the differential-.equations of R and T. Hence we may pass through this preliminary reduction stage with relatively little explanation.

The reductions center principally on the operators P and T making up the exponent operator X in, (9) . The form of X i s given by (6), and p and t in turn are defined in (3). and (4) of sec. 7.1. For the purposes of the present reduction we may drop references to the depth

variable; however, the directional variable E in the integral form of p and T, must be explicitly exhibited:
(1 7)

Thus, the homogeneity of X(0,w) allows a convenient suppression of the depth variable y in a,p,a, and T. When p is applied for example to the upward radiance distribution N+(Y), we use . E_+ in (I6) along with C in - : By means of the general
partitions of ~+ and :_ established in (1) , (2) of Sec. 7.7
we can replace p by an nxm matrix whose general element in the ith row and jth column is:

6(~~ -) R(B-)
ain which i is 'in B i (C +) and' E - is in A j and "v i" denotes 1 ~1'k j . Q (Bi) is the s~I id ~ angle content of H i .
We denote this matrix by "p_+". A similar matrix p_ can be manufactured 'Suth that it has elements of the form:
where ti, is in Ai (C E-) and & j is in B j (C +) , and . rhe tt "u i" denotes I ti k j . Hence p .. has dimension mxn, Fu;r,t~. er,,

use*
as the ij-th element in an mxm matrix denoted by "T_". An nxn matrix T_+ is constructed in a similar manner for upward radiance, its ij-th element being
a Chi;) R(Bi) I.
,r..S. _.....
The preceding mode of reducing the operators p- and T is the most simple and direct mode. Alternate modes of- d more sophisticated type (such as those using various quadrature formulas for a) are possible; however, the 'structure of the main
formula (25) below is independent of the choice of such modes. Reassembling these _matrices into one grand matrix X where we have written:
T+ p+
and writing:
##N_ (Y) it
for [N(y,&j), ... ,NNAY1,CIr;)).

when t, in A_l

the approximating counterpart to (9) is seen to be:

N(y) = N(0) exp {X y)
(25)
and which shall stand as our base of operations for the remainder of this section.
Y

Clearly N(y) is an (m+n)-component vector and X is a square matrix of order (m+n).

As noted' above, the general form of (25) is invariant under the choice of mode of reduction of the operators P,T'and radiance functions $N_+(y)$, N-(y). Therefore what we have to say about (253 below will hold above and beyond the details of the reduction procedure leading from (9) to (25). ' .

Equation (25)~ as it. stands can form the basis of perhaps the simplest and most direct of all techniques of solution of radiative transfer problems in homogeneous plane-parallel me
dia with stratified light fields. For by simply raising the matrix X to the first p integral powers and constructing the 5 uIR

where p is, perhaps as small as 5 or 6, one obtains a reasonable approximation-to exp (,f y), so that when applied to N(O), we have

as -a correspondingly reasonable estimate of N(y). Observe that knowing N(n) means knowing all m+n components of N(0). Hence we can predict N (y) once N(O), the surface or boundary lighting conditions are known. More generally, in view of (10), N (z) is computable whenever N (x) is known, where z and x are any two depths. This most remarkable fact points up in sharp clear detail our rather general assertions about the "strong inner structure" of natural light fields discussed in Sec. 3.7 (cf. Ex. 7 of Sec. 3.7).

An alternate scheme to that just discussed is based on the matricial counterpart to (2):

-L = N(Y) X(Y)
dy
(26)

where ?((y) may now depend on y (hence X(0,°°) may be non homogeneous but stratified). Thus we work with (26) directly and integrate that system of linear ordinary differential equations on a general purpose computer. The initial condition on N(y), namely N(O) is assumed known.

It is unlikely that any computation techniques could be simpler in concept or in execution than those based on (25) or (2G) (or on the decomposed versions of (2b)) using N in
the manner just explained. This points up one of the earmarks of invariant imbedding techniques, i.e., the ability to replace some of the classical and somewhat numerically cumbersome eigenvalue techniques by relatively simple initial value or one-point boundary value techniques, and which may be handled generally by the tools of semi-group theory.

The Characteristic Representation of N(y)

In deep homogeneous media with stratified light fields, such as those we are studying in this section, the exponential law (25) for radiance distributions N(y) can be cast into a particularly instructive form using the Jordan canonical form of J'IC . The Jordan canonical form of a matrix is defined in most works on modern algebra, and in some texts on ordinary differential equations such as [47], and we therefore need not digress to discuss the details of its computation. However, we shall define the canonical form and discuss its physical interpretations in the radiative transfer context. Our purpose in casting X into its Jordan canonical form is two-fold. First, we shall be able thereby to fulfill our promise, made at the outset, to show the special place of Chandrasekhar 's theory of solution of the equation of transfer within the general theory of solutions as given by the invariant imbedding and interaction principles of radiative transfer, Second, the characteristic representation o f N (y) , as we shall call the resultant equation obtained below, deepens our understanding of the exponential structure of light fields in natural optical media by showing explicitly the delicate interplay of the various streams of radiant flux as they penetrate the body of an extensive optical medium, each stream with a characteristic mode of decay. In particular, we shall be able to explicitly observe the eventual dominance of a characteristic radiance distribution at great depth within the medium, the shape of the characteristic distribution being determined solely by the volume scattering function a of X(o,w) and being independent of the directional structure of N(0), the radiance at the boundary of the medium. All of this knowledge is possible without explicitly solving the equation of transfer for X(0,w), as we shall now see.

To begin, we recall from the theory of linear algebra that the Jordan canonical form of the (m+n) x (m+n) matrix X can be obtained by the construction of a suitable (m+n) x (m+n) invertible matrix P and performing the operation:

$P^{-1}X\ P$

Let us denote this resultant matrix by "of ". It follows immediately that:

X = P1 P-1

exp {1(y) = exp {P j P^{-l} y}

= P exp { ~yl P-1

The latter equality may be verified by using the definition of exp { Y1.

ow, the general gestalt of the Jordan canonical matrix is as follows

D	...	o	D
I	...	0	0
0	...	x_q+i	I
0	...	0	x q + i

for i = 1,
s,
and where Yi. is an r i x ri matrix, s a that

"
r_i + q = m+n. Now, exp { $)_i$t) is of the form etXq+i A,
i=1
where all elements of A below the main diagonal are zero, and the elements of the upper jth diagonal,, counting the main diagonal as zero, are of the common form yJ / j ! , j = D , . . . , ri -1.
The numbers A. j 1 $_e$.., q, q+l, ..., q+s are the distinct =
Characteristic (or eigen) values associated with -&. The X. from j = 1 to j = a have multiplicity 1, those of the form ~ +i, 0 < i :~.s, have multiplicity ri. Hence, altogether, count Q multiplicities, there are m+n characteristic values Xi, as expected. 5o much for the abstract algebra of Jordan canonical forms.
Let us turn now to the particular matrix at hand, namely 7f, and attempt to block out the salient structure of its canonical Jordan form. Imagine the operator X , as given in (6) far the present context, to be replaced by its matricial

approximant. The main outlines of (G) will persist and we will have, according to (22):
T
For the purpose at hand, namely to deduce the special form of Chandrasekhar's equations, we adopt a special partit i o n of 'E, as follows: Let =.- be partitioned in an arbitrary,
manner. Then reflect W-, in its partitioned form, in a horizontal plane. The result is a partitioning of + which will be a mirror image of that of ~_.. In particular, we number the partition elements Ai, Bi such that Ai and Bi are mirror images of one another. The effect-of this type of partitioning on the Jordan canonical form of the resultant matrix 1'C can be seen by examining typical entries of X ¯as given in (18)-(21). Thus we find that, under the mirror image partition of E:
T_+
P_+"P
and)(becomes
where each indicated block 'Matrix is an m x m matrix (since
m = n, by virtue of the mirror partition). Now it is an elementary fact of matrix theory that a matrix such as X , in its newly obtained form, has eigenvalues which come in signed pairs. Thus, if A is an eigenvalue of X, then so is -X* For example, consider the following+ 2 x 2 matrix made up of the numbers a,b:
-
-a b wb a - The characteristic equation for this matrix is:
-a-a b
det = 0 -b a-a

The X's which satisfy this equation are the required characteristic values. The preceding equation simplifies to:

tat ~Ca~r~ + bz _ u

so that, a is required to be: a + or X- where a + _ + Paz -bz)

that is X+ = -X_. Furthermore, it may be shown, on physical grounds, that the component matrices 4i of . (-v:ther than ~?O) do not occu in the case of X.

Briefly, i > o, to appear in it is necessary_ that there exi:s.;t.- .components of .N (y) such that they can have scattering orders "of at most finite order ri, No component o N(y) has this property, so that the i do not occur in ,. Hence the Jordan canonical form of must then be such that s = 0, i.e,, ~ consists only of o, The resultant form of (25) is then quite simple: 9

N(y) = N(O) exp {P ~0 P-1Y1

N(0)P exp {i_oYIP^{-1}

It is easy to see that:

exp { toy)

We define the characteristic radiance vector by writing:

"t (Y) " f or

N (Y) P

Then (25) can be written:

-W AV

N(y) = N(0)

Equation (27) is the requisite equation jor the characteristic radiance vector N(y).

Observe that if N_j (y) is the jth component of N (Y) , then we have

X_J-Y

f~ ow

N i (Y) = N J .(0) e

(28)

Hence each component N-(y) of the characteristic radiance vector has a specific rat9 of growth (if A_j> 0) or decay (if a - < 0) , In infinitely deep media such as X(0,,*), wherein here are no internal sources and the only incident radiance is at the upper boundary, the components Nj(0) associated with the positive valued eigenvalues are set to zero,„, The eigenvalues and components can be renumbered so that Nj (0) 0 for m + I s j ~E 2m. T o see the effect of this on the physical radiance vectors N(yj , let the elements of P be of the form ai j , and those of I?- be of the form b_{ij} , then from (27)

e

a2 MY

e

IX 2Y w a_my

N (y) (N,j (0) e ,N_2(0) e ,..., Nm(0)e ,0,0,...,o}
N(Y)P
Hence:
~" X I Y ,. amy
N (Y) (N1 (0) e ,..,, N_M(0)e ,0,.,,,O)P
Therefore;
1-0
~-Y
N - (Y)m
~ . N_i (o) 'b_i -e i J
...,2m
(29)
This is the desired characteristic representation of N(Y). Each Xi is non positive, i.e., ai ! O for i = 1,...,m. Observe that each of the 2m quantities Nj(y) is completely determinable, knowing the 2m quantities N (O), the entries aki and b-- of the matrices P and P , and o course the m eigenvalue s~ ai, By retracing the steps leading to (29) and assuming X(O,w) to be replaced by a finitely deep homogeneous medium x(O,d), d4 a*, we see that (29) changes only slightly:
the upper limit of the i-sum becomes 2m and the non negative eigenvalues Xi, $_m$+l, i!:- i2m can enter the representation. Equation (29) or its counterpart for x(O,d) is representative of the general form of Chandrasekhar's equations in his classical work (43]. The salient difference between them rests in the manner of representing N and a over` , thereby fixing the associated values of ati, b-- and Ai. Chandrasekhar uses Gauss' method of representing N and Q functions by Legendre polynomials, whereas the present method appeals directly to the observable partition of the radiance function as given in (1), (2) of Sec. 7.7 and (23). (24), In this way we have arrived at the first goal of the present discussion, namely, the illustration of the place of Chandrasekhar's mode of solution of the equation of transfer in the general scheme of radiative '` transfer theory, as seen from the invariant imbedding point of view.

Asymptotic Property of N(y)
The final topic for discussion in this section is 'the matter of the asymptotic property of radiance distributions in deep homogeneous media. The property states that the shape of the radiance distribution .N(y,•) approaches a limit as y-o-w in X(0,°°), and that this limit is determined solely by the structure of the volume scattering function v on X(0,°°); and so, in particular, this limiting form of.N(y, •) _ is independent of the radiance distributions at the surface of X (0 ,*o) . We shall discuss this matter in detail in Chapter 13. However,, there exists a simple instructive proof of the asymptotic radiance property using the general system of equations (29), i.e., the characteristic representation of N(Y), and while the momentum of the present discussion is still high, we shall give a demonstration of the asymptotic radiance property using (29) as a bases
Our present goal,_ therefore, is to show that the 2m-component vector N(y), whose j th component is given by (29), approaches a 2m-component vector N(oo) as a limit ,.that R(ao) is

determined only by a, and that N(w) is independent of N(0) . Now, the first thing to notice is that the m .numbers Ai are, in real media, all negative,. so that N(y)~generally goes to the zero vector o (I.e.; the 2m-component vector with all components zero). This, of.course is not the vector N(co) we are seeking. The decrease in size of N(y) as y+c* is distracting as one .seeks its asymptotic shape ,'and this decrease can be erased by normalizing 'N(y) with respect to some factor which decreases to zero with y at the same 'rate as N(y). The. graphical interpretation of this normalization is quite simple:
the radiance distribution at each depth y is magnified in size so that one of the radiance components, say that representing vertically downward radiance, is of unit magnitude. Then all other components arrange themselves in size relative to this unit component. I f N (y) , so plotted, approaches a fixed vector, as y-+ao, then we say that the limit N(c*)' exists.
In the present cast the 'normalization factor' may conveniently be chosen as e Y where k is the smallest of the numbers -Ai, i 1,...,m. Specifically, we write; ad hoc: for Min'. -p -Am} .
This implies that $_A e^- ky$ goes to zero with + least speed of
the iY Ai Y
all factors e . In. particular e goes to D as y goes to w for every i, except for when Ai = -k. To be specific suppose Ai = -k. Armed with this factor, we multiply each side o f (29) b y e ky and let Y-.°°
2m
lim N (y) $e^k Y$ = N (0) a b - (30)
Y+W i k k i 17 k=l
for j
seep 2me
1,

Let us write a" for the th column of P, and "bl" for the lth row of P-'.[1] Then the system of 2m limits (29) can be written:

N (ca) (N(O)*a)bl
(31)
where in turn we have written:
f or 1 im N . (y) $e^k y$

$Y^- s^{00}$
Equation (31) shows clearly that the directional structure of N N(co) i s simply that o f the Y th row o f P^{-1}, The I th row of P^{-1} is determined solely by the matrices p and t which are manufactured from a. Observe that the directional structure of N(O) is wiped out by the taking of the dot product of N(0) and the I.th column of . Hence the asymptotic directional structure of N(y) can be so determined solely by computing P-1, and this is independent of N(O). Looking back on the trail we have traveled traveled , we recall that P is the matrix which %aps JIC into its Jordan canonical form. Thus we can find N(co) by purely algebraic operations on X which, as

we have seen, is the infinitesimal generator of the group r2(0,[w]) of invariant imbedding operators associated with the medium X0,[40]) .

Asymptotic Properties of Polarized Radiance Fields

l We conclude the discussion of the characteristic form of the radiance solution by noting tha t,the techniques just used for the unpolarized context can equally well be applied to polarized radiance distributions. This means,- in particular, that the theoretical questions of the asymptotic properties of polarized radiance fields raised in Sec. 4.6 and still earlier in Chapter 1 can be fully resolved using the preceding technique. Equation (31), as it stands, has the gestalt of the corresponding equation for polarized radiance, differing from. the polarized version only in the dimensions of the vectors and matrices involved. This difference is precisely determinable: all vectors in the unpolarized context go over into the polarized context with a four-fold increase in components, and all matrices go over with a corresponding four-fold-increase in their linear-dimensions. However, beyond these quantitative differences, the two theories of polarized and unpolarized radiance distributions are algebraically alike. [See, e.g., Section 114 of Ref, [251],) Some-experimental work on the asymptotic polarized light field-has been done by Herman and Lenoble (147]. Otherwise, there exists at present very little experimental study of the asymptotic polarized light field.