Media

Homogeneity, Isotrop, and Related Properties of optical

In this section we collect together some special knowledge that has been gathering during the development of this and earlier chapters, knowledge concerning the properties of homogeneity, isotropy, polarity, and related concepts associated with optical media. This accumulation of facts is timely in that it will play an important role in rounding out the theory of internal-source generated light fields in natural optical media to be considered in the following section, and in Example 14 of Sec. 5,7.

As we shall see, the problem of internal sources in optical media requires for its solution no new concepts beyond those presented in Example 3 of Sec. 3.9, However, this battery of concepts gives rise to some relatively complex (but highly instructive) operations with the standard reflectance *and transmittance operators* for optical media. Any insights into the reduction of the number of the participating operators and their assemblies in the final formulations will correspondingly reduce the amount of labor required to effect specific numerical or theoretical answers to the source problems.

One of the classical means of simplifying radiative transfer formulations is the use of "symmetry principles", chief among which are various reciprocity principles governing *the R and T* functions. It is one of the purposes of the present section to define and discuss these symmetry properties, outline their extensions to general media, and to indicate when the extensions are or are not helpful. Perhaps the most important outcome of this discussion, at least from a practical point of view, is the unpleasant fact that most of the "symmetry principles" of the classical theory no longer hold in the general settings of arbitrary optical media. In other words, many of the "symmetries" that arose in the classical settings arose because the settings themselves were symmetry-cal and generally quite idealized, and not because there subsisted some inherent invariant character of the symmetry,

For example, by graduating from the use of irradiance or from scalar irradiance (or radiant density) within infinite or semi-infinite homogeneous isotropic media, to the use of radiance in such media, at least one important reciprocity theorem falls by the wayside. By making the space inhomogeneous, but still isotropic, an important symmetry property vanishes into the void. By making the space *finite,* inhomogeneous and irregular in geometric structure essentially all but one of the classical symmetry properties (reciprocity for radiant density) leave the investigator with handfuls of functional equations whose associated analytic difficulties must be squarely faced without any essential help forthcoming from the lone surviving symmetry principle. In short, the moment one steps from the nice one-dimensional spaces with their nice one-dimensional radiometric concepts and enters the representer of the real world, namely euclidean three-space, and attempts to describe radiant flux in that setting in terms of

radiance rather than radiant density, then, except in the most singular cases, the classical symmetries no-longer subsist and hence are no longer available to facilitate numerical and theoretical activity.

Despite the predominantly negative features of the following discussion it will still be instructive for the reader to have consolidated and clarified some of the more frequently used "symmetries" and "uniformities" in the processes, of constructing models of natural optical media, To this task we now turn.

Throughout this section let X *be an* optical medium in euclidean Throughout this E3 to which is associated a volume attenuation function a, a volume scattering function a and an index of refraction n. The domain of n is X, the domain of a is X x W, that of a is X x x ~, the values a(x, &) and a (x; &' ;`C) are non negative real numbers. It seems best to proceed by" making formal *definitions* and following them with appropriate comments that illustrate their physical meanings and in them. We begin with the local concepts, i.e.,' concepts associated with the points of X. Following this the global concepts are introduced (those associated with subsets-of X) and an attempt will be made to define the global concepts analogously to the local concepts whenever possible. One of-the main problems of the present area of radiative transfer theory is to determine whether a valid local concept in a given medium X carries over to the global context. We shall indicate, by theorem and example, some instances of this problem as the discussion proceeds.

Local Concepts

Definition I . X is said to be *homogeneous if* the values n(x) , a (x , 4) and a (x ; g' ; 4) are independent of x for every ' , in .

Since a depends generally on x and E, the values a(x,E) in a homogeneous space, while independent by definition of x, may possibly depend on &. Thus, e.g., while a [x, E] = a (x' , ~) for every x, x' in X, this common value may depend on E. Per haps this is an academic point in the sense that homogeneity is rarely found in such a general form in nature. Be that as it may, the present definition, being necessarily framed with n, a and a as the basic concepts at hand, some decision must be made as to the &-dependence of n, a and a in the homogeneous case. The decision adopted above imposes the least restrictions on the functions while capturing the basic idea behind homogeneity: the *uniformity in the spatial domain of* the values of n,a and a.

Homogeneity helps simplify the equations of radiative transfer in many ways. The most immediate effect is in the structure of the beam transmittance function. In general, for a path $6^{:2}{}_{r}[x_o$,Q we have

r

T_r (x_o *3* [n2 (x) *In' tx$_o$~] exp* I- a (x' , E) dr'} fo

n (x') is the index of *refraction at x' = x_o + r' E , a distance r' along 6Pr(xo,E)* from the initial point 'x_o. When X is homogeneous, the index of refraction function n is independent of location in X so that in particular /r(xo,0 is a straight line segment with direction E. Further, n (x) = n (x_o) , along with a (x' , E) a (x_o, E)-. Thus under homogeneity, Tr (x09 E) becomes:

-aCx$_o$r

T_r*Cx$_o$*~~) (Z)

where a (xo, E) is the fixed value of a in X associated with the direction E, and r is the length of ~?r(xo,o .

It may be possible to have the index of 'refraction essentially constant on X without having a or a independent of *location. When this is the case we have*

restricted inhomogeneity - of X. *Such* inhomogeneity is ideal for the theorist in radiative transfer: he has the opportunity of studying the main problems of radiative transfer without the annoying and distracting possibility of curved or broken paths *,Cx 9*
~ 3,
and of varying radiance values in an otherwise clear medium (cf. Sec. 21 of Ref. [251]), or in the beam transmittance function (cf. (1)). Therefore, throughout this section, when we consider X to be inhomogeneous it will be understood to be a restricted inhomogeneity of X.

We conclude this discussion of homogeneity by rephrasing the definition in terms of the notion *of a displacement transformation* on X. A function D on X with values in X is a dis*placement transformation* if, and only if, there exists a fixed point y such that:

Here we are *using the* fact that the points of E3 (and hence those of X) are ordered triples of numbers, as in analytic geometry, so that there is an algebraic basis for adding them together. The main part of Definition 1 may now be phrased analytically as follows. "X is homogeneous" means:
whenever D (x) is in X, then n (D(x)) = n (x)
and a (D(x) , E) = a (x, E) and u (D(x) ! C' ; C) '~ v (x; &' ; 4)
.
A less restrictive notion than homogeneity but one that still permits all the analytic blessings of homogeneity to be enjoyed by the theorist is the notion of separability of X:

Definition 2. X is said to be separable if the index of refraction function is constant and a(x, ~) is independent of E and if a (x; E' ; g) ja (x) is independent of x for every &',E in f .

The reason for the name "separable" becomes clear on fixing x in X and writing:
It p(x;t'; 0 +1
for 41Q (x; &' ; &) /a (x) (3)

The *function p on X* x = X so defined is called *the phase function* in astrophysical optics (cf. Ref. (43 j) and by means *of it a (x-4[1];g) map be written:*
afix ') *1* *a(x)P(x; V; 9)* *(4)*

Hence in separable media a(x;E';E) may be written as the prod*uct of two functions:* one which is free of x and the other which depends on x--so that the spatial dependence is uncoupled or separated from the main directional dependence. In particular, in a separable medium the spatial dependence of
is carried by a, while the directional dependence of a is carried by p. The utility of the separability assumption becomes clear on examining, e.g., the definitions of the matrices r_ (a) , t_ (a) , etc., occurring in (9) of Sec. 7.7. If the medium were assumed separable, then r_(a), t_(a), etc. would be independent of a, while still allowing a measure of inhomogeneity of the medium to be present.

In separable optical media, the natural measure of distance is not geometric distance but optical distance, in the following sense: If 6' (x,~) is a path in a separable medium,
then its optical length is the number r , , the integration being taken along the path. $\sim_o$ a[x) dr This number is usually designated by "T(r)" and enters into the theory via the equation of transfer when a transition from r to T (r) is made. Thus the equation:

dN
a -aN + N*
dr

becomes:

1 dN
,~ N + (N*/a)
a dr

Since:

dT

a a
dr

we have:

dN „ (N - N) dT q
Beam transmittance in separable media becomes:
T r (X,4) n '-r(r)
(6)
If the dependence of T on r is suppressed and T is made the basic measure of distance, then the medium X'is homogeneous, in the sense of Definition 1 with respect to the distance measure T. Furthermore, the volume attenuation function oin such a separable medium with optical distance T is replaceable by a unit-valued function at all points of X. In other words, in a separable optical medium one can normalize the *volume attenuation function* and effectively remove it from

the scene, and the volume scattering function is replaceable by the phase function.
Definition 3. An optical medium X is said to be isotropic at x if the values a(x,E) are independent of t and the values a(x;g';4) depend only on the scalar product Cot of E' and
X is isotropic if it is isotropic at every point.
From this we see, first of all, that while homogeneity of X is constancy of a and a on X, isotropy of X is a con*stancy of a on* B along with a certain special constancy of a 04 on w x Specifically, "X is isotropic at x" means
for every &',~ in , a(x,&') = a(x,E), and (7)
for every 91, C Z , 9 s , ~ 4 in E, if then

Q(x;Ei;E2) = a(x;Ea;W 0 (g)
Isotropy of X can be characterized by means of rotation transformations of E3. *Let T be a rotation of E* 3 at x. Then the preceding isotropy conditions may be rendered as:
and
a (x, T () = a (x,)
a(x,T(&),T(~')) = T(x;&';&)
for every ~;,&
in z. and every rotation T at x.
Definition 4. A scattering process (or a) is said to be isotropic at x in X if a (x; E' ; E) is independent of E' , E in .
An attenuation process (or a) is said to be isotropic at x in X if a(x,E) is independent of E in W. A scattering or attenuation process is isotropic if it is isotropic at every x in X.
The distinction between the medium X being isotropic and the scattering process on X being isotropic is thus clear, The connections between the two ideas are as follows: If a *and a are isotropic, then X is isotropic.* Can the other hand, *if X* is *isotropic then a is isotropic, but a need not be iso*tropic. This anomaly of symmetry in the isotropy properties stems from the fact that a has two spatial variables while a has only one. Hence nailing down isotropy of X fixes that of a but leaves a a margin of variability, a margin, incidentally, which has been found most useful in the classical theory.
Observe that if a is isotropic at x, then
a (x; E' ; E) = s(x)/4w (9)
where s (x) is the value of the volume total scattering function of x. Furthermore, if X is separable and a is isotropic, (4) and (9) combine to yield:
s (x) = a (x) p W ; 0
so that:
P(E' ; 0 = s (x) Ion (x) (1D)

From this we see that the phase function value p(&';&) is independent of E' and E and is a real dimensionless number between 0 and 1, *a number* which we have called the-scattering attenuation ratio.
Definition S. A scattering process (or a) is said to be re*versible at x in X* if the following property of a holds: for *every 9',9 in , a(x;9';C) = a(x;ˉ ;- ']*. An attenuation pro*cess (or a) is said to be reversible at x in X if*
a (x, &) _ *a (x, -E)* for every & in,"". A scattering or attenuation) process is reversible if it is reversible at every x in X.
It is clear that if a medium is isotropic at a point x, then a is reversible at x, for, indeed, since E' • E a
the reversibility follows from (8). However, the converse need not be true: reversibility of a at x does not logically imply isotropy of X at x, and the reader may devise theoretical examples which show this.
We summarize the four main local properties of an optical medium in Fig. 7.21 which shows the class of all optical media in $E_\$$ grouped into families which are homogeneous, separable, isotropic, and reversible. Observe how the class of *homogeneous* spaces is included in the class of separable spaces, and of how the class of reversible spaces (i.e., spaces with reversible a) includes the isotropic spaces as special cases. The

classes partially overlap in the Figure, showing that generally a space may have several, one, or none of the four general uniformities.

Global Concepts

We shall now show that the local concepts of homogeneity and isotropy can be carried over, after suitable modifications, to the global description of the scattering properties of ex
tended media. To keep the introduction to these ideas simple and intuitively meaningful we shall at first consider only stratified plane-parallel media, i.e:, media whose a and a are independent of location on planes parallel to the boundaries. Later in the discussion more, general media will be briefly discussed.

Now the counterpart to a in the global context is the reflectance function R(a,b; &' ; &) and the transmittance function T(a,b; &' ; &) associated with a plane-parallel medium X(a,b).

These pairings are intuitive and not to be taken in a formal sense. They suggest various analogous properties of the global functions that one may seek. For example, the analogous global *property to homogeneity* is the condition that R(x,z;&';9).and *T (x , z ; .&' ; &)* depend only on the difference z -x , where, e , g . , &' is in W-, and & is in E+, as the case may be. It is easy to see that, if and only if X(a,b) is homogeneous or separable, then this property holds for R and T, either directly, or after shifting over to the optical length parameter.

The next concept which may be profitably extended to the *global* setting is that of isotropy of the medium X at a point x.

Polar [non separable or
non reciprocal
Separable
Homogeneous

Isotropic
Reversible
Set of all optical media in euclidean Euclidean space

FIG. 7.21 The four principal categories for local properties of optical media and their general logical interdependence.

Instead of a point we now have a general subslab X(x,z) in X(a,b), and instead of the condition that &'•& be fixed in magnitude, we require that the directions be related by means of a reflection in a plane parallel to Xa thus:

Definition 6. A stratified plane-parallel medium X(a,b) is said to be symmetric if the following properties hold for every subs lab X (x, z) of X (a, b)

R(x, z; E' ; E) = R (z fx;M(E') ;M(E)

and:

T(x, z; E' ; E) , = T (z,x;M(E') ;M(E)) (12) for every reflection transformation M of in a plane parallel to Xa.

FIG. 7.22 The reflection of directions E,E' in plane P, as used in describing polarity of an optical medium.

The opposite notion to that of symmetry in the present context is polarity. Thus X(a,b) is *polar or exhibits polar*ity if it is not symmetric; and this, by Definition 6, means that there exists a subslab X(x,z) of X(a,b) such that either

or:

R(x, z; t' ;&) 0 R(z,x;M(&') ;M(P))

T(x, z; g' ; &) 0 T (Z,x;MW) ;M(9)

for some reflection transformation M of ' in a plane P parallel to X_a (see Fig. 7.22 for the case of reflectance). The main theorem about polarity is the following:

Polarity Theorem: Let X(a,b) be a stratified plane-parallel medium. (a) If $X(a_l b)$ is separable and isotropic, then X(a,b) *is symmetric; (b) If X(a,b) is non separable and isotropic, theca X (a, b) is polar.*

The proof of the theorem maybe made to devolve on the differential equations for R(a,b) and T(a,b) in-Sec. 7:3, but will be omitted here. The main point of the theorem is that symmetry of X(a,b) may be lost by the presence of essential inhomogeneities in X(a,b); by "essential inhomogeneity" is meant that the medium is not just separable, but rather such that o (z; 4' ; t) /a(z) depends on depth z in X(a,b). A proof of the polarity theorem along with examples for discrete spaces is given in some detail in Sec. 57 of Ref. [251]

Going on now to the global counterpart of reversibility, we have:

Definition 7. A stratified plane-parallel medium X(a,b) is said to be reciprocal if the following properties hold for every subslab X(x,z) of X(a,b)

R(x, z; t' ; t) = R(x, z; -t; -t')

and:

T(x, z; t' ; E) a T(z,x; -g; -E')

and:

and:

T(z,x;t';&). s T(x,z;-~;-C)

Examples can be given which show that symmetry and reciprocity of X(a,b) are generally independent notions. Thus X(a,b) may be symmetric but not reciprocal; and conversely, ~a,b) may be reciprocal but exhibit polarity.- That this is plausible may be seen without too much preliminary work by letting X(x,z) approach zero thickness so that symmetry of X(x, z) becomes a manifestation of isotropy of a; and reciprocity of X(x,z) reduces nearly to reversibility of u. Since reversibility and isotropy of a are partially independent, this independence can be inherited at least by very thin slabs X[x,z]. *The main* theorem on reciprocity is the following:

Reciprocity Theorem. Let *X(a,b) be a stratified plane-parallel medium. If a on X(a,b) is reversible, then X(a,b) is reciprocal.*

Observe that this theorem, which can be proved using the differential equations for R(a,b) and T(a,b) in Sec. 7.3, holds in particular for non-separable media. The theorem was first
stated and proved for separable plane-parallel media X(a,b) by Chandrasekhar in Ref. (43]. A proof of the reciprocity theorem for general isotropic media is sketched in Ref, [40].

Summary

To summarize the main results of this section so far we *may say that in going from* the local to the global level in stratified plane-parallel media one generally can carry over the concept of reciprocity but not symmetry. More precisely, and in terms of the defined concepts above, a locally reversible medium is always reciprocal, but a locally isotropic medium may exhibit polarity.

The loss of symmetry (where we use the term in the sense of Definition 5) is a phenomenon that arises because of the adoption of radiance as the basic radiometric concept rather than irradiance or alternatively, radiant density. ..Had. we used the latter concept, then symmetry would hold (in the scalar irradiance context) ¯for inhomogeneous isotropic plane-parallel media. Symmetry would be-lost in such a context only when isotropy was lost. By adopting radiance over irradiance we reap the benefits of a more detailed description of the light field at the expense of the classical symmetries possessed by irradiance. Furthermore, the reflectance and transmittance functions R and T in the scalar irradiance context, - being scalars, commute; i.e., symbolically, RT = TR. By adopting radiance, R and -T become integral operators or matrices,
and these objects are notoriously noncommutative, thus block*ing still further* the passage of certain symmetries of the ,. scalar formulations to the field of. operator formulations.

Conclusion

In conclusion, then, the elevation of the local notions of homogeneity, separability, isotropy, and reversibility to the global settings in plane-parallel media is quite possible. However, only the local concept of reversibility is generally inherited by the space on the global level tin the form of reciprocity). But this inheritance is precarious and can conceivably vanish on graduation to arbitrarily *shaped anisotrop*ic media in which the radiometric concept used is radiance rather than irradiance or scalar irradiance. Thus all the' classical symmetries are in principle left behind in the - search for general invariant properties of scattering--absorbing media. The general principles of invariance, the invariant imbedding relations and their various semi-group properties are *important examples of* general properties of optical media which are invariant under the transition from local to global formulations within those media. This has been shown in. detail in Chapter VI of Ref, [251], for general discrete spaces.

Further study of the problem of the extension of local symmetries to he global level are best handled by means of the standard J-operator x:a b . A detailed stud of such *extensions has yet* to be made. It would be of interest to formulate the appropriate counterparts to homogeneity, and isotropy for general media using V (x; a,b) , and then to find theorems, if possible,. which. are the appropriate generalization of the Polarity *and* Reciprocity theorems.