

## 8.2 General Irradiance Equations

The global description of the irradiance field in a plane parallel medium  $x(a,b)$  as given by the invariant imbedding relation (14) of Sec. 1.8.-I will now be supplemented by a local description in the form of a pair of differential equations for the irradiances  $H(z,\pm)$  as a function of depth  $z$  in  $x(a,b)$ . The approach we shall take at present is through the principles of invariance <sup>(1)</sup> and <sup>(2)</sup> of Sec. 8.1.

The idea of the derivation is quite simple: We isolate for attention the subslab  $x(x, z)$  of  $x(a,b)$  and form the difference quotient:

$$H(z,-) - H(x' - (1) z x$$

We then let  $x$  approach  $z$  and determine the associated limit of (1). The physical meaning of this activity should be carefully noted at the outset, as it will repeatedly suggest the subsequent moves in the sequence of explorations below. Thus (1) is the average rate of change of the downward irradiance field over the depth interval from  $x$  to  $z$ . As  $z - x$  is made smaller and smaller (and hence  $x(x, z)$  thinner and thinner) we increasingly localize the factors governing this average rate of change until, in the limit, we should have a completely local description of the change of the downward irradiance field at depth  $z$ .

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Following the program just outlined, we set  $y = z$  in (2) of Sec. 8.1, the result being:

$$H(z,-) = H(x,-)T(x,z) + H(z,+)R(z,x).$$

This representation of  $H(z,-)$  is used in (1) to obtain:

$$H(z,-) - H(x,-) \frac{H(z,-) - H(x,-)}{z - x} = H(x,-) [T(x,z) - 1] + H(z,+1) R(z,x)$$

Now as  $x$  approaches  $z$ , the difference  $z - x$  approaches zero, and the slab  $x(x,z)$  becomes increasingly thinner, so that its downward transmittance  $T(x, z)$  approaches 1 and its upward reflectance  $R(z,x)$  approaches zero (cf. (9)-(12) of Sec. 7.3). Therefore the quotients on the right side of (2) have a chance of going to well-defined limits. Indeed, our discussion in Sec. 7.3 (see e.g., (9)-(12) of that section) prepares the ground for the following definitions; we write:

$$\lim_{x \rightarrow z} T(x,z) = 1 \quad (3)$$

$$\lim_{x \rightarrow z} \frac{H(z,-) - H(x,-)}{z - x} = -\frac{dH(z,-)}{dz} = -H'(z,-) \quad (4)$$

Then, if we write as usual:

$$-dH(z,-) = H'(z,-) dz \quad (5)$$

$$dz = z - x$$

equation (2) yields:

$$\frac{dH(z,-)}{dz} = T(z,+)H(z,-) + p(z,-)H(z,+)^{(6)}$$

This is the equation governing the downward irradiance field  $H(z,-)$ . In a similar way, setting  $y = x$  in (1) of Sec. 8.1, forming the difference quotient:

$$\frac{H(x,+) - H(z,+)}{x - z} = \lim_{x \rightarrow z} \frac{H(x,+) - H(z,+)}{x - z} = T(z,+)H(z,+)$$

and writing:

$$\lim_{x \rightarrow z} \frac{H(x,+) - H(z,+)}{x - z} = T(z,+)H(z,+)$$

$$\lim_{x \rightarrow z} R(x, z) = R(z, z)$$

and

$$\lim_{x \rightarrow z} \frac{dH(z,+)}{dz} = T(z,+)H(z,+)$$

for  $\lim_{x \rightarrow z} z = z$

$$H(x,+) - H(z,+)$$

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the preceding equation yields:

$$z = T(z,+)H(z,+) + p(z,-)H(z,-)$$

This is the equation governing the upward irradiance field  $H(z,+)$ .

The notation in (6) and (7) is designed to point up the fundamental similarity of these equations to the principles of invariance (1) and (2) of Sec. 8.1. The basis of this similarity (6) and (7) are also called the local forms of the principles of invariance, and  $T(z, \pm)$  and  $p(z, \pm)$  are the

local transmittance and local reflectance factors, respectively for upward (+) and downward (-) irradiance. The unity of the invariant imbedding approach is underscored when (6)

and (7) are compared with (5) and (6) of \*Sec. 7.1. In fact on the basis of this comparison, we are moved to write:

" $H(z)$ "

$$\text{for } (H(z,+), H(z,-)) \quad (8)$$

$$- T(z,+) P(z,+) \sim$$

$$f Q r \quad (9)$$

$$L - p(z,-) T(z,-)$$

so that (6) and (7) can be written:

$$d\{\sim\} = H(z)\sim C(z)$$

(10)

Equation (10) is the vector form of the general irradiance equations, and is the irradiance counterpart to (9) of Sec. 7.1.

The practical distinction between (10) above and (9) of Sec. 7.1 should be kept firmly in mind: (10) is a vector equation whose components are numbers while (9) of Sec. 7.1 is a vector equation whose components are functions and thus one level higher in the conceptual hierarchy. However, both the functional and numerical components obey many of the same algebraic relations and, generally speaking, whenever a functional equation deduced from (9) of Sec. 7.1 is valid, then there exists a corresponding valid counterpart deducible from (10). It is almost as if the local forms of the theorems of irradiance fields are the one dimensional shadows of the corresponding theorems of radiance fields; similarly for deductions from the global forms of the principles of invariance and, their irradiance correspondents. Caution should be exercised in attempting to extend results the other way, i.e., from the irradiance level (10) to the radiance level (9) of sec. 7.1, or from (1) and (2) of Sec. 8.1 back to the principles in example 3 of Sec. 3.7. While no universal rule exists to guide extensions from the irradiance to the radiance level,

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It is clear that if essential use is made of the commutativity of the numerical factors R and T while gaining a result, then the associated result need not exist on the radiance level, since commutativity of the R and T operators does not hold in general. Some examples using this observation were discussed in Sec. 7.13 (see (91) of Sec. 7.14). We shall turn to the applications of (10) in the discussions of Sec. 8.7; for the present we continue to explore its analytic structure. Before going on to do so, we pause and note one rather interesting similarity between (10) above and the fundamental dynamical equation of quantum mechanics:

$$i\hbar \frac{d}{dt} \sim \sim H \quad (10a)$$

Where  $\sim$  is a state vector which pairs with our  $H(z)$  and

$H$  (or  $-(i/\hbar)H$ ) is the Hamiltonian matrix operator which pairs with our  $\sim(z)$ . As a result of this pairing we see that the mathematics of time-dependent atomic systems is homomorphic to (i.e., of the same kind as) that for steady irradiance fields in stratified media (cf. also (46) of Sec. 8.6 and the remarks following (91) of Sec. 5.7). One more connection between (10) above (and its generalization (46) of Sec. 8.6) and the mathematical structure of different fields of physics may be noted. This concerns the formulations by Brillouin \* of the transmission line equations for two phase and polyphase electric fields. The use of Pauli and Dirac matrices to compactly represent circuit equations for such fields can evidently be carried over with only slight modifications to the radiative transfer context. However, in the present work we shall develop the algebra of radiative transfer on the basis of the invariant imbedding point of view introduced in Chapter 7.

!1 7 r" T'y r1 tw 7 y 1~