8.5 Two-D Models-for Irradiance Fields

We arrive now at the heart of the theory of irradiance fields in natural optical media, namely the two-D models in such media. The "two--D" aspect of the models refers to the radiance distributions over $^{7:}_{+}$ being assigned fixed shapes so that in turn the distribution functions D(z,±) are assigned two arbitrary fixed values: D(±) for all z.

As a - result, the exact two-flow equations (19) of Sec. 8.3 and (12) of Sec. 8.4 have known depth-independent attenuation functions and so the solution procedures of those equations reduce to straightforward applications of the theory of second order ordinary differential equations with constant coefficients. The routine solution procedure of the equations can be enriched with digressions into the physical meanings of the various basic terms arising in the procedure, and we shall expend most of our efforts in the present section in such activity.

On the Depth Dependence of the Attenuating Functions

The first matter we shall take up is the depth dependence of the functions f(z,±), b(z,±), a(z,±), and s (z,±) , in natural optical media. The observations we shall make are designed to lay the ground work for the two-D theory. Thus our present goal is to show that the attenuation functions listed above vary relatively little with depth in homogeneous media. To begin, we consider the depth behavior of a(z,±) and s(z,±), as this behavior is relatively simple to analyze into its physical and geometrical components. The case of a(z,±) is typical, so that we can limit our attention to it. According to (6) of Sec. 8.3 a(z, ±) is the product of two factors: a (z) and D (z , ±) . Hence the depth-variation of a(z,±) is tied to that of a(z) and D(z,±). The depth variation of a(z) constitutes the physical component of the depth variation of a(z,±) and the distribution function D(z,±) constitutes the, geometric component of the depth variation, of a (z , ±) . If the, medium X (a , b) is homogeneous, then a (z), is independent of, z , so that any depth variation of a(z,±), is contributed by, that of D (z , ±) .,

Now it is intuitively clear that there is generally a variation of the shape of the radiance distribution with depth z in natural waters or altitude z in the atmosphere. This variation in shape is reproduced by D(z,±) in its own characteristic manner. It turns out that in most natural hydrosols, for example oceans and lakes, the depth variation of D(z,±) is quite small, and what variation exists is quite regular or mild with depth. Table 1 exhibits a typical set of cases of the depth-variation of D(z,±) on clear and overcast days. These values were computed from some experimental data collected by J. E. Tyler (Ref. [298]) taken in a homogeneous part of Fend Oreille Lake, Idaho, for wavelength 480

TABLE 1

Experimentally Determined Distribution Functions

Clear Sunny Day			Completely Overcast Day		
Depth (meters)	D(z,+)	D(z,--)	Depth z (meters)	D(z,+)	D(z,-)
4.0	2.67	1.25	3.0	2.75	1.22
10.0	2.70	1.26	12.0	2.82	1.32
16.0	2.79	1.28	24.0	2.85	1.31
28.0	2.76	1.31	36.0	2.93	1.33
40.0	2.78	1.31	49.0	2.86	1.33
53.0	2. 77	1.30			

± 64 m⁻ p. In particular, we used the data recorded in Table 1 of Sec. 1.4, and rounded depths to integral values. The main observation we can make about the data of Table 1 is the relatively small amount of depth variation in D(z,±) under both sunny and overcast conditions. It is also to be noted that D (z , +) is approximately twice that o f D (z , -) and that their sum hovers in the immediate vicinity of 4. These interesting numerical relations are quite universally observed under the stated conditions and we shall study and apply such numerical regularities in Chapter 10. For the present we rest with the fact that there is an empirical basis for the two-D assumption about natural hydrosols to be made below.

It remains to make some observations on the depth behavior of the forward and backward scattering functions f (z , ±) , b (z , ±) . According to (7) and (8) of Sec. 8.3, the depth dependence of both of these functions is a complicated composition of a physical component contributed by a(z;E';C) and a geometric component associated with N(z,~'). We thus cannot separate the geometric and physical components of the depth variation of f(z,±) and b(z,±) as simply as in the case of a (z . , ±) and s (z , ±) . However, we can still make a few , observations that will be of help in building the two-D theory.

First of all we note that if the shape of the radiance distribution N(z,•) is arbitrary but fixed as z varies with depth, then D(z,±) are independent of depth. Thus suppose:

$-f(z)g_+(\sim)$ if is in

f (z) g_ ( ) if is in

N
$g_{\pm}$(~) I ~ ' k I dQ (~)
This shows that, under the fixed-shape assumption (1) on N(z,~) the distribution functions are independent of depth.
Now what about the converse of this observation: If the distribution functions D(z,±) are independent of depth, are the associated radiance distributions fixed in shape as a function of depth? If this were so, then Table 1 would supply the empirical evidence necessary to assert the depth independence of the shape of radiance distribution. To answer the preceding question, let us consider what constraints are imposed on N (z , •) when we require D (z , -) to be independent of depth, say with fixed magnitude D(-).
All we need know about- --D(-) at the moment is that it is not less than 1, as a perusal of its definition would show. Thus we have, directly from (5) of Sec. 8.3:

fz N(z, E) I ~ • k I dQ(E)
For D(-) to b e fixed is evidently a rather restrictive condition on N(z,•). This condition can be rewritten in the form:

This form makes it quite clear that, despite the condition imposed on it by (3), N(z, Q- can still vary in shape with depth. For suppose that we partition ~_ into m pieces Aj , i = 1) . . . , m , over each o f which N (z, ~ i s essen
tially constant. Then (3) becomes:
m i=1
c_i Ni _ o (4)
where Ni is the constant value of N(z,-) over Ai and where we have written:
11_C
Tt
1
for $_{JA}$[D(-) k .
i
It is evident from equation (4) that there is an infinite number of ordered m-tuples, (N,, $N_{2)}$ *0-$^*{}_{)}$ N_m) which satisfy it, even if the m-tuples are constrained to have nonnegative components, as required in the present case by the nonnegativity 'of N. While this may cut down on the number of multiples which satisfy (4) we certainly cannot work with the residual infinite number of possible solutions for which the constant property of D(z,±) generally holds.
However we still have at least one trick to play in the, present algebraic game with (4), one that is based on ,the fact that the numbers Ni in (4) are not to be drawn at random from the real number system but are to represent physical radiances typical of those

found in natural waters. Therefore it is fair to impose a further condition on the Ni , other than that of nonnegativity. This additional condition is what we shall call a monotonicity y condition and i t may be stated as follows: Let " Wi " denote the collection of all multiple solutions of (4). Then we say that $??_m$ obeys the monotonicity condition if the members of It_m can be arranged in a sequence, ordered by real numbers, such that if (N,(z),
. . . , $N_m(z)$) and (N_l (y) , . . . , N (y)) are two members of n
with y < z , then Ni (y) > Ni (z T for i = 1 , . . .
, m. The physical origin of the monotonicity condition is clear, the real number indexing the multiples corresponds to depth in
x (a , b) . The greater the depth z , the smaller the m
radiance components of the multiple (N, (z) , ... , $N_m(z)$). In fact the components are to .decrease monotonically with depth.

We now return to (4) armed with the monotonicity condition and require the collections 71m of solutions of (4) to obey this additional condition. Toward this end observe that the coefficients c- partition into two groups: those that are positive, and tkose that are negative. If any of the ci are zero, clearly a partition {A_j , &*.;, A_m } of - can

be re-chosen with only minor changes so the associated _ ci is not zero. The sum in (4) is rearranged so as to collect together all positive terms in one group and all negative terms in another. Hence as depth z is increased the monotonicity condition requires the m-tuples in the positive group to uniformly decrease and (4) requires the sum to be zero. Hence the members of the negative group must also decrease and in such a manner as to preserve the balance of (4). A moment's reflection will show that this still leaves many solutions satisfying (4) but it is clear that the variation of the shapes of these distributions has been severely restricted by the imposition of the monotonicity condition. It therefore appears that, on a practical level, the depth independence of D(z,±) entails that of the shape of N(z,•) in real optical media.

We shall rest the matter of the converse property of the distribution functions at this stage, having made it plausible that the observed depth independence of D(z,±) in natural waters will imply a corresponding depth independence of the shape of the radiance distribution in natural waters because of the conditions such as nonnegativity and monotonicity imposed on the radiance distributions which are based on auxiliary physical reasons. Another condition on N(z,•) which may be imposed is that of convexity of the shape of the distribution.

The preceding discussion has shown that the problem of the converse property of the distribution function is not fully resolved and it is left to interested students of the subject to pursue. Briefly the problem is this: What conditions on N(z,•) in addition to (3) and the monotonicity condition must be imposed so that the shape of N(z,•) is to be depth independent? A more practical problem of comparable mathematical difficulty is: Describe the limits within which the shape of N (z , •) may vary when N (z , •) i s subject to condition (3) and the monotonicity condition. Our preliminary analysis above showed that these limits may be quite narrow.

Returning now to the question of the depth independence of f (z , ±) and b (z , ±) , we see that in pursuing this question we are led along essentially the same analytic and algebraic path as in the case of the distribution function just concluded, so that any solution to .the converse distribution problem defined above should shed light and be directly applicable to the associated depth independence problem of f(z,±) b(z,±). In particular, we can reach the corresponding conclusion that a relatively small depth dependent variation in the shape of the radiance distribution can be expected if the depth variations .of f(z,±) and b(z,±) are small, whenever we are in homogeneous natural hydrosols.

It turns out that the converse distribution problem outlined above is needed only for relatively shallow depths in homogeneous media (on the order of three or four attenuation lengths) for below such depths the asymptotic radiance distribution begins to take hold (cf. (3) of Sec. 7.10, and Chapter 10) and the distribution function becomes essentially depth independent along with f (z, ±) and b (z, ±) . For if (1) holds, then in addition to (2), we can conclude that in homogeneous media

f :7

f (z, ±~

g+() I '

. f= k dQ( )

and

_

g+(V)Cr(z;V;0"(0

M

dQ (~)

which follows from (7) and (8) of Sec: 8.3. Since the medium is homogeneous, the "z" may be dropped from

to emphasize the homogeneity assumption and the fact that f (z , ±) and b (z , ±) are then depth independent.,

We have now arrayed enough evidence, both empirical and theoretical, to make plausible the working assumptions of the two-D theory, namely that D(z,±) along with f(z,±) and b(z,±) are essentially depth independent in natural optical media over the greater part of the depth ranges of such media. Furthermore we have shown that the distribution functions are workable indices of the shape of radiance distributions in natural waters, so that the relatively complex changes in the shape of radiance distributions can for many practical purposes be succinctly compressed into and represented by the pair of numbers D(z,±).

Two-D Model for Undecomposed Irradiance Fields

The two-D model for undecomposed irradiance fields takes as its foundation the general two-flow equations (19) of Sec. 8.3 and adopts the following additional assumptions. Let X(a,b) be a plane-parallel medium such that:

(i) a = 0 , b = z with 0 < z

I

(ii) X(0, z,) is separable and its boundaries are transparent.

(iii) The radiance distributions in X(0, z,) satisfy condition (1).

(iv) H(0,-) is an arbitrary irradiance, and H (z, , +) =0.
Assumption (i) merely sets the stage on a convenient slab in euclidean Euclidean space with a terrestrial coordinate frame for hydrologic optics. The upper boundary is at depth 0 and the slab

may be either finite or infinite in depth. Assumption (ii) says several things at once; first of all, separability means that the ratio a / a is independent of depth (cf. Sec. 7.12) over the range of depths $0 < z < z_i$. Therefore, by a change of the depth variable from geometric to optical depth, x(O,z~) can be rendered homogeneous in the optical depth coordinate system. We shall assume that this is done so that a and a may be independent of depth in what follows. Other than (ii) no restrictions are made on the relative magnitudes of a and a , or on the directional structure of a. The required transparency of the upper and lower boundary planes x_o and Xz~ eliminates the need to consider interreflection effects between these planes and the body of X(0, z_l) .

Such an interreflection calculation is neatly dispatched by means of the interaction principle and is reserved for the latter stages of the present discussion. The assumption (iii) immediately implies, via (5)' through (7), and (13) of Sec. 8.3 that all the coefficient functions in (19) of Sec. 8.3 are constants with respect to depth. We shall emphasize this consequence of (iii) throughout this discussion by writing " a (+) " and " t b (+) " for the coefficients of the two= flow equations (cf. (7)) and "D(±)" for the fixed values of the distribution functions as given by (2). Finally, assumption (iv) limits the incident source flux to an arbitrary irradiance on the upper boundary of X(O,z,). The solution of (19) of Sec. 8.3 obtained under assumptions (i) through (iv) is known as the first standard solution of the two-D model for undecomposed irradiance fields.

Equations (19) of Sec. 8.3 reduce, under the above conditions, to:

+ dH[z,± = - [a(±) + b(±)]H(z,±) + b(+)H(za+) dz

The general solution of system (8) may be cast into the form:

H (z , ±) = m+g+ (±) ek+z + m-g_ (±)e k- z

where m_+ are two constants which will be determined by assumption (iv), and where we have written:

" $g_+(\pm)$ " for

tt g- (+) tt for 1 ± a ~+
k (11)

and:

tt k+ tt for [1] [a (+) + b [+) - a -) - b (-) 2
[(a +) + $_{b\ t+)}$ + a(-) + b(-)) 2 - 4b $_{(+)b}$ (_) $_1$ 1/2)
(12)

It is easily seen from (12) that

(13) (14) (15)
)

and k_+ = 0 (16)

if a > 0, t en $k_- < 0 < k_+$

if a = 0 and b (+) = b (--) , then $k_- = 0 = k_+$

i f a = 0 and b(+) > b(-), then $k_- = 0$ and

$k_+ = b(+) \_ b(\_)$

i f a = 0 and b(+) < b(-), then $k_- = b(+) - b($-

In all cases, then, we have $k_- < 0 < k_+$. This permits us to view the term in (9) with k_+ in the exponential as a general growth term, while that with k_- may be viewed as
a general decay term, when z is measured positive downward into X(O,z_l).
We shall henceforth assume the medium to be nondegenerate, i.e., k_+ ~ k- , which limits our considerations to media with properties (13), (15), and (16). Degenerate media have trivial radiative-transfer properties, as may be inferred from the solutions displayed below.
Putting assumption (iv) to work, we require of (9) that:
H(Q,From (17) and (18) we
where we have written

for
-) =
$m_+ g_+(-) + m\ g\ (-)$ (17)
$_0\ k\_$
= (18)
$m+g+ (+)\ _e k_+ z_i + m\_g\sim (+) e\ z,$

find that:

$+ H(0s^-) g\_ (+) e^{k_+ z'} (Z')$ (19)

:
$g_+ (+)\ g\_ (-)\ ek+z, \_$ (20)
$_{g+} (-)\ g\_\ (+)\ _e k\_ z,$

Therefore the first standard solution of the two-flow equations for x(O,z,) consists of~the following two equations:
H(z,±)

(+) g_ (   ) ek+ z'   + k- z - g_ (+) g_+ (±) $_e$k+ z + k   z
(21)
Writing the upward and downward irradiance equations separately, we have:

H(z,+) = H 0 0 z

g_+ (+) g_ (+) I e^{k+z} I+ k ,,-z _ $_e$k+z + k_ z $_I$
(22)

and

(23)

Observe how the boundary conditions (iv) are now .built into (22) and (23). For by settingz = z in (22), we obtain
H (z $_I$, +) = 0. Setting z = 0 in (23} yields an identity; as expected. For all intermediate depths-.0 < z <- z, , equations (22) and (23) give the values of the irradiance fields at those depths.

It is of interest to examine the standard solutions (22) and (23) in the light of the invariant imbedding relations (7) and (8) of Sec. 8.1. Thus in (8) of Sec. 8.1, set x = O , y = z and z = z, ,so that

Furthermore, in (7) of Sec. 8.1 with the same substitutions:
H(z,+) = H(o., _)6~(O, z, z_1) (25)
From (24) and (23) we have at once an explicit representation of Y(O, z , z $_j$) for x(0, $z_?$) and from (25) and (22) we have an explicit representation of q(O,z,z,) for x(0,z,). Furthermore, since:
PL(O,O,z_1) = R(O,z1)
it follows that:

R(0, z)
g_+ (+) g_ (+) k+z I _
A (z e
$_e$k_ z $_I$
(26)

Similarly, since:
T(O~z~,z~) = T(O,z~)
it follows that:

T(0= z) w z g_+ (+) g - (-) - g_+ (-) g - (+
(27)

We .note that for infinitely deep media, i.e., for the case z, = oo , (26) and (27) become, respectively:

(-k-) - a(--)
R_{CO} = (-k$^-$_) + a(+) ~ T

o

A question that often arises in practical applications of the principles of invariance to plane--parallel media concerns the polarity of the R and T factors (or R and T operators, if radiance is used) (cf. Sec. 7.1_2). It turns out that the two-D theory allows very detailed studies to be made of this question in the irradiance context. Thus, in the present setting the R and T factors by definition possess polarity if:

$R(4, z_1)$ ~ R(z

or

$T(O,z_1)$ ~ T(z110)

As we saw in Sec. 7.12 the phenomenon of polarity can arise when X(Q,z,) is anisotropic or nonseparable. Since the present medium is separable, the only way polarity can arise is in the case that the inherent optical properties of x(O,z-,) exhibit anisotropic structure. We have as yet made no essential use of the isotropy of $y(Q,z_{.1})$. Suppose that we now assume the medium to be isotropic. From the polarity theorem of Sec. 7.12 we know then that the R and T operators-do not possess polarity. But what of the factors

$R(o, z_1)$ and $T(o, z_1)$? Since a(±) and b(±) play the same roles for H(z, ±)as a and a do for $N_+ (z)$, we should examine this question with specific reference to a(±) and b(±).

In preparation for the answer, let us generate the second standard solution by replacing (iv)- above by the condition:

(v) $H (z_1)+)$ is an arbitrary irradiance, and H (o , -) =0.

Therefore we are to irradiate $y (o, z_l)$ from below. Since the procedure for fixing m_+ is now clear, we merely state that under condition (v) with. (i) through (iii) above in force, we have:

m_+ = + $H(z_1$ +) g_+ (-)/A (z_1) (28)

It follows that the second standard solution is:

H(z '+)k+z 1) 19

g+ (-) g_ (+.) $_e$k_ z

(2s)

H(z 1 1+) g, H g- H [e k+z

_z

(30)

Being guided by the invariant imbedding relation once again, we arrive at:

g+(-)g_(-) k+z1- k_z_1

0(z.) [e e

(31)

1 A(z1) 19 + + -.

(32)

It is now of interest to compare (26) with (31) and ¯(27) with (32). In the case of the reflectance factors we see that:

$R(O, z_1)$ = $R(z_1 , 0)$ if and only if g_+ (-) g_ (--) = g+(+) g_ (+) (33)

and that:

$T(O,z_1) = T(z_1 ,0)$ if and only if $k_+ + k_- = 0$ (34) The conditions o n the right of each statement are each implied by a single statement; namely:

a(+) = a(-) and b(+) = b(-) (35)

In this way we can find an answer to the question about polarity of the R and T factors.

Toward this ends, we shall agree that the medium $x(O,z_1)$'is anisotropic with respect to the irradiance field if (35) does not hold, i.e., if its negation:

a(+) ~ a (-) or b(+) fi b(-) (36)

is true. Thus, if $X(0,z_1)$ is anisotropic with respect to irradiance, we are generally to expect polarity of the R
and T factors, so that under standard conditions (i) through (v) a full description of the radiative transfer process in $x(O,z_1)$ by means of the irradiance field requires the four factors: $R(O,z_1)$, $R(z_1 .0)$., $T (O, z_1)$, $T(z_1.0)$, as given by (26), (27), (31), and (32).

The basic equations of the two-D theory for undecomposed source-free irradiance fields have now been derived. The solutions of the special forms of the two--flow equations (8) giving rise to the two-D equations are conveniently grouped

VC L. V

into parts: the first and second standard solutions given, respectively by (22), (23) and (29), (30). These two groups can be assembled into one compact package by means of the invariant imbedding relation. Toward this end and in view of (24) and (25) we deduce:

_)'(0· z I z 1)
g - H + k- z + k_
g+ (+) e k_+z, k_+z z
d z - 1
e

(37)

1 I + k- z
- g+ (+) g_ (-) ek+z 1
z 1
+ k_
_ (38)
g_+ (-) g_ (+) $_e$k+z z,

In a. similar way we deduce the present forms of the complete reflectance and transmittance factors from (29) and (30).

As a result we have:

6~(z ~Z'0) g+(¯)g_ (-) k+z k_z
I t'(z) le

(39)
(z z,0) = [1] [g+(+)g_(-)e.k+z - g+(¯) g_(+)e^{k-z}(40) o(z_1)
The preceding expressions can be condensed slightly by noting that the square-bracketed quantities in (38) and (40) are generalized forms of A (x) for suitable x. Furthermore the bracketed quantities in (37) and (39) can be represented in terms of hyperbolic functions, after suitable rearrangements. However such condensations are relatively complex and,-in the final analysis, quite inessential, since the true algebraic structures of the irradiance field in plane--parallel media are those given by the invariant imbedding factors and the invariant imbedding relation to which they belong. Thus the right sides of (37) through (40) serve merely to supply the numerical magnitudes o f.the q and 7 factors. It is immaterial how simple or complex these symbolic arrangements are. In view of (7) and (8) of Sec. 8.1, the essential algebraic structures are the following:
H(z,+) = H(z_1 7+)Y(z_1, z,0) + H(0,-)6.(O,z, z_1) (41)
H(z,¯) = H(O,-)Y(O, z, z_1) + H(z_1 ,+)6~(z_1 ,z,O) (42) or their matricial form:
(H(z,+),H(z,-)) = ((H(z_1 +),H(0,-))N(0,z,z1}~
(43)
Therefore, once a tabulation of the entries of the matrix ~(O.,Z,Z,) is made for a medium X(0,z,) , (43) supplies the

irradiance field H (-z,±) at each' depth z, in x(O, z_l) in terms of the incident irradiances on boundaries X_o and xz . A convenient means of tabulation is given by the set (31 through (6) of Sec. 8.1. One can first build up an independent tabulation of four standard R and T factors for x 0, z,) using (26), (27) and (31), (32). Then the (~, and Y." factors may be obtained in any detail as desired. The= particular forms of (3) through (6) of Sec. 8.1 required in the setting of X(O,z_l), are obtained by making the substitutions x } n , y -* z , and z + z . However, before extensive tabulations are considered., the observations throughout Sec., 8.7 should b e studied, especially those which show how the R and T factors and their invariant imbedding counterparts and generalizations, can be obtained by direct integration procedures and semigroup calculations.

Two-D Models for Internal Sources

The basic equations for the two-D theory of irradiance fields as given in (8) are written explicitly for source-free media. It is a simple matter to extend these equations to include continuously distributed internal sources in x(O,z,), and we shall now devote some attention to this matter. Before doing so, we note that the case for a finite number of discrete internal sources in X(O, z) is readily solved using the results of Example 3 of Sec. 3.9. (See (38) of Sec. 3.9, and replace (N_+ (y) , N- (y)) by (H (y, +) , H-(y, -)), etc.) These results, though written for the radiance functions, adapt immediately to the irradiance case. The R and T factors (26) ,(27) , (31) , and (32) and the P,. and .[1] factors (37) through (4) are now used to construct the T - factors by means of (20)¯(23) and (3l)-(34) of Sec. 3.9.

The equations(8) are adapted to the continuous internal source problem by simply adjoining the source terms h_n(z,±) to the respective equations, thus:
dH z,±,[(+)H(z,+) + p(+)H(z,+) + h (z)±) (44) z
Here we have used (11) and (12) of Sec. 8.3 to help write the present compact version of the two-D equations. The present version is the irradiance counterpart to the local forms of the principles of invariance. The connection of h_n(z,±) with the source function in y(0,zJ is simply this: we have written:
t1h_n($_z$:+)" for 1W Nn(z,~)do(~) (45)
and in the process of going from the equation of transfer to the two-flow equations, as explained in Sec. 8.3, +
N_n (z, C) is converted to h_n (z We may note in passing that we have- used the field irradiance interpretation asof N_n in (45) so as to use h_n. The alternate interpretation of N_n as a surface radiance would have resulted in

w_p (cf. (19) of Sec.2. 7) .
To solve (44) we can make use of the solutions obtained earlier in the source-free case. The homogeneous solutions of that case will be used. In order to find the particular solutions of (44) we proceed as follows. First observe that the syst6m (44) may be converted into two second order differential equations in H(z,±). Thus, let us write:
T t D I T
for
d dz
Then, (44) becomes
[D + T(+)]H(z,+) = - p (-)H(z,-) - h_n (z ~+) (46)
[D - c(-)]H(z,,-) = p (+)H(z,+) + h_n (z ~-~ (47)
Operating on (46) with the operator [D - T(-)], we have: [D^-T($^-$ [D^+T($^+$)
]H(Z~$^+$) '~ $^-$p(_) [D-T(-)]H(Z,, [D^-T( )]h .$_1$ (Z:$^+$)
$^-$p (-) [p (+)H(z,+)+h .$_1$ (Z,$^-$)]
- [$_D$_ T (-)] h_{fl} (z g +)
The second equality follows from use of (47). Simplifying this result we have:
n
(48)
Next, operating on (47) with [D + T(+)], we have [D^2 + (T(+) -- T(-))D + p(+)p(-) - T(+)T($^-$)]H(Z.-) = [D + T (+)]$h_{r)}$ (z,,-). - p (+)hn (z i+) (49)
[D^{2+}(T($^+$) _T(-))])+p(+)P(-)-'C(+) 'r($^-$)]H(z,$^+$)
Let us write:
"Y (Z : 4 1 1 f or
+ [D + T (+)]h T1 (z ,±.) - p (+)hn (z ,+) (5©)
and
1 0 1 1 for [D^2 + ('r (+) - 'r (-)) D + p (+) P (-) - T (+) T ( )] (51) Then (48) and (49) can be written as

DH(z,±) = Y (Z,±)

(52)

The source-free, two-D equations (8) are a special case of (52). Thus (8) is equivalent to:

(53)

and is obtained by setting h (z.,±) = 0. The characteristic equation associated with the differential operator. ~) is:

k^2 + [T(+) - 'C(-))k + [p(+)p(-) - T(+)T(-)] ' 0 (54)

whose solutions are:

k± 2 [T (-) --C(+)]

[(T(+)+'C(-))2-4p(+)p(-)]1""2

(55)

These roots k_+, k- are precisely those given in (12), but are now represented in terms of the local transmittance and reflectance factors -r(+), p(+). The earlier forms can be recovered by means of (11), (12), and (18) of Sec. 8,.3.

We are now ready to find the general solutions H (z, ±) of (52). First we observe that two linearly independent solutions of the homogeneous equations (53) are available in the forms:

$_e$k±z

Ind __ed, evz function satisfies (53) and the Wronskian W(e +z , e } of these two functions is of the form:

$_e$k+z $_e$k-z

W (e^k+ z $_e$k- z)

k e^k+ ^{z} k ek- z

-- (k- -. k_+) exp ((k_+ + k_) z)

for every z in the interval [O,z,]. Since we have agreed to work in nondegenerate media (cf. (13)-(16) of Sec. 8.5), it follows that k_+ ~ k- , and so the Wronskian does not vanish on [O,z,], thereby indicating the linear independence of the solutions.

By means of the method of variation of parameters, particular solutions of (52) are found to be of the form:

1 f^z [ek+(z-s) - $_e$k- (z^-s) y (s ±)_ ds (56) k_+ k_ 0

which we will denote by "H_p (z, ±)" .

It follows that the general solutions of (52) are of the form:

H(z.,±) = m+g+(±)ek+z + m_g_ (±)e^{j-G} + H p (z I ±) (57)

where m_+ are constants of integration,. These constants may be determined by suitable choices of the values H(y, ±) at any two distinct levels in X (0 , z,) ,- or by knowing H (y , ±) at a given depth and dH (y, ±) /dy at the same, or generally some other depth. Still another and quite general requirement would be imposed by simultaneously specifying H(0,-), H (z $_I$, +) , exactly as in the source-free case considered above (re: (19).,.(28)). This we shall do, our ultimate goal being a representation of H(z,±) by means of an invariant imbedding relation, as in (43), but now taking cognizance of source terms in (44) and hence in (57). Thus, we require:

H(0,-) m+g+(`) + m_g_ (-)

H (z +) = m+g+ (+) ek+z l + m_ g` (+) ek- z +H p (z , +)

The requisite values of m+ are:

m

1 LH (z +) - H (z P +) j g. H (0 g _ (+) e kT_ z

0 z~ p 1 + +

(5,8)

These values are now-returned to (57) and some algebraic reductions made, keeping in mind the present goal. The results may be written as:

(H (z,+) ~H(z.-)) = (H (z1 1 +) ,H (0)-))7~(O,z:z,)

- (Hp(z l t +) 20fopz~z,) + (Hp(z,+),Hp(z,-)}

(59)

This is the desired set of general solutions of the equations (52). Observe that we return to the source--free case (43) if h, (z , ±) = 0 for every z in [0 , z l] ,. The new parts comprising the present solution are generated by the presence of sources in x(0, z,) . The members of the 2 x 2 matrix k(0, z, z,) were evaluated in (37) through (40), so that, as it stands, (59) is ready for numerical. computations.

Equation (59) can be recast. into an alternate form so as to achieve greater symmetry of form and also to bring out explicitly the intuitive features of the role played by the continuous sources $h_p(z,\pm)$ within x(O,z,). The concept of the continuous Y' -operator, introduced in (37) of Sec, 3.9, guides the reformulation of (59) towards this new goal.

First we observe that each particular solution H (z,±) may be written as a sum of two terms, using the explicit form of Y(x,±) given in (50)

z z

Hp(z,+) _ Q(z-s) [D-T(-)] h . l(s +) d s - Q (z -S) P(-)h n(s,-)ds

0 0

(60)

z z

HP(z,-) = Q(z-S) [D+T(+)]h .1 (S,-)ds - f Q(z_s)p(+)h.i(S)+)d5

a v

(61)

Note in particular how the derivative operators [D-T (-)] and [D+T (+)] are to be applied to h., (s ,+) and h (s,-) in the first integrals in each of the representations (60) and (61). For brevity, we have written:

VtQ (z_s)11 for l Lek+ (z-s) - e k- (z-S) . (62)

The two equations (60) and (16) can be succinctly written in matrix form and in such a way as to make contact with the

- operator concepts of example 3, Sec. 3.9. Thus:

(HP(z,+),Hp(z,-))

z
(hn(s,+) , h ' n (S , -))   Q(z-S)ds
j   -   P ~- 3   [D+T (+)]
0
Let us denote the matrix in the preceding integrand by "6(s)" . The variable s is used in the notation of e(s) to indicate that the derivatives are to be taken with respect to s. We agree that in the integrand the operator 4c~(s) acts on (h2 (s,+) ,h (s! -)) (and not Q (z-s)), and that the result is multiplied y Q(z-s). Hence:

z
(HP(z,+) ~Hp(z'-)) _ f   (h11(s.+) ,hn(s~_))d(s)Q(z-s)ds 0
(63)

Now, in a similar manner we can reformulate the term (Hp (z l , +) , 0) in (59). Indeed, we observe first that
(Hp(z1 $+) 20) = (Hp(z1 ~+) sHP(zl s-))C+
where C+ is one of a pair C + of 2 x 2 matrices defined in (4) and (5) of Sec. 7.4. Now the I+ occurring in C+ reduce to the number 1 in the irradiance context. Hence, by the same reasoning as before:

z
(H p (z~ s+~s0)   ~'   fo 1 (h n (ss+),hT1(5s-~)L.(s)C+Q(zi-s) ds
(64}

The integral in (63) _ can be given z, as an upper limit by adopting the function X defined on the real line, where
0 if x<0

1 if x>0
so that:
z1

(Hp(z,+),HP(z,-)) = f   Chp(x,+),h'n(x,-))4f(s)X(z-s)Q(z-s)ds 0

With these preliminaries established,-we return to-(59) and cast the particular solution terms into the form:
(Hp(z,+) 3,Hp(z,,-)) - (HP(z ~+) .90)?k(D,z,z) =
z1
(hn(s,+):hn(s,-))E(s) LX(z-s)Q(z-s)-C+?~(.0,Z,z1)Q(z1- s)] ds

Let us write:
"T (s, z) ~~   for   e,(s) CIX (z-s)Q (z-s) -C+k (O P Z s Z1)Q(z 1-s)   (65)
where I is the 2 X 2 identity matrix, so that (59) becomes:

CH(z,+)H(z,-)) = CH(z1 ,+),H(0,-))?~Ca,z,z1) + z
1
+   f   [h .1[s,+)} .hT1(s,--))T(s,z)ds 0
9 < z < z

(66)

Further connections of irradiance field theory with the functional relations of invariant imbedding theory can be sought, especially with those of Sec. 7.13. Some of this research may be guided by the query: What are the specific connections between the four components of the 2 x 2 matrixT (s , z) defined in (65) and those of T (s , z : 0,z,) studied in Sec. 7.13? Evidently the theory of Sec. 7.13 is an independent means of arriving at T (s, z) constructed above. By relating the two methods of approach to T(s,z), new understanding of invariant imbedding techniques should be forthcoming. A beginning is made in Example 10 of Sec. 8.7. Some recent developments for media with-internal sources are cited in §10.

SEC. 8.5 TWO-D MODELS 16

Two=D Model for Decomposed Irradiance Fields

The two-D model for decomposed irradiance fields, to which we now turn, was the first two--D model constructed (Ref. [2211). The prime motivation for the model was the need for a more accurate and detailed irradiance model than what was available in the classical one-D theory of irradiance fields (Sec. 8.b). The present model has the advantage of working with the diffuse irradiances generated by arbitrarily shaped radiance distributions.

It is well known that the shapes of the diffuse radiance distributions in natural media change with depth at a rate much slower than the shapes of the undecomposed distributions, thereby making the two-D assumption (1) a still more faithful reflection of the actual radiometric state of affairs in the present instance. A further advantage of the decomposed irradiance model is that the shape of the incident radiance distributions on X(O,z,) can be chosen independently of the shape of the diffuse radiance distributions throughout y(©,z_l), thereby constituting a greater flexibility than the one-D model. As. a result the decomposition:

$$H(z,\pm) = H^0 (z,\pm) + H^* (z,\pm)$$

contains two independent terms which together are more flexible in representing actual irradiance fields. A third advantage of the two-D model for decomposed irradiance fields is that the mathematical procedures for solving the model's equations are simply special cases of the solution just studied in the case of the undecomposed irradiance field with internal sources. Indeed, the "internal sources" are now the upward and downward primary scattered irradiances originating from the residual irradiances H°(z,±). This may be seen by comparing (2) of Sec. 8.4 with (44). Hence replacing "$_h$ ($_{z:\pm}$)" i_n (44) formally by 1tfo(z,+)H°(z,+) + bo(zs+)H0(z,+)11, (b9) yields the requisite general solution of the present two-D model equations.

Actually, such generality as that just noted is not usually needed in practical hydrologic optics or meteorologic optics computations. The following radiometric conditions are found to be sufficient for most purposes, and we hereby explicitly adopt them in the present study:

(vi) No (0, ~) = $N^0$6 (~-~°) for C° in

and

N° (z_1, ~) = 0 for in + . Further,

$H^*(0,-)$ = o and $H^*(z\_,+)$ = © .

The remaining conditions for the present discussion are (i) through (iii) as given for the undecomposed irradiance field equations (8). Now condition (iii) applies of course to N*(z,•), 0 < z < z,,. Therefore X(O,z,) has a collimated source of radiant flux incident on its upper boundary along 'the direction & ° in 7--, No other .sources are incident.

on or in X(0, z l) . As stated in (vi), the boundary conditions H*(0,-) = 0 and H*(z ,+) = 0 are also in force. In regard to related uses of these conditions, see (30) and (31) of Sec. 8.4.

With the preceding assumptions in force, (20) of Sec. 8.4 reduces to:

+ dH*(z,±)

-

dz [a* (±) + b* (±)]H* (z ±) + b* (+)H* (z, +)

+ N^o exp { - az/$P_0$16+(u_0) (67)

where we have written:

It
CF tr
0 (68)

I I o + ( ) •t
f o r (69)

and in which we write:

,t11 t t for I C^o - k (70)

0

Therefore a+ (p) are simply forward and backward :scattering functions, slightly modified so as to allow the collimated radiance condition (in the form "NO") to remain explicitly before the eye (cf. (42) and (43) of Sec. 8.4). However, in numerical work and other theoretical investigations, it is easy to see, by inspection of the various results below, how to return to the fo and b° concepts, when desired.

The general solution of (67) is

H^* (z , ±) = ^{m}+g+ (±) e^{k+z} + m_ g_ (±)e $^{k- z}$ - N^0C (u 0)±)e- az/uo (71)

In this set of equations, the $g_\pm$(±) are given in (10) and (11) with "a* (±)" replacing "a(±)" and "b* (±)" replacing "b(±)" everywhere (including in k_+), since we are now working with the diffuse irradiance field. For simplicity of notation, the star superscripts will be kept behind the scenes in (71). Furthermore, in (71) we have written:

"c(u~ ±)"' for

6+ (uo) b * (+) + Q + (uo) [a* (+) + b* (+) + (a/uoA

(7 2)

The quantities C(u0,±) clearly depend on the direction of incidence of the radiance distribution N°(0,~), ~ in E_ . I t is this term in (71) which gives the dependence of H* (z , ±) on uo, and which permits the simulation of general incident lighting conditions.

By tabulating the values C (uo, ±) for

a few typical choices of a and a , a useful set of tables of H*(z,±) can in turn be constructed, and these can be used via superposition calculations to simulate general incident conditions. The constants m+ in (71) are as yet the only undetermined quantities in (71). However the remaining lighting conditions on H* in (vi) rigidly fix the structures of m+ and m-. Thus it may he shown that:

m+

0

N ± g_ (-)C(u ,+)e"azil v + g- (+)C(11 '-)ek+z~

0(z1) + o + °

(73)

Explicit expressions for H*(z,±) or for H(z,±) are now derivable from (71) using (73). (See, e.g., (1) and (2) of Sec. 10.3.) In particular, expressions for q(0, z, zi) and ~ (0 , z , z ,) are now readily forthcoming. However, we shall, not take the space here to display such representations of the complete reflectance and transmittance factors. We shall be content to exhibit, for later purposes, the standard reflectance and transmittance factors R (O, zl) and T* (O, z,) . Toward this end, by means of (30) of Sec. 8.4 and suitable substitutions, we have:

H* (0;+) = HO(O,-) R(O, z1)

Since

H0 (02-) = N011

(71) and (73) combine to yield:

C (11 o ~ -) g+ (+) g_ (+) k+z, _

R(O'z~) u A (z) e 0

C (u o' +) o (0) az, lu0

Uo [A z1) e

(74)

Further, from (31) we have:

which, with (71) and (73), combine to yield:

There is generally one other reflectance-transmittance pair of factors for upward incident flux on X(O,,z1). The procedures used to find (31) and (32) may serve as a guide to find the factors for the present case.

Inclusion of Boundary Effects

We conclude this section with a description of how to include the effect of reflecting upper and lower boundaries on the medium X(O,z ~. The general problem of interreflections between the body of a medium and its boundary, including the possibility of internal reflecting interfaces, was discussed in Example 6 of Sec. 3.9 and applied to the unified atmosphere-hydrosphere problem in Example 7 of that section. We shall now repeat the essential ideas of those examples, but in the irradiance context, and, so as to keep the discussion simple, we shall assume no internal interfaces.

The simplest case will be considered first: the medium X(O,z,) has only one reflecting boundary namely that at level 0, and which we shall denote by "X0 it,The -boundary at level z_1 will be assumed transparent. In practice, when z_1 is relatively great, the present case may be freely used even though X(O,z_1) has a reflecting lower boundary. Furthermore, the only~ source in X(O,z,) will be the downward irradiance H (0,-), incident on the upper boundary. We will work with the undecomposed irradiance field.

With these conditions, we have established the present interaction problem as that between two media: a plane X_a and a slab X(O,z_1). The reflectance and transmittance factors for X_e are developed in their full generality in Sec. 3.3. In particular we use r_+ (x) and t_+ (x) as developed in (19) of Sec. 3.3. Examples of the use of these factors in the irradiance context are given in Sec. 3.4. Hence we can employ the interaction method in the present problem without further explanation.

We direct attention first to plane X_o, and enumerate all incident irradiances:

A : all irradiances like H (0)

A_2: all irradiances like H_+(O)

The set A, is the set of all irradiances on the upper side of X_o. The set A_2 consists of all irradiances on the underside of X_o. The set of response radiant emittances of X_0 are:

B

all radiant emittances like W (0)

B

2

all radiance emittances like W_+(O)

The four associated interaction operators for X_o are simply the reflectance and transmittance factors $r_\pm$(O), t_+(O) associated with X_Q

s -------- t_(0)

1 1

s ------- r_ (0)

1 2

2 1 _______ r+(O)

We next direct attention to the slab X(O,z_1). The class of all incident irradiances on X(O,z_1) is:

A_1 : all irradiances like H(0,-)

The class of all response radiant emittances is:
B_1: all radiant emittances like W(O,+)
B_2 : all radiant emittances like W (z
1 , - -]
The requisite response operators s and s_{12} are in this case the numerical reflectance R(011 given by (26) and T(O,z_1) given by (27):
According to.the interaction principle applied to X_o , we have

W_+ (0) =
H_ (0) r_ (0) + H_+ (0) t_+ (0) (76)

_(0) = H_ (0) t_ (0) + H_+ (0) r_+ (0) (77)

The interaction principle applied to X(O,z_1) yields:

W(O,-$_+$) -
H(O,_)R(O,.z1)
(78)
W(z_1 .--) = H(O,-)T(O, ^{z}1)

The auxiliary equations for the present ..problem are:

H_- (0) = Ho
. (0, -) (79)

W_(0) = H(0,1-). (80)

H_+ (0) = W(0,+) (81)

We are interested in the responses of X_o and X(O,z_1) as a result of their radiometric interaction induced by Ho(0,-),.and so use the auxiliary equations (79) through (81) to remove as many incident quantities as possible from (76) through (78). The results are:

W_+ (0) = H^0 (0 $_s$ _) r_ (0) + W(0,+) t_+ (0)(82)

W_ (0) = H°(D, -)t_ (O) + W(O,+) r_+(O) (83)

W(O,+) `
W_- (O)R(O, z_1) (84)

W(z_1 t^-) = W- (^{0})T (0 , z 1)

Equations (83) and (84) are autonomous, so that:

W- (0) ^ Ho (0 2 -) t_. (0) + [W- (O) R(0 , z 1 r+ (0) and we have:
-
W_(O)= H^O (0, -) t- (0)
1-R(O,zl r+ (o
(a5)
From- this and (84)

H0 (0, -) t_ (0)R(O, z,)
(86)
From this and (82) :

(87)
These results may now be used to obtain the internal irradiances H(z,±) in y(O, z,) . Indeed, by the invariant imbedding relation (43) with H (z , +) = 0 and H (0, -) = W_ (0) now given by (85), we have
[H(z,+),H(z,-)) = (O,H(O,-))7~(O,z,z1)
In particular:
1-R(O, z).r 0 ~(, ?)
(88)
H z - -~ Ho (0, -) t- (0) O z z1-R O,z r+ 0 ~~ > > ~)
(89)
It is interesting and instructive to pause here and view (88) and (89) in the light of the semigroup relations (52) and (53) of Sec. 3.7 which certainly apply in the ir-radiance context. Toward this end, observe that the factors before R and ' in (88) and (89) comprise basically a complete transmittance factor. Thus supp-os e we write
f l y (- 1 y 0 9 z }t ~ for -(t- (0)
1 R O,zl)r+ 0)

and
"H(-l,-)" for H°(O,-} Then (883 is of the form:
H(z s+) = H(-1 .9 ~-1, 0 . z1 A(0,

(90)
and (89) is
H(z._) = H(-1,-)3i(-1,0,zd7(O)z,zd (91)
The semigroup relations show that we can write (90) and (91) as

H(z,,+) = H(-1~-)q(-l., z"z1)
H(z,-) = H(-1j-)D1"(_l)z~z1)
(92) (93)
The conceptual unity afforded by the invariant imbedding point of view is illustrated quite strikingly by this conversion of (88) and (89) into (92) and (93). As a consequence we can see that the interaction problem between the boundary Xfl and X(O,zi) does not differ algebraically from that between any two subslabs X(O,y), X(y, z,) of X(O,z,). In view of this, we could

rework the preceding analysis so that we need not count depths in X(0,zl) starting with --1 , but rather with some other fiducial depth. However, the-present notation has been found convenient and workable in the discrete space setting (Ref. [M]), and shall be retained.
The reader who has followed the pre-ceding derivation of (92) and (93) can now readily extend these results to the medium X(O,z,) which is endowed with two interacting boundaries X_o, X_{z1} . Indeed, the invariant imbedding relation for one-parameter source-free media can be invoked at this juncture without the necessity of a fresh application of the
interaction method. We merely use the semigroup relations (12) of Sec. 3.9 and the invariant imbedding relation (l0) of that section applied to the one parameter medium made up
of the union of Xo , X(O, z,) and $X_{z,}$.
To see how such an application proceeds, let us first establish some notation. The interaction factors for X are-as listed above. Those for Xare r, (Z,), t_+ (z, Y,
as established generally in Sec. 3^Z.1. Those for X(a,z,). are the four standard R and T factors found in detail above. Thus each of three interacting entities X_o, X(O,zl) and X_{ZJ} is generally assigned four interaction operators: two ,reflectances and two transmittances. The three media, together, as a class, will be denoted by"X_3 (O, zl)" . If, further, we write "H(-1,-)" and ''H(z,+l-,+)" for the down
ward and upward incident irradiances on the space X_3(O,z,)$_7$ then the irradiance field H(z,±) at any depth z , 0 < z < z, is given by:

(H(z,+) sH(z,$^-$)) = (H(z_1 +l,+) ~H(-l,-))7~(-l, z, z~+l)
(94)
where the four components of the matrix 7~(-l,z,z +l) are found by decomposing them using the semigroup properties (12) of Sec. 3.9. Two examples of such decompositions have already been given in (90) and (91) for the one-boundary downward flux case, . In the present two--boundary case, we have for example:
1,z,z_1 +1) _ Y(-1'0'z~+1)J0*f"(01z'z1+l)
(95)
The geometric setting for (95) is depicted in Fig. 8:2. Using this figure as a guide, and turning to (40) through (43) of Sec. 3.7, we see from (42) of Sec. 3.7 that
--(-110sz,+1) = ^{T}($^{-.1}$~^{0}) [1-R(09z,+l)R(02-1)]-l
(96)

FIG. 8.2 Geometric scheme for including boundary effects of boundaries X_0 and Xz, , of medium X (O , z ,) . Hypothetical levels labeled "-l" and ^{11}z ~. + 1 " are introduced as places
external to X(O,zl) from which radiant flux may irradiate X(O,z,) or at which emergent radiant flux may be mea.5ured.
 Here, of course,

T (-1:O) - t_ (O) (97)
R(0,-1) = r.+, (0) (98) Furthermore, by (15) of Sec. 7.-3, we have
$R(O,z_1+1) = R(O,z_1)$ + $COOz_1$' z $_1^{+1}$)T(z_1 M (99) By the semigroup property of [6]?:
~(O ? z $_3$,Z +1) jr(O S z Z +1) R(z s Z +1) (100)
1 1 1 1 1 1
Here we have:
R(z 1 ~ z 1 +1) r- (z 1 (101) and once again by (42) of Sec. 3.7:
7(O, z. , z +1) = T (O, z) [1-R(z: ,O) R-(z Z +1)]$^{-1}$ (102)
? 1 1 r 1 1
In this way we can completely, and systematically analyze the factors on the right in (95) using only the main -semigroup properties of .X and R and the composition formulas, (40) through (43) of Sec. 3.7. The remaining three components of . 7t(-l, z, z $_1$+1) may be analyzed similarly.
The reader interested in pursuing the radiative transfer theory of "mixed spaces"---i.e., collections of simultaneously interacting surfaces, slabs, and general media, in three and higher dimensional settings--is referred to Ref, [251], which systematically develops this discipline, known as discrete-space radiative transfer theory. The preceding activity is a particular instance of application of discrete-space theory. We shall ,return to this matter in Example 5 of Sec. 8.7 and systematize the preceding boundary effects procedure for irradiance settings.