9.4 General Analytic Representation of the Observable Reflectance Function

The concept studied in this section is the observable reflectance function R(•,-) whose value at a depth z in an arbitrarily stratified plane-parallel optical medium is given by:

R(z.,-)- H(z,+) H (z,-)

as usual, the quantities H(z,±) are the observed upwelling (+) and downwelling (-) irradiances at depth z in the medium (re: (16) of Sec. 9.2). Several representations of the function R(*,-) are established which will, (a)

explicitly exhibit in terms of differential equations and definite integrals the dependence of R(-,-) on the inherent optical properties of the medium, as far as this is possible; (b) illustrate the dynamic equilibrium-seeking tendency of R(',-) which appears to hold in all plane--parallel media; and finally, (c) suggest some methods of solving the problem of predicting the depth-structure of R(-,-) in general media. To place the present discussions in their proper perspective for the general reader, we prefix the following observations.

The reflectance function R(*,-) is one of a set 'of seven main apparent optical properties introduced in Sec. 9.2. This, set consists of the functions R(•.,+), K(s)±) = k(. Y±) y and k , and is defined along with D(•,+-) in terms of the four directly observable radiometric functions: H(•,±)., h(•,,+-),, where h(",+-) are the upwelling (+) and downwelling (-) scalar irradiance functions. The theory of the measurement of these latter radiometric quantities and a discussion of the salient physical characteristics of this extremely useful set of apparent optical properties was briefly sketched in Sec. 2.7 (see Fig. 2.18). Further discussion of these properties is given in Chapter 13. Section 9.6 contains a classification of the optical properties of an optical medium into the classes of inherent and apparent optical properties, and the necessary distinctions that must be made between them, in both experimental and theoretical procedures.

The main fundamental set of local inherent optical properties of any scattering-absorbing optical medium consists of the functions a and u-, the volume attentuation and volume scattering function, respectively. These functions are by definition independent of the ambient light field. The apparent optical properties, however, depend jointly on the inherent optical properties and the ambient light field. Specifically, the apparent optical properties depend .on a , o , and the radiance distributions N(z,•) in the medium.

Despite this dependence o f the R , K , D , and k functions on ephemeral lighting conditions,- they exhibit a behavior in both space and time of such a strikingly regular and generally predictable kind ,.that each is dignified with the appellation: "optical property." However, we point out the fact that this is a matter of first appearances only,

and that, under incisive analytical and experimental scrutiny, their regularities are seen to be at the mercy of variable boundary lighting conditions and the internal distribution of the values of a and a in an optical medium. To emphasize this fact, the qualification "apparent" has been put before "optical property."

Detailed examples of the regular behavior of the apparent optical properties are given in Secs. 9.2 and 9.3, in the following sections, and in the remaining chapters of

Part III. The present section adds to this store of knowledge of the apparent optical properties by developing in detail the exact differential and integral representations of the reflectance function R(•,-) and drawing some theoretical and practical conclusions from them.

The Differential Equation for R(-,-): Unfactored Form

The physical setting for the derivations that follow is a plane-parallel source-free optical medium whose inherent optical properties spatially depend only on depth, and this depth dependence is assumed arbitrary. Further, the medium may have either finite or infinite optical depth with arbitrary boundary reflectance properties and arbitrary incident lighting conditions on its upper and lower boundaries.

Assume that an irradiance probe sweeps through a range of depths between z' and $z^{4'}$, where 0 < z' < z" < O , and that at each of the depths in this range readings H(z,±)

are taken, z' < z < z^1 f. Then the reflectance R(z,-) at each depth is determined by

- H(z,+) H(z,-7 J'

and its depth rate of change dR(z,-)/dz for downward motion or for upward motion is easily found and is given as (re: (32) of Sec. 9.2)

dR (z,-) - R(z,-) [K(Z,-) - K(z,+)] dz

We may now introduce the exact representations of K(z,±) established earlier ((18) and (19) of Sec. 9.2)

+ K(z,±) = [a (Z,±) + b (z,±)] - b(z, +)R(z,±) ,

where a (z , ±) W a (z) D (z , ±) , and b (z , ±) are the values of the absorption and backward scattering functions at depth z for the upwelling (+) and downwelling (-) streams of radiant flux. Substituting these representations of K(z,±) in the above derivative, the result is:

- dR z,-) b(z,+)R2(Z,-) - c(z)R(z,-) + b(z,-) z

c(Z) = Ca(z,-) + a(z,+) + b(Z,-) + b(z,+)J

where

This is the desired differential equation for R(•,-) in unfactored form, and is the basic differential equation governing the observable reflectance function. It is an exact equation within the presently chosen general physical setting and forms the basis for all our subsequent deductions of the properties of R(•,-). The mathematical structure of (1) is that of a general Riccati equation, which generally has non-elementary solutions. The reader should not fail to observe the striking resemblance between (1) above and (39) of Sec. 8.7, keeping in mind (11) and (12) of Sec. 8.3.

We shall return to consider this resemblance later, in the equivalence theorem.

The Differential Equation for R(•,-): Factored Form

The basic differential (1) may be factored by observing that its right-hand side is a quadratic in R(z,-) for each depth z. Thus for a given z, the roots of the quadratic equation:

b(z,+)t2 - c(z)t + b(z,-) = o

are

R z - - c(Z) + [c2 (z) - - 4b (z,-)b(z,+)] 1/2 2 z,+)
Rq (Z, _) = c (z)
[c2(z) - 4b(z,-)b(Z,+)] 1/2 2b z,+)
Hence (1) may be written:
b (z,,+) [R(Z,ˉ) - R_a(Z,ˉ)] [R(Z,ˉ) - R_q(z,`)]
Equation (2) is the factored form of the differential equation for R(.,-). The function R_q ~ is the equilibrium function for R(*,-), and R,(*,-) is the attenuation function for R(-,-ˉ). The kinship of R,(-,-) and R (ft,-) with the two-D theory's k_+ and k- , as given in (A) of Sec. 8.5, may be noted. Equation (2) will be cast into deep perspective within radiative transfer theory when we study the universal transport equation in Chapter 11 [see (15) of Sec. 11.2 and (1) of Sec. 11-3).
Second-order Form of Differential Equation for R(-,-)
We now introduce a new function Q defined on the depth interval of interest, and having the property:

= b (z ,+)R
I T_ (z,ˉ)
where the prime denotes differentiation with respect to z.
It follows that Q satisfies a linear homogeneous second-order differential equation whose coefficients are functions of c (z) and b (z, ±) . To see this, we differentiate each side of the above relation. Theˉresult is
Q(Z)QIQ' (z)I~ W
Q^2 (Z)
b(z,+)R' (z,ˉ) + b' (zI+)R(z,-)

Then using (1) for the expression equivalent to R'(z,-):
V z _ t (z = b (z,+) [-b (z,+)R^2 (z,-) + c(z)R(z,-) - b (z,-) I (z) Q. z)
+ bI (z,+)R(z,-)
so that:
Q1 z) - b^2 (z,+)R2(z,-) (z)
+ [ˉb^2 (z, +) R^2 (z,--) + c(z)R(z,-)b (z,+) - b (z,--)b (z,+) I
+ b' (z,+)R(z,-)
Hence,
?r z Q z
= [c (z)b (z,+) + b' (z,+)]R(z,ˉ) - b (z,-)b (z,+)
Using the defining relation for Q once again, this becomes

V (z) - Ec (z) + b (z + Q^f (z) + b(z,-)b(z,+)Q(z) z,

Equation (3) is the desired homogeneous second order differential equation for Q , Upon obtaining its solution, the defining relation for Q is used to obtain the expression for R(•,-).
The Equilibrium-Seeking Theorem for R(-,-)
Preliminary Observations

We turn now to study one of the deeper properties of the observable reflectance function, which states that the observable reflectances R (z, -) at each depth z is tending,
with increasing depth, to some well-defined equilibrium value, and that it will attain that value i f the medium is sufficiently uniform in structure. To prepare the way for this theorem, known as the equilibrium-seeking theorem for R(•,-), we shall make some preliminary observations on the nature of "equilibrium concepts" in general radiative. transfer theory.

Equilibrium theorems abound in the theory of radiative transfer. Perhaps the earliest explicit and unmistakable instance of an equilibrium theorem was given by means of Koschmieder's equation (Sec. 4.3) which describes the apparent radiance Nr. of an object of inherent radiance N_o as seen along a horizontal path of length r along which both the inherent optical properties of the medium and the ambient lighting conditions are constant:

-ar N -ar

Nr ; N_o e + a 1 _ e

Here the quantity N_* (the path function) is the (constant) radiance per unit length of path, in the direction of the path at every point of the path, and which is generated by scattering of the ambient light. The quantity a is the fixed value of the volume attenuation function for the medium along the path of sight. Assuming that this type of path maybe extended indefinitely in the direction away from the object, the equation implies that, for any medium with a > 4 ,

N_*

1 imr -*m ^{N}r - a

The quantity N_*/a , usually denoted by "Nq" , is the equilibrium radiance of the path of sight, and in this case is simply the observable horizontal radiance. It is dependent on both the local inherent optical properties of the medium (a Q) and the lighting conditions along the path (N*). The term equilibrium radiance is understood in the following sense: For any initial choice of N_o , N_Y.. tends
toward and eventually attains the value N_q. Thus if N_o exceeds N_q , then N. decreases from N_Q to N_q as r
goes from o to Co. On the other hand, if N_o is less
than N_q , then N_r increases from N_o to N_q as r goes from o to C* .

This phenomenon of the equilibrium-seeking tendency of the apparent radiance actually holds for an arbitrary path of sight in an arbitrary optical medium along which there are no sources and along which a > o. This may be seen by taking the general transfer equation for radiance:

dN car

aN+N*

and writing it in the form:

a [N - N_q]

where we have written " N_q " for N_*/a , as in Sec. 4.3. It must be emphasized that this equation is completely general; hence a may change from point to point along a path, N_* (and hence N) may depend on direction about a fixed point, and

the angular dependences of N_* at two different points may be quite distinct. Now select any path of sight in the medium, and choose an initial point of the path. At this point suppose the value of N is given. If then N > Nq,
the above equation immediately shows that dN/dr < D , so that N tends toward the value of N_q at this point as r in
creases. On the other hand, if N < N , then dN/dr >0., and N tends toward N_q once again a~ r increases. Now
it is quite possible that N may change from point to point
along the path. But the important fact to observe is that

at every point of the path, regardless of the relative sizes of N and N , the tendency of N at that point-is to change its value so as to decrease the absolute value of the difference $N - N_q$ at that point.

The phenomenon of the equilibrium-seeking tendency of radiance in an arbitrary medium gives rise to a host of equilibrium theorems for various other radiometric quantities and even for the apparent optical properties.

These equilibrium theorems are explored in detail in Chapter 11 where it is shown that no less than 34 radiometric and related concepts are subject to a single equilibrium principle.

The following discussion of the equilibrium theorem for R(-,-) is patterned after that exhibited for N above. Furthermore, this special discussion will pave the way for some interesting observations of the properties of the reflectance function as seen in the light of the principles of invariance. These observations will be given later in this section.

The Equilibrium-Seeking Theorem for R(•,-)

To establish the present equilibrium theorem for R(•,-), consider an arbitrarily stratified source-free plane-parallel medium over the depth interval © < z' < z" < oo in which scattering takes place, i.e., a 1⁻4. The medium may be optically shallow or deep; its boundary reflectances are arbitrary, as are the boundary lighting conditions. The present setting, therefore, is of maximum generality. Imagine a reflectance meter at depth z in the medium. The reading R (z , --) gives the complete reflectance of the material between the level z and the lower boundary, inclusively. This number is a complex combination of the effects of the standard reflectance of the medium in that depth interval (i. e: , the standard reflectance R(z,z") of the slab X(z, z")) , the interreflections between X(O, z') and X(z',z"), and the angular structure of the downwelling incident flux at level z. The angular structure of the downwelling flux at level z in turn depends on the inherent optical properties of the medium throughout its extent. However, despite this complex situation, there exists at every level z along with R (z , -) , the values R_a (z , -) and R (z,-) of the attenuation function and equilibrium function associated With R(•,-), which guide the evolution of R (z , -) as _z increases. In view of the active role played by R (z,-) and R (z,-) in determining the depth behavior of R~z,-}, we pause to examine their structure. It turns out that R,,(z,-) is of central interest. We shall first establish the fact that R,(z,-) > 1 for all z .

We deduce the fact that R_a (z , -) > 1 for all z on strictly analytical grounds, starting from the defining equation:

R z - - c(z) + [C2 (Z) - 4b(z,-)b(z,+)] 1/2 a(s) 2b z,+

Observe first that the value R,(z,-), when considered as determined solely by the magnitude of c(z), monotonically increases with c(z), in the sense that we hold b(z,±) fixed and let c(z) increase. Thus in particular if for some special value c_o of c (z) we can show that R

then for all c (z) > c we will certainly have R_{ot}zz , -) L I . Now c(z) = a(z,--) + a?z,+) + b(z,-) + b(z,+). Hence (since all a ' s and b ' s are nonnegative)

c (z) > b (z,+) + b (z,-)

in fact the strict inequality holds in all real media. Let us denote b (z,+) + b (z,-) by "c_o ". Then

co + [$_{C0}$ - 4b(z,-)b(z,+) 1 ?/2

It follows that R,,,(z, -) ? I for 0 < z < z" plane--parallel optical medium.

in every

We recall at this point the fact that:

in all real optical media. (See (29) of Sec. 9.2; in fact the strict inequality holds in such media.) From these inequalities we deduce the fact that the difference R(z,-) - R_a (z , -) in all real media is negative for all z.

Continuing with the development of the theorem, suppose that we now measure R (z , -) , z > 0 , and then move the reflectance meter a small distance in the upward direction

(maintaining, of course, its horizontal collection-orientation throughout the move). What we are in effect doing by such a move is increasing by a small amount the material of the medium below the level occupied by the meter. It turns out that this upward motion is the natural direction of motion one should go in order to discern the equilibrium-seeking behavior of R(•,-), just as the natural direction of motion of the-observer in the equilibrium theorem for N was such that it increased the amount of scattering-absorbing material between the observer and the initial point of the path (Figs. 9.2-9.3). In this connection see also the discussion of the contravariation of K(z,+) and D(z,+) presented in Sec. 9.3. Therefore, to analytically describe

the result of this motion the derivative term of equation (2) is now read as dR(z,-)/d(-z). The final steps in in the proof may now readily be taken. Suppose that R(z,-) < R (z,-) at the depth under consideration. (See Figs. 9.2., 5.3.) Hence, R(z,-) - R ($z_)$--) is negative. By the preceding observations, it is know that R(z,-) - R,(z,-) is invariably negative in all real media. Thus ,the derivative dR (z , -) /d (- z) is positive, indicating that R(z,-) tends toward the value of the equilibrium reflectance R_q (z, -) at this depth, as z decreases. On the other hand, if R(z,-) > R (z,-), then R(z,--) - R_q(z,-) is positive, and since R(z,-y - R,(z,-) is invariably negative it follows that in this case dR (z, -) /d (- z) < 0 , so that

FIG. 9.2 As one moves upward through the hydrosol, in the direction of decreasing z, observe how the slope of R(z,-) is always directed so as to decrease the gap between

R(z,-) and the equilibrium reflectance R (z,-). This is the process referred to by the equilibrium-seeking theorem for R(z,-).
once again R (z , -) tends toward R_q (z , _) . This completes the proof of the theorem.
We summarize the equilibrium-seeking theorem symbolically as follows:

dR (z,-)
sign z] = sign [R (z,-) - R(z,-)
T(-) q
We may now make several observations on this equilibrium seeking property of R(*,-).
Observation 1
By returning to the basic premises of the present discussion, we observe that the condition a 1 o was imposed. This condition has both physical and mathematical relevance to the conclusion (4). Mathematically, R,(z,-) and R_q(z,-) are prima facie undefined for the case a E D . Physically, the reflectance of a purely absorbing medium or a vacuum is

FIG. 9.3 Further variations on the equilibrium-seeking theorem for R (z , -).
trivially zero. To lift the veil of mathematical indeterminacy of R(z,-) in the case of a = 9 , we return to equation (1). Under the present conditions (1) reduces to:
dR z,-) _ _ c(z)R(z,-)a (z)
where in this case c(z) = a(z,-) + a(z,+) > D . Hence if the initial value of R(* , - at z" is R (z" , -) _> o , then clearly:
R(z,-) = R(z",-) exp
f $_z$t I z
c (z')dz'
By letting b(z,±) + o in the quadratic equation governing R (z,-) we see thatR_q (z, -) = a. The preceding formula for
R~z , -) shows that, for media with a(z) > Q for all z , and a =_ 9 on. 1z' ,z"] , 1 im $_z$tl _ ., ,,R (z , -) = © . Hence the equilibrium seeking tendency of R(-,-) is borne out for this case also.
Observation 2
What about the opposite case to that just considered? Namely that o (z ; V ; 0 a for all z , E' , E , and a - 9 ? I t follows that R_a[z,-] = 1 and that R_q(z,-) = b(z,-)/b(z,+)

for each z. Now suppose that R(z,-) < 1 at some depth z in the interval [z' ,z"]. This implies that H(z,+) < H(z,--) and hence that the radiance distribution N(z,-) over (the upward directions) is on the average less than the radiance distribution over 7-- ,(the downward directions). Now using the scattering functions defined in (31) and (32) of Sec. 9.3 and the definitions of b (z,±) as given in (8) of Sec. 8.3, we see that in general,
1 b(z,-) R(z,-) b(z,+) _
f

o_+ (z; ~')N(z, ~') dQ (~')
o+_(z19V)N(z .E')cM(~')
But if the preceding supposition, namely that R(z,-) < 1 holds, then it follows that:
7
1 b (z 1-)
R z, T, ~ z,+
or in other words:

By the equilibrium-seeking theorem, it follows that dR (z, -) /d (- z) is positive. Hence as depth is decreased in the purely scattering medium, the values R(z,--) increase monotonically. But since the preceding supposition, namely R(z,-) < 1 , was for an arbitrary R(z,-) magnitude less than unity, it follows from (2) that for every z ,
lim(z„_ z) } ,, R(z,-) = 1
Thus we have in observations 1 and 2 above proved (or outlined proofs for) some outstanding folklore about the elementary properties of R(•,-) in plane-parallel media. These proofs were arrived at by reasoning strictly from the various exact differential equations governing R(-,-). In this way we hope to illustrate the power inherent in that approach to radiative transfer problems under development in this chapter, which discards particular mathematical models and which concentrates on the study of directly observable quantities of the light field. It is to be emphasized that the reasoning in this approach proceeds directly from the exact forms of the equations of transfer.
The Integral Representations of R(z,-)
Starting with the factored form (2) of the transport equation for R(-,-) we make use of the separation of variables that exists within it, and we write:

dR(z,-) = b (z,+) d(-z)
R (z,- R z ,,_ _ Rq z~_
If we formally integrate each side, we obtain the desired integral representation by letting --z range from some
value z_j, to $z_2 > Z_1$6 Let the corresponding values of
R(-,-) at these depths be distinct: R(z,, -) R(z2,-] ; the depth range of interest can always be partitioned into pieces so that this is true; otherwise the problem of R(-,-) is trivial over this depth range. In other words, the range of integration can be subdivided into intervals so that over each, R(•,-) is monotonic and so that there is a one to one correspondence between the values of R(-,-) and the points of the interval With these observations we may then write:
R(z 2 ~-) dt z 2
= , +} dt
t-R t - t-R t
a ~ z 1 b (t
which is the desired integral representation of R(*,-). The variable t in the integrals acts as a dummy variable of integration, oriented as in the equilibrium-seeing theorem.
An alternate integral representation of R(*,-) may be obtained from (1) in which the variables are also conveniently separated. The same general arguments used to establish (5) may now be directed to the equation (1). The result is:

R(z2s-~
dt
_ b(t,+)t2 _ c(t)t2 + b(t,_)

Applications

We now discuss two methods of evaluating R(-,-) by means of its differential and integral equation representations given above. We illustrate the use of (5) for a very simple case, which is a useful approximation to reality, namely the case in which R_a (• , -) and R (*,-) are constant functions. The second method is based d1rectly on (1) or (3) and promises to yield a means of determining R(•,-) under realistic conditions.

Special Closed Form Solution

I f over some depth interval the functions a(•,+-) and b (• , ±] are constant, then the functions R,(•,-) and R (• I-) are constant functions over the same arbitrary depth interval,,

say [z j , z,,], z_2 > z , in a plane-parallel medium. Under these conditions, (~). is immediately integrable, and the definite integrals take the forms:

1 $^R q^{-R} a$

From the definitions of and in fact:

t..Ry R(z_2,-)

= b (+) (z_2- z,)
R(z_1 ,-)
R_q and R_a ,
we see that R.- R_q > o
R -R = a q
[c^2 - 4b (-) b (+) 1/2 b (+)
where we have written:
for
a(-) + a(+) + b(-) + b(+)
and where a(±) and b(±) are the the functions a(•,+-) and b(•,+-) [z I , $Z_2$1. In the present method may be distinct.
assumed constant values of over the depth range
all four of these quantities
Applying the
limits to the left integral, we have:
R(z_2,-)-R R(z ,-)-R
In IR(z - -R
2' a

[c2-4b(-)b(+)]1 /2 (z_-z.).

Hence,

R(z_,-)-R, -T R(z ,-)-R

_ j exp -_ [c2-4b(-)b(+) 'f2(z -z)

z , - R, 2

LKz ,-2)__-_R7 . If we write:

" c(z 3 ,-) 't for

R(z~ ~-) - Rq

R(z~ --_) - Ra

then:

R(z2'V-)

R -R a^c (Z, , -)exp { - [c^2-- 4b (-) b (+)].1 i2 (z 2- z 1) }

1 - c(zl ~-)exp{- [c^2-4b (-)b (+)l '/"(z2-zl) }

Hence if the four constants a(±) and b(±) are known or estimable over an interval [z_l,z_2] and R(z_l,-) is known, then R(z_2,-} is determinable. Observe that if we let (Z.-Z,)} o , thereby simulating an infinitely deep layer, then R(z_2,-) 2J%-)_)A. Hence R_q in this instance is the R - quantity of he classical theory.

The points of contact with the classical theory may be increased by observing that if we set: a(+) = a(-) = a*, and b(+) = b(-) b* , then

[c^2- 4b (-)b (+)] 1/2 = 2 [a*+2b*)] 1/2

2k

where k is the diffuse absorption coefficient of the classical, one-D model of the two-flow theory (see, e.g., (8) or (32) of Sec. 8.6).

By partitioning an inhomogeneous medium into essentially homogeneous contiguous layers, successive applications of (7) will yield a useful practical formula for the reflectance of the entire medium. The solution (7) automatically includes the effects of interreflections between the partition pieces. Thus suppose the medium, which extends over an interval [z_o,,z_n], is partitioned in n homogeneous layers defined by the depths -9 [z o , z'] , [z , z21, ... , [z_n- 1 , z_n] . If R(z ,-) is known (this may Le the reflectance of the bottom Boundary of the layer [zn-l,zn]), then by (7) we find R (z_n- l , -) . Another application of (7) with R (z_n- l , -) as the initial reflectance then yields R(zn-2,-)$ and so on to R(z_o,-) which then is the reflectance associated with the medium over the depth interval [zoPzn].

Differential Analyzer or Digital Solutions

Equations (1) and (3) as they stand, are suitable for determinations of R(*,-) by means of differential analyzer (and analog) or digital techniques, especially when the functions a(•,±) and b(-,±) vary extensively over the medium.

Series Solutions

By means of series solution techniques, equations (1) and (3) may also be used to solve the difficult problem of determining R(-,-) over some interval [z, , z] when a(-,,±) and b(-,±) are nonconstant and known over tRis interval. By expanding the coefficients of R^2 (z,-) and R(z,-), and the b(z,-) term in (1) in terms of infinite series in z, recursion formulas may be obtained for the coefficients in the infinite series expansion of R(z,-) over [z11Z21

Equivalence Theorem for R(-,-)

Comparison of the differential equation (1) with (39) of Sec. 8.7 shows that the observable reflectance function R.(*,-) and the standard reflectance function R (• , z_l) , both defined in a given slab x(©,z,), satisfy the same differential equation within x(o,z_l). This is a somewhat arresting fact since the interpretation of the two numbers R(z,-) and R(z,z_l) are quite different conceptually. Briefly,

R(z,-) is obtained as a ratio of H(z,+) to H(z,-) deep within X(O,z,) ; and R(z,z_l) is the ratio of H(z,+) to H(z,-) when X(z,z_l) is thought of as an isolated slab of X(0,20. This is not to assert that these functions need always agree in value at each point. Indeed, for z = z_i , we have R (z_j , z,) = 0 , since the reflectance of a slab of zero thickness is zero. On the other hand R(z_l,-) is generally not zero: its magnitude being the reflectance of the lower boundary of X(O,z_l). However, if X(O,z_l) has no lower boundary and no sources there or one whose reflectance and transmittances are 0 and 1, then R (z_j , -) = O and we would .expect R(z,--) = R(z, z_l) for every z in [0, z j ,since both functions now satisfy not only the same differential equation but also the same initial condition.

The principles of invariance for irradiance help clarify this somewhat unexpected relation between R(z,-) and R(z, z_l) . From (1) of Sec. 8.1 we have for the present medium X(z, z_l)

H(z,+) = H(z, +)T(z_1 .9z) + H(z,,-)R(z'z1)

If X(z,z,) has no upward irradiance at level z , so that H(z_l,+) = 0 , then:

H(z,+) = H(z,-)R(z,z1)

on the other hand, we have, by definition of R(z,-):

H(z,+) = H(z,-)R(z,-)

whence follows the equality of the two functions R(•,-) and R(-,z) over [O,z_i]. We shall summarize these observations as follows:

Equivalence theorem for reflectances : Let X (o, z be an arbitrary stratified source-free plane-parallel optical medium with arbitrary boundary irradiances H (O, -) and H (z_l , +) . Then the observable reflectance function R(•.,-) and the standard reflectance function R (• , z_i) for a general subslab X (z, z_i) o f X (o, z_i), satisfy the same differential.' equation [M above, or (39) o f Sec. 8. 7]. I f H (z , +) = o , then R (z, -) = R (z, z_l) for every z in [0-,z,].

The reader may gain still further insight into the connections between R(•,-) and R(•,z,) by contemplating the connections between R(•,-) and the complete reflectance functions 6(z, z' , z] , z < z' < z;~ in X(O, z,) ; and also the relation (35) of Sec.¯7.5.

Connections with the Two-Flow Theory

The equivalence theorem cited above permits a simple bridge to be constructed between the two-flow theories of Chapter 8 and the directly observable quantities H(z,±)

of the present chapter. The classical one-D model of the two-flow theory of the light field describes the irradiances in a boundaryless , sourceless, isotropically scattering

homogeneous slab over an interval [O,z_i] irradiated at the upper level (z = 0) by a directionally uniform radiance distribution and with H(z, +) = 0 (Sec. 8.6). The theory proceeds on the assumption that b(z,-) = b(z,+) = b* and a (z,--) = a(z,+) W a* (i.e., that the backward scattering and absorption functions for each stream are pair-wise identical and have the constant starred values over the slab). We can immediately deduce the values R(0,a) and T(q,z) associated with this slab on the basis of the present general theory. To do this, we recall the statement of the equivalence theorem for reflectance equations proved above. This allows us to use the expression for R(z_2,-) given in (7). We need only observe that, under the present setting, we should make the following pairings of variables:

z -~

z
2

R(z_ ,-) = (

R R =
a_q 1

0 z
$R_{a+}R_q$= c/b*
and finally we observe that:
[c^2- 4b (-) b (+)] 1/2 = $_2$ [a* (a*+2b*)] ' /2 = 2k
where k is the diffuse absorption coefficient of the model. Hence (7) reduces to:
one--D
[1--expl-2k$z_1$1]
R(0,z 1) R -R expt- z (8)

b* sinh kz
(a + sin kz, + cash z 1
Equation (9) gives the usual form for the reflectance of a slab of depth z, . (As a check, let z l -}c , and compare result with (8) of Sec. 9.2.)
The remaining form for T (0 , z l) can now be deduced immediately from (42) of Sec. 8.7, but the point of this discussion has essentially been made: The classical two-flow theory is an elementary special degenerate case of the present theory of directly observable quantities in real light fields. For convenience of reference, the transmittance T (0 , z l) i s given by
k (10) T(O,z_1) -- a + sit 11:k in z + cos z

This section develops several differential and integral formulas governing the observable reflectance function R(*,.-). Methods of using the formulas are outlined. Thus R(-,-),may be obtained directly without first solving for the irradiance functions H(•,+-) as has been necessary previously. The methods discussed are general enough to allow the determination of R(• ,-) if the absorption and backward scattering functions a(•,-+), b(-,+), respectively, for each stream in an arbitrarily inhomogeneous stratified medium are known. A general equilibrium-seeking theorem for R(-,-) is also demonstrated, the substance of which is the fact that the derivative of R(*,-) at each depth z invariably has an algebraic sign so as to decrease the absolute magnitude of the difference R(z,-) - R_q(z,-) between R(z,-) and the value R_q(z,-) of the equilibrium reflectance function. An equivalence theorem is proved which shows that the observable reflectance function R(-,-) and the generally nonobservable standard reflectance function R (- , z_l) for slabs within X (Q , z,) both obey the same differential equation, thereby establishing an important link between these theoretical (R(•,z)) and empirical (R(-,-)) concepts.