

9.5 The Contrast Transmittance Function

The list of transmittance concepts associated with a path $O_r(z,0)$ in an optical medium will be completed in this section with the introduction of a companion transmittance concept to the beam transmittance function introduced in (3) of Sec. 3.10, and the radiance transmittance function introduced in (13) of Sec. 4.5. This new transmittance concept is called the contrast transmittance function.

By way of introduction to the contrast transmittance function, we review briefly the general types of transmittance functions studied so far in the present work. Suppose X is an arbitrary optical medium with boundary S . Then

associated with X itself is the standard \circ -operator: $4(X;a,b)$, introduced in Sec. 3.8, whose physical significance is depicted in (a) of Fig. 9.4. When the incidence region a on the boundary S coincides with the response

region b , then $v(X; a, a)$ is interpreted as a general reflectance operator. When a and b are disjoint, $ry(X;a,b)$ is interpreted as a general transmittance operator. Three general transmittance operators may be associated with the light field in X

$$4^\circ(X; a,b) \quad * (X; a,b) \quad 4(X; a,b)$$

The circled and starred operators are associated with transmitted residual and diffuse radiance respectively; the unadorned \circ -operator describes the undecomposed or directly observable light field. In Sec. 3.8 $4(X;a,b)$ was defined in detail. $JO(X;a,b)$ is manufactured readily using the geometric structure of X , a Dirac-delta function, and the beam transmittance function Tr for X , much in the way $TO(x, z)$

SEC. 9.5 CONTRAST TRANSMITTANCE FUNCTION 163

FIG. 9.4 Three settings for transmission operators:

(a) general media, (b) one-parameter media, (c) plane-parallel media.

is defined in (3Z) of Sec. 7.1 for plane-parallel media. follows by subtraction of 4° from (cf. (41) of Sec. 7-1). It follows that:

$$J(X; a,b) = j^\circ(X;a,b) + J^*(X; a,b) \quad a \quad (1)$$

Much in the same way the complete transmittance operator $J(x,y, z)$ associated with a subset $X(x, z)$ of a one parameter medium $X(a,b)$ can be rendered into its residual and diffuse parts:

$$J(a,y, b) = X^\circ(a,,y,b) + j^{*-}(a,y ,,b) \quad \$ \quad (Z)$$

and which is depicted in (b) of Fig. 9.4. Similarly, the decomposition of the standard transmittance operator:

$$164 \quad \text{GENERAL THEORY OF OPTICAL PROPERTIES} \quad \text{VOL. V}$$

$$T(a,b) = T^\circ(a,b) + T^*(a,b) \quad (3)$$

is but a special. case of (1) or (2) and is depicted in (c) of Fig. 9.4.

All of the preceding examples pertain to transmittances of three dimensional subsets of an optical medium. One dimensional subsets can be endowed with similar sets of transmittances. Thus if $Q, (x, o)$ is a path in X , we have in our earlier work associated with it the beam transmittance $T_r(x, \sim)$ (Sec. 3.10). In order to emphasize, during the present discussion, that the beam transmittance is associated with residual radiant flux, i.e., flux transmitted over $Q_r(x,\sim)$ without being scattered or absorbed, let us place a circle superscript on the transmittance symbol so: " $T^\circ(x, \sim)$." Furthermore, corresponding to $J(X;a,b)$,

$g'(a,y,b)$, and $T(a,b)$ above, that is, corresponding to the transmittance operators for undecomposed (i.e., observable) radiance, we have the radiance transmittance operator associated with $Cd_r(x,U)$, as introduced in Sec. 4.5. Let us denote this operator by " $T_r(x,0$." The fundamental roles

played by $T_r(x, \sim)$ and $T_r(x, E)$ may be seen by comparing their effects on the initial radiance $N_0(x, \sim)$ of the path $O_r(x,0)$ a

$$N_r^0(z, \sim) = N_0(x, \sim) T_r^0(x, \sim) \quad (4)$$

$$N_r(z, \sim) = N_0(x, \sim) T_r(x, 0) \quad (5)$$

where $N_r(z,0)$ is the radiance at end point z of $c?r(x,0$ in the direction \sim . We shall consider only straight paths $O_r(x, C)$ in this discussion, so that we have the simple representation: $z = x + r\sim$. We interpret (4) as usual by saying that $N_r(z, \sim)$ is the residual radiance of $N_0(x' U)$ transmitted over $Q_r(x, E)$. We interpret (5) in the sense that $Tr(x,0$ is simply the number which when multiplied

into $N_0(x, 0$ gives $N_r(z, 0$ where $N_r(z, \sim)$ and $N_0(x, 0$ are the two radiances as they exist at points x and z in the actual light field in X . In other words, the current definition of $T_r(x,0$ consists precisely in agreeing to write:

$$N_r(z) = N_0(x) T_r(x, 0) \quad (6)$$

with respect to a path $Q_r(x, Q$ in X . Thus (5) is, an elementary consequence of (6). It is obvious that $T_r(x,U$ is immediately obtainable as a consequence of the interaction principle applied to $Q_r(x, 0$.

We can now readily round out the roster of transmittance concepts, associated with a path $(Pr(x, Q$ in X , and in such a way as to be uniform with the other basic transmittance equations (1)-(3).. Thus let us write:

$$T^*(x, \sim) = T_r(x, \sim) \quad (7)$$

SEC. 9.2 CONTRAST TRANSMITTANCE FUNCTION 165

Then:

$$T_r(x, \sim) = T^0(x, \sim) + T^*(x, \sim)$$

By the work of Sec. 3.11 and Sec. 4.5 we have convenient integral representations of T^0 and T_r . Thus, by (3) of Sec. 3.11:

$$T_r(x, \sim) = \exp$$

and by (2) of Sec. 4.5:

$$T_r(x, \sim) = \exp$$

In terms of the compact notation of Sec. 4.5, these may also be written:

$$T_r = T_r[-a] \quad (11)$$

$$T_r = Tr[-C, K] \quad (12)$$

It is clear from the definitions that T_0 is an inherent optical property of $g_r(x, \sim)$ and T_r (and hence Tr) is an apparent optical property of $[?r(x,0$.

The Concept of Contrast

We are now ready to formulate the concept of contrast and develop some of its basic properties. To begin, we can state that the sense in which we use the idea of contrast in radiative transfer theory is in the relative difference of two radiances. For example if one

directs attention to two points 1, 2 on a distant mountainside or submerged scene which have apparent radiances N_1 and N_2 , then the contrast of the first point relative to the second is given quantitatively by the difference quotient: $(N_1 - N_2)/N_2$. If $N_1 = N_2$, then there is zero contrast, radiometrically speaking

between the two points. If it happens that $N_1 > N_2$ then we say that there is a positive contrast of point 1 with respect to point 2.

Conversely, since $(N_2 - N_1)/N_1$ is now negative, we say that point 2 has negative contrast with respect to point 1. One of the fundamental problems in visibility theory in natural optical media is the prediction of the contrast of a visual target against its background.

The problem is solved if the contrast transmittance of the path of sight from the observer to the target is known. We shall now develop the theory of the contrast transmittance of an arbitrary path of sight in a natural optical medium.

156 GENERAL THEORY OF OPTICAL PROPERTIES VOL. V

By extracting the salient features of the intuitive notion of contrast given above, we see that, at its core, radiometric contrast is characterized in terms of two radiances which in turn are associated with two paths of sight in an optical medium. Therefore we can quite generally state the following:

Definition 1. If $O, (x_1, \sim)$ and $G_{A_s}(x_2, \sim_2)$ are two nontrivial paths (i.e., $r \sim 0, s \sim 0$) in an optical medium

(Fig. 9.5), then the difference quotient $[N_r(Y, \sim) - N_s(y_2, \sim_2)]/N_s(y_1, \sim_1)$ is the apparent contrast of $N_r(y_1, E_1)$ with respect to $A_s(y_2, \sim_2)$. If $r = s = 0$, then the difference quotient is the inherent contrast of $N_o(x_1, \sim_1)$ with respect to $N_o(x_2, E_2)$.

Regular Neighborhoods of Paths

The notion of contrast attains virtually all its use in situations where the two paths of sight are in some suitable sense optically close neighbors of each other. The following definition isolates the essence of the requisite concept of "closeness" of paths:

Definition Z. Let A and B be two subsets of an optical medium \mathcal{S} , and let $C(A, B)$ be a collection of paths in \mathcal{S} such that the initial points of the paths are in A and the terminal points of the paths are in B. (See Fig. 9.6.) Then $C(A, B)$ is called a regular neighborhood of paths

FIG, 9,5 Two paths and their associated contrasts as in definition 1,
SEC. 9.5 CONTRAST TRANSMITTANCE FUNCTION 167

FIG. 9.6 A regular neighborhood of paths.

in \mathcal{S} if and only if the paths in $C(A, B)$ have a common length r , a common beam transmittance T_o , and a common path radiance N_r .

Here are some examples of regular neighborhoods of paths: Let A and B of the definition be the two boundary planes of a stratified plane-parallel medium $x(a, b)$, and let $C(A, B)$ be the set of all paths parallel to a given direction. Then with stratified lighting conditions in $x(a, b)$ $C(A, B)$ is a regular neighborhood of paths in $x(a, b)$. Observe that two paths of $C(A, B)$ may be extremely remote, spatially. As another example, imagine two points x and y in a natural hydrosol, say the sea. It is clear, on intuitive grounds at least, that we can always find two spherical regions A and B or two small parallel plane

regions, A and B, about x and y as centers, respectively, such that the associated set C(A,B) of all paths with initial points in A and terminal points in B is, for all practical purposes, a regular neighborhood. Our intuition in this matter is grounded in the general appearance of spatial continuity (but not necessarily uniformity) in the inherent optical properties and light fields of real media. Finally, it is clear that if A and B consist of any two single distinct points, then C(A,B) is trivially a regular neighborhood of paths.

The importance of the idea of regular neighborhoods of paths in an optical medium rests in the following observation.

Observation 1. If C(A,B) is a regular neighborhood of paths in an optical medium X, and $O_r(x_j, C)$ are two paths of C(A, B), with terminal points z_1, z_2 , respectively, then:

168 GENERAL THEORY OF OPTICAL PROPERTIES VOL. v

$N_r(z_2, z_1) - N_r(z_1, z_2) = N_0(x_2 | E_2) - N_0(x_1 | E_1) = T_r(x_1, z_1)$

$N_r(z_2, z_1) - T_r(x_1, z_1)$

or more briefly:

$T_r(x_1, z_1)$

$C_r(z_1) = C_0(x_1, z_1) = T_r(x_1, z_1)$

where $C_r(z_1, z_2)$ and $C_q(x_1, 0)$ denote the apparent and inherent contrasts occurring in (13). This observation follows readily from use of the fact that if both $N_r(z_2, z_1)$

and $N_r(z_1, z_2)$ are decomposed:

$N_r(z_2, z_1) = N_r(z_2, z_1) + N^*(z_2, z_1)$

and their difference taken, then, since C(A,B) is a regular neighborhood:

$N_r(z_2, z_1) - N_r(z_1, z_2) = N_r(z_2, z_1) - N_r(z_1, z_2)$

$N_r(z_2, z_1)$

$(N_0(x_2 | E_2) - N_0(x_1 | E_1)) = T_r(x_1, z_1)$

Furthermore:

$N_r(z_1, z_2) = N_0(x_1 | E_1) T_r(x_1, z_2)$

so that (13) follows. On the basis of this observation, we are led to make:

Observation 2. If C(A, B) is a regular neighborhood of paths in an optical medium X, then to each path

$Q_r(x, z)$ there is assignable the quotient $T_r(x, C) / T_r(x, z)$.

where x is in A and r and z are such that x + r is in B, and this quotient is an apparent optical property and is generally dependent on x, r and z.

Contrast Transmittance and Its Properties

The preceding two observations point up the fact that the quotient of transmittances in (13) is a number which can be assigned to each member of a regular neighborhood C(A,B)

of paths. Since the quotient incorporates the apparent optical property $T_r(x, 0)$, it is also an apparent optical property in general. In view of this and observation 1, we can state:

SEC. 9.5 CONTRAST TRANSMITTANCE FUNCTION

169

Definition 3. Let $C(A,B)$ be a regular neighborhood of paths in an optical medium X .

The quotient $T_r(x, \sim) / T_r(x, \sim)$, where x is in A and $x+r$ is in B , is called

the contrast transmittance of the path $r(x, \sim)$ in $C(A, B)$, and shall be denoted by $J_r(x, \sim)$ or J_r^{-1}

Thus with the preceding definition (13) may succinctly as:

be written

$$C_r(z \in C) = J_r(x, \sim) \sim J_r(x, \sim) \text{ or } C_r = C J_r$$

Observation 3. If $G_r(x, \sim)$ is any path in a general optical medium x , with integrable a and K then:

$$J_r = T_r / T_r = T_r - (a - K)$$

(14)

The proof is immediate, using (11) and (12) and the multiplicative property (8) of Sec. 4.5. Therefore J_r is determined for a path of sight once a and K are known over that path. Another result which devolves on the multiplicative property (8) of Sec. 4.5 is

Observation 4. If $G_r(x, \sim)$ and $Q_s(y, \sim)$ are any two

continuous subpaths of a path $(r+s)(x, \sim)$ (see Fig. 9.7),

and J_r and J_s are their respective contrast transmittances, then: J_{r+s}

$$J_{r+s}(x, \sim) = J_r(x, \sim) J_s(x+r, \sim)$$

(15)

or, briefly:

In other words, the contrast transmittance, enjoys the semi-group property along with T^0 , T_r and the complete transmittance operators $\sim V(x,y,z)$. The semigroup property (15) holds even for paths along which the index of refraction is nonconstant, and even discontinuous (see, e.g., (25) of Sec. 12.2). The value of (15) lies in the fact that the contrast transmittance of an extended path of sight is determinable once the contrast transmittances of its parts are known.

Some important special cases of (14) are found in the case of stratified plane-parallel media. According to (18) of Sec. 4.5, we have

$$J_r = T_r [-(a + K \cos \theta)] \quad (16)$$

in such settings. Moreover, if we adopt a two-D model of the light field in an infinitely deep medium $X(0,\infty)$, then (16) reduces to:

170 GENERAL THEORY OF OPTICAL PROPERTIES VOL. v

exp

$$-(a - k \cos \theta) r \quad (17)$$

where K now becomes the diffuse absorption coefficient k of the two-D theory (Sec. 8.5). In the case of one-D theory $k = -k$ (Sec. 8.6).

Observation 5. If X is an arbitrary optical medium with integrable a and l then the canonical form of the apparent radiance N_r associated with a path (P_r) in can be written in the form:

$$N_r^0 = N_0 T_r + a - \dots \kappa [1] \quad (18)$$

where J_r is the contrast transmittance of (P_r) .
 This observation follows at once from observation 3 and [15] of Sec. 4.5.

FIG. 9.7 The semigroup property for contrast transmittance.

SEC. 9.5 CONTRAST TRANSMITTANCE FUNCTION 6

Alternate Representations of Contrast Transmittance

In view of the fact that the component transmittances making up contrast transmittance (definition 3) are themselves expressible as ratios of radiances, it follows that the contrast transmittance of a path may also be represented as such. Indeed, from definition 3 and (4) and (5):

$$T_r^0(x, 0) = \frac{N_0(x, 0)}{N_r(z, U)} \quad (19)$$

for every path $O(x, 0)$ with terminal point $z = x + r$. Hence T_r^0 may be usefully thought of as the ratio of the residual radiance to the apparent radiance of the associated path. The manner in which N_r is generated along r may be through either scattering or reflection or both. The latter mechanisms occur when br passes through interfaces, such as the air-water interface. Since $N_r(z, 0)$ is decomposable into the sum of $N_r(z, \sim)$ and $N_r^*(z, \sim)$, ((5)' of Sec. 3.13), (19) yields up still other useful forms, such as the following:

$$N_r(z, C) = \frac{(x,) 1}{N_r(z, \sim)} \quad (20)$$

or:

$$N_r(x, \sim) = \frac{N_r(z, \sim)}{1 + N_r(z, \sim)} \quad (21)$$

or:

$$N_r(z, \sim) = \frac{N_r(z, N^*(z, \sim))}{1 + N_r(z, N^*(z, \sim))} \quad (22)$$

$$N_r(z, \sim) + N^*(z, C)$$

or:

$$J_r(x, 0) = 1 \quad (23)$$

$$N_r(z,)$$

$$N_r z, 1J$$

or:

172 GENERAL THEORY OF OPTICAL PROPERTIES VOL. V

$$43_r(x, \sim) = 0$$

$$N_r(z)Q$$

$$N_r z, \sim - N_r(z \sim U)$$

$$N_r z, +$$

$$(24)$$

In virtue of the fact that radiance values are invariably nonnegative, and that decomposition property (5) of Sec. 3.13 holds, it follows that:

$$N_r(z, 0) < N_r(z, 0)$$

$$N_r(z, E) < N_r(Z, O)$$

for every path $P_r(x, \sim)$, so that (19) or any one of the preceding representations shows that:

$$(26)$$

for every path $O_r(x, 0)$ in a general optical medium X .

Some necessary and sufficient conditions that $J_r(x, \sim) = 0$ or 1 are clearly discernible from the preceding representations. Observe in particular that:

if $K = 0$ over Q , as is readily apparent from (14). Condition (27) holds for example when the path P_r is a horizontal uniformly lighted path in an extensive homogeneous portion of the sea or atmosphere. Furthermore,

$$\sim 0 = 1$$

if and only if $N_0 = D$. It is clear that $N_0 = a$ if and only if the index of refraction over the path is continuous at the point of consideration of N^* (for an example where 1, see (29) of Sec. 12.2). A contrast transmittance

μ_{00} which is distinct from 1 is called a singular contrast transmittance, and the associated path O a singular path.

Contrast Transmittance as an Apparent Optical Property

It was observed above (observation 2) that contrast transmittance is an apparent optical property, i.e., that it depends on the radiance field in the medium of interest. If the lighting along the path O_r is uniform, then we obtain such a corresponding extreme case as (27). On the other hand if the lighting along \sim_r is irregular, for example when O_r is directed through shadowed and sunlit regions, then the

SEC. 9.5 CONTRAST TRANSMITTANCE FUNCTION 8

values of J_r manifest these shadowings and lightings in a regular and predictable way, as the following two examples will show. To fix ideas we shall initially choose a very simple setting.

For the purposes of the first example, part (a) of Fig. 9.8 depicts a path O_r in a plane-parallel medium $X(a,b)$ which is parallel to the boundaries of $X(a,b)$ and at some depth y . The medium is stratified and has a stratified light field except in the shaded region shown, which simulates a shadowed part of the medium. For example, such a shadow may be produced by an isolated cloud over the ocean, or - a ship shadow, etc. It is of interest to relate the contrast transmittance of O_r before and while it is shadowed. It is also of interest to compare this situation with that of

FIG. 9.8 The effect on the contrast transmittance of a path when shadows fall within and behind the path.

174 GENERAL THEORY OF OPTICAL PROPERTIES VOL. V

part (b) of the figure which depicts the shadow falling beyond the path O_r .

It is at once clear from (19) that if " $N_g(z, \sim)$ " denotes the shadowed apparent radiance for Q_r , then since:

$$N_r^\#(z, \sim) < N_r(z, \sim)$$

and $N_Q(z, E)$ is unchanged, we have:

$$\frac{N_r(z, \sim)}{N_r(z, \sim)} > \frac{N_r^\#(z, \sim)}{N_r(z, \sim)} \quad (28)$$

(is shadowed internally)

for a shadowed path O_r as in (a) of Fig. 9.8. In the second example which is depicted, in case (b) of Fig. 9.8, both $N_r(z, \sim)$ and $N_r(z, C)$ are affected and exhibit a decrease. However $N_r(z, \sim)$ is decreased more than $N_r(z, C)$ so that:

$$\frac{N_r^\#(z, \sim)}{N_r^\#(z, C)} > \frac{N_r(z, \sim)}{N_r(z, C)} \quad (29)$$

(Q_r has shadowed background)

for the path O_r shadowed as in (b) of Fig. 9.8. This may be seen by inspection of (20).

For, by hypothesis, $N^*(z, \sim)$ is unaffected (no shadow from x to z) and $N_r(z, C)$ is decreased, so that the quotient in (20) is increased, the difference in (2D) thereby decreased, which was the effect to be shown.

The apparent radiance formula (5) of Sec. 3.13 may be used to obtain quantitative estimates of the increase or decrease of J_r due to shadowing studied in the two preceding examples. To see how the formula is used, let us (for brevity) write: " $N(z)$ "

for $N(z, \sim)$ in (a) of Fig. 9.8 and " $N(v)$ " for $N(v, \sim)$, and so on. since z is fixed all along the path. Further " NS " will briefly denote $NS(z, C)$, and

"TS" will be short for T° (v.0). Then for the portion of path O_r in (a) of Fig. K8 extending from v to z , we have from (5) of Sec. 3.13:

$$N(z) = N(v)T_s + N_s$$

Further, for the segment from u to v : before the path is shadowed:

$$N(v) = N(u)T_t + N_t$$

where we have written " $N(u)$ " for $N(u, \sim)$. # If the path is now shadowed, then N_t decreases to, say $N_t^\#$. Let us write (ad hoc):

$$\text{for } N^\# / N_t t$$

SEC. 9.5 CONTRAST TRANSMITTANCE FUNCTION 9

We can now estimate the magnitude of the shadowed apparent radiance $N(z)$ of G_r as follows: Let " $N(v)$ " denote

the shadowed apparent radiance at v . We then have:

$$\begin{aligned} N^\#(v) &= N(u)T^\circ + N^\# t t \\ &= N(u)T^0 + cN_t \\ N(u)T_t + N_t + (c-1)N_t \\ N(v) + (c-1)N_t \end{aligned}$$

Returning to $N^\#(z)$, we have:

$$\begin{aligned} N^\#(z) &= N^\#(v)T_o + N^* s S \\ [N(v) + (c-1)N_t] T_s + N_s \\ r N(z) + (c-1)T^\circ N_t \end{aligned}$$

Hence the desired representation of $N(z) - N^\#(z)$ is"

$$N(z) - N^\#(z) = (1-C) T^\circ N^* \quad (39) s t$$

Now in general, the smaller t is, the smaller N_t will be and so by (30), the less effect the shadow will have. Further, the farther away the shadow is (i.e., the greater s is) from the point of observation, the less effect the shadow will have, since TS decreases as s increases in real media. A similar analysis can be made for part (b) of Fig. 9.8, and it is left to the reader to show that, in this case:

$$N(z) - N^\#(z) = (1-c) T^\circ N^* \quad (31) s t$$

From (30) and (31) we see that both cases are covered by the same type of formula. It is interesting to observe that if the shadow region from u to v in Fig. 9.8 straddles point x in just the right way, the contrast transmittance of the path $r(x, 0)$ is unchanged, even though, as (30) and (31) indicate, there are definite changes in the lighting conditions. An examination of the preceding arguments would show that no essential use was made of the plane-parallel structure of the medium depicted in Fig. 9.8, nor of its stratified light field. Furthermore, the shadowing factor c may just as well have been a lighting factor (i.e., $c > 1$), without affecting the algebraic structure of the resulting equations (30) and (31). We use these observations to generalize the results and to summarize our findings as follows.

FIG. 9.9 The general setting for Fig. 9.8.

Let $r(x, 0)$ be a path in a general optical medium x , and let $T_t(u, \sim)$ be a proper $co 2$ Zz near path with respect to

$Q_r(x, 0)$ i.e., for some positive or negative number s ,

$u = x + s\sim$ and $r > s + t$ (see Fig. 9.9 in which distance is measured positive from x to z , and negative from z to

x). Suppose that the path radiance $N_t(v, \sim \text{ of } \sim t \sim u, \}$ (where $v = u + tE$) changes by a factor c . This change generally affects the apparent radiance $N_r(z, \sim)$, where $z = x + r\sim$. Let " $N_r(z, E)$ " denote this new apparent radiance, then:

$$N_r(z, \sim) = (1/c) T_{r-}(s+t)(v, E) N^*(v, \sim) \quad (32)$$

If the proper collinear path $(P_t(u, E))$ is wholly contained within $Q_r(x, C)$ (i.e., $s > 0$) then the contrast transmittance

SEC. 9.5 CONTRAST TRANSMITTANCE FUNCTION 10

$J^{\#}(x, \sim)$ corresponding to the factor c is greater or less than $J_r(x, \sim)$ according as c is > 1 . On the other hand, if

the proper collinear path $(P_t(u, E))$ is behind and disjoint from $Q_r(x, C)$, $s < 0$, and $s + t < 0$ then the contrast transmittance $J^{\#}(x, \sim)$ corresponding to the factor c is

less or greater than $J_r(x, \sim)$ according as c is > 1 . A quantitative estimate of $J^{\#}(x, \sim)$ may be based on (32) by

solving for $N_r(z, \sim)$ and using any one of (19)-(24). In

case $s < 0$, then $N_r(z, \sim)$ is also changed, and must be computed accordingly, using (32), and (5) of Sec. 3.13. In general three cases should be distinguished for $J^{\#}(x, \sim)$:

the proper collinear path $(P_t(u, E))$ is either (a) contained in $Q_r(x, C)$ or (b) behind and disjoint from $Q_r(x, C)$ or (c)

$(P_t(u, E)) \sim Q_r(x, C)$ behind and overlapping $Q_r(x, C)$

or (d) $(P_t(u, E)) \sim Q_r(x, C)$ behind and overlapping $Q_r(x, C)$, i.e., $s < 0$ and $s + t > 0$

On the Multiplicity of Apparent

Radiance Representations

One final observation may be made at this point on the use of (5) of Sec. 3.13 in deducing the preceding properties of contrast transmittance, and which helps cast some light on an interesting general fact about the concept of apparent radiance. The attentive reader will have noticed that there appears to be an infinite number of choices in the manner of representing a radiance $N(x, E)$ as an apparent radiance associated with some path $Q_r(x, 0)$. This fact was essentially observed in Sec. 3.13 in the discussion of (5) of that section. However, the present applications of that equation

in deducing (30), (31), and (32) bring home the multiplicity of the representations with renewed force; and we shall now formalize this fact for future reference. The

observation may be phrased generally as follows. Suppose $N(z, \sim)$ is the radiance along the direction \sim at some point z in an optical medium X . If now we place a path (an imaginary construct) in X with direction C so that x is its initial point and $z = x + r\sim$

is the terminal point (Fig. 9.7), then we may immediately reorient our conception of $N(z, E)$ from that of a primitive radiance in X to that of an apparent radiance of the medium at x associated with the path

$Q_r(x, 0)$. That is, by (5) of Sec. 3.13, we can write:

$$N(z, \sim) = N(x, C) T_0(x, C) + N^*(z, \sim) \quad (33)$$

and to point up the fact that $N(z, E)$ and $N(x, \sim)$ are viewed in the framework of the path

$Q_r(x, \sim)$, we may, as is customary, attach 'r' and 'l' subscripts to them, respectively. By imagining still another path $Q_s(y, 0)$ with direction

and initial and terminal joints. and $z = y + s E$, we can
 represent the same $N(z, C)$ in (33) as
 $N(z, C) = N(Y \sim E) T S(Y s E) + N_5(z, \sim)$ (34)

This phenomenon of the multiplicity of possible representations of a given measurable radiance $N(z, Q)$ with respect to

178 GENERAL THEORY OF OPTICAL PROPERTIES VOL. v imaginary paths in an optical medium is reminiscent of, and actually logically related to, the freedom of choice of the parameters x and z (Holding y fixed) in the statements of the principles of invariance (e.g., in Example 3 of Sec. 3.7) or the invariant imbedding relation (e.g., in Example-4 of Sec. 3.7).