12.4 Harmonic Analysis of the Dynamic Air-Water Surface We shall devote some attention in this section to the topics in harmonic analysis required for our present studies of radiative transfer across the dynamic air-water surface. The' battery of concepts of harmonic analysis, as they are applied to the air-water surface, are relatively new, having been intensively applied during the past decade by increasing numbers of workers in mathematical and experimental oceanography. A survey of the history of the subject is out of place in this work, but it can be begun by consulting the references, [320] , [307].,,.[19l]., and others listed during the discussion below.

Our primary aim in the discussion below is to prepare . the ground for answering some of the initial basic questions raised by researchers entering this domain of ideas for the first time. The most frequently occurring questions are: What are the sources of the ideas of harmonic analysis? What is the difference between harmonic analysis and synthesis? Sometimes one sees Fourier integral representations of an analyzed function, and other times a Fourier series representation. Is there some way of deciding between these two modes of representation for a given context? Is there some special justification for choosing the tools of harmonic analysis for use in describing the sea surface and the dynamic surfaces of natural hydrosols in general? Even if such harmonic analyses of the dynamic air-water surfaces can be made, why is the energy spectrum singled out for so much attention? What are some of the things one can do with the notions of harmonic analysis in pursuing the studies of hydrologic optics?

The Roots of Harmonic Analysis

Modern harmonic analysis may be said to have begun with the seminal paper by Wiener [320] in 1930 which was concerned with the rigorous mathematical .foundation of the method of periodogram analysis. The immediate mathematical basis of wiener's work rested on that of Plancherel [207] a mathematician of the early twentieth century. Periodogram analysis was begun by Schuster [278], [27-7] in 1897 in his studies of geophysical optical and magnetic phenomena. Schuster's concern was with the detection of "hidden" periodicities in these physical processes as they evolve in space and time. One of the principal optical conundrums of that day, to-which Schuster addressed himself and his periodogram analysis method, was the nature of the composition of white light. The physicist Gouy [loo] used Fourier's series (which in turn date back to the first decade of the 1800's in Fourier's studies of heat [93]) to represent white and other composite flows of light. Implicit in Gouy's analysis was the belief that white light did in fact consist of a composition of distinct individual flows of colored light. However, in those early days, the sophisticated way of viewing "white" and other "mixtures" of light, namely as superimposed quantized electromagnetic fields, had not yet been evolved, so that for a time the natural philosophy of light as envisioned by Gouy and others was partially beset by Schuster's conclusion that when "white" light was analyzed by a diffraction grating, the monochromatic components were manufactured on the spot by the special geometric arrangement of matter in the grating. This is only a partial statement of the present view, and it turns out that Gouy and Schuster both saw only one facet of reality. In the present view, light may be conceived as having in reality a composition synthesized of predominantly monochromatic wave trains of light which manifest either particulate or wave structure (and hence color) which then may be observed or not depending on what mode of observation is used. Thus the present view is that the actual observation can either select or manufacture the appropriate radiometric component from the composite field, depending on the state of the observed field and the state of the observing instrument. (See, e.g., [150], [151].)

The physical basis for the harmonic analysis of light, as begun by Schuster at the turn of the century, currently rests in Maxwell's equations (in classical or quantized form) whose solutions may take the form of sinusoidal functions (plane waves) and which in turn, because of the linearity of Maxwell's equations, may be synthesized as linear combinations of those sinusoids. Most of the modern field theories, while differing from Maxwell's equations in form and content, nevertheless show these two essential features eigen functions and linearity) with their logical ancestors, so that these important physical bases for harmonic analysis still stand today. As we have seen in [99] of Sec. 12.3, and as we shall see later in more detail, the physical basis for the harmonic analysis of the air-water surface rests in the linearized equations of hydrodynamics.

Harmonic analysis, then, originated principally in the work of Fourier, Plancherel, Wiener, Schuster, and then fanned out into the work of a host of physicists and mathematicians since Wiener. For a historical sketch of Fourier analysis, see [38]. Some modern mathematical references on harmonic analysis are [170]. [269], [162]. Working references on harmonic analysis are [313],.[21], [321],-[38], [24],.[294], and [191]. Two especially useful reference texts on harmonic analysis are [152] and [29]. The powerful concept of distributions promises to become a working tool in the harmonic analysis of the physically more realistic functions (nonsummable over noncompact domains). The studies [891, [1_59]$_1$, 151J.9 [327] make a beginning in this direction, and serve as supplements to Wiener's pioneering studies in [320].

Harmonic Synthesis vs. Harmonic Analysis

The term "harmonic synthesis" denotes the addition (or superposition) of a finite or infinite number of sinusoidal functions to form a new function. An example of a harmonic synthesis is given in (99) of Sec. 12.3. In that case the elevation function ~ describing the dynamic air-water surface was the result of the synthesis.

The term "synthesis" thus denotes a building-up or construction of an object from simpler components. "Analysis," on the other hand denotes the breaking-down or taking apart of an object into simpler components. "Harmonic analysis" therefore denotes the analysis of an object into its harmonic or sinusoidal components. Beyond these simple definitions lies an interesting and relatively deep distinction between "analysis" and "synthesis." Of these two ideas, that of synthesis of an object appears to be a relatively straightforward process: given certain components, we can put them together in a prescribed manner and end up-with a composite, synthesized end-object. Thus, a synthesized object presents no mystery about what went into its structure. On the other hand, when confronted with a given object, (say white light, or the sea surface)which comes to us from nature as an unanalyzed apparently indivisible whole, there seems to be an element of artificiality and arbitrariness in the subsequent analysis of the given primitive object into the preselected components (say sinusoidal functions). That is, while conceptual synthesis of objects seems to be straightforward, conceptual analysis of objects on the other hand, raises the question of the reality of the analytical components in the original object.

Some thought shows that in the case of the dynamic air-water surface, just as in the historic case of the analysis of white light, the recording or observing of various harmonic-or sinusoidal--components, that is, the observed presence of waves of a given wavelength, depends jointly on the mode of origin of the waves and their mode of observation. Thus if one drops a pebble into an otherwise still air-water surface,

an instant later there is a set of small circular ripples spreading out from the point of entry of the pebble. If we choose to analyze this dynamic surface by means of the plane
wave components cos (k:.., r. - at + e) introduced in Sec. 12.3, it is clear on the one hand that there is not a single plane wave in sight in the system of circular ripples so that a visual analysis into plane wave components is impossible. On the other hand, it is a simple matter (and a stroke of genius) to mathematically analyze this disturbance into a Fourier series of plane wave components. This analysis can be accomplished with arbitrarily great precision in practice, and exactly in principle, and therein lies the importance of Fourier analysis: It is a convenient tool with simple analytic properties. Here, then, is an instance where the observing instrument (the mathematical theory or one of its hardware realizations) can manufacture plane-wave harmonic components and endow the analysis of the object (the circular ripples) with elements not inherent in the object. However for many practical and theoretical purposes of conducting scientific work this forced and willful analysis is satisfactory.

In this way those readers coming on the notions of harmonic analysis for the first time can be prepared to , view-harmonic analysis as a useful powerful tool which, in its resultant representations, may or may not' use components inherent in the original object. The importance and worth of the analysis therefore rests ultimately not in the conceptual objects it resolves physical data into (for these, as we have just seen, may have no natural counterparts) but whether the analysis can be uniquely reversed by a synthesis which faithfully yields all the features of the original physical process selected for representation. .

Another simple example from natural everyday activity may serve to emphasize the preceding viewpoint of harmonic analysis. Imagine that an observer is standing on a hillside overlooking a quiet sunlit meadow in the center of which is a small pond. He spies a hawk suddenly swoop straight down out of the sun and alight on a small furry creature several yards to the right of the pond' ,s edge. Notice now how the sun and the pond played central roles in helping the reader to visualize the scene. They serve as primitive coordinates relative to the observer in locating the preceding activity in the space above and on the meadow. This descriptive activity is formalized in theoretical discussions by replacing the hawk's instantaneous position by a vector v- Once a coordinate frame has been selected, v'can then be analyzed into components along the x,y and z axes in the customary manner. Thus:

v ,b _91)1t (v ,J) J + (v .. k) k

is the representation of v. relative to the components (i, j ,k) . The main point now being made is that the choice of the components (i,j,k) is completely arbitrary, and that neither these components nor any others are inherent in the structure of v and in the original pastoral scene. Yet it is manifest that for all practical purposes this kind of analysis is useful and occasionally indispensable in painting faithful symbolic pictures of realty. The preceding analysis of v is of the simplest kind of geometric analysis. Yet, as we shall see,

it shares the algebraic heart of the concept of harmonic analysis with the most awesomely complex examples ever, or yet to be manufactured. For, tho dot products v • i, v • j , v • k are a form of analysis of v. and as such are analogous (or correspond) to taking the Fourier transform of v, or Fourier integration over a finite interval of the function v. The adding-up of the components(v • i)i, (v • j)j , and (v • k)k is a form of synthesis and is analogous to the sum of a Fourier series, or the inv6rse of the Fourier transform of v.

Integrals vs. Series in Harmonic Analysis

The question of whether one uses Fourier integrals or Fourier series to represent a given function f is often resolved by deciding on the domain of f. If the domain of f is (closed and) bounded, then the representation of f may use Fourier series. If the domain of f is (not closed or) not bounded, then the representation of f .may use Fourier integrals. Here are some examples of closed, bounded domains: [0,27r], [-27r, 27r], [a,b], where a,b are finite numbers. Two dimensional closed, bounded sets are: [0,27x] x [Q,27r], [a,b] x [c,d], where a,b,c, and d are finite numbers, etc. In each of these examples the endpoints of the intervals are included and the sides of the rectangles are included in the domain,
and this is what makes the intervals closed. We have included the condition "closed" in parentheses above, since this condition is of secondary importance in geophysical practice.

Examples of unbounded domains are: [-oo, 0} , [-w,co], [Q,°°] , [a,°°] , a finite. Further: ~ [-oo,oo] x [--oo,oo] , [a,°°] x [b,°°] are examples of two-dimensional unbounded regions. These domains are closed or open depending on whether the endpoints or .sides (as the case may be) are all included or all omitted, respectively. The mathematical basis for the preceding criteria may be found in [269] or [170].

It is clear that if f is defined over some finite plane rectangle D which represents an extensive portion of the sea, then the whole infinite plane containing D may be partitioned into a checkerboard using copies of D over each copy of which the structure of f is repeated. It is clear also that, while f can be so extended to an infinite domain, its essential domain is finite and consequently its representation may be effected by Fourier series.

Fourier series representations of the dynamic air-water surface may then suffice for all conceivable practical situations in hydrologic optics. Fourier integrals are not excluded, however, but their principal role, except in the simplest cases, will be that of a tool to be used in theoretical studies of the air-water surface prior to or instead--of numerical studies. In the present exposition, therefore, we may and shall use Fourier series or integral representations as the need for each type of tool arises. The practical criteria for the use of one or the other mode of representation are thus clear: to study a finite (albeit large) region of the air-water surface, one can use Fourier series- to study an infinite region (i.e., E] of the air-water surface where most of the energy of the disturbed surface is contained in a finite region (e.g., a storm center) outside of which the energy of the surface can be made arbitrarily small, one can use Fourier integrals; and finally to study the surface in an arbitrary steady state over an infinite region one uses the Fourier-Lebesgue-Stieltjes representation, to be described below. The Fourier series representation thus is adapted to handle what have been customarily called the periodic functions (because they may be prescribed essentially over a finite region and the remainder of their extent is obtained by mere replication); the Fourier integral represents what are called transient functions or aperiodic transient functions; and finally the Fourier-Lebesgue-Stieltjes representation is relatively recent and applies to what are called random functions or aperiodic stationary functions (mathematically they are simply bounded measurable functions). Interesting parallels of the present representation problems with those arising in communication engineering may be studied in [152].

Fourier Series Representations of the Air-Water Surface

A formal representation of the dynamic air-water surface by means of Fourier series may be obtained as follows: Let f be a real or complex valued function defined on the closed interval [a,a+2p] of the real line whose real or imaginary parts describe the surface along a given directed line. Here a and p(>4) are arbitrary finite real numbers fixed throughout this discussion. It was originally' shown in essence by Fourier [Art. 171 of [93] et seq.) that the value f (x) of f at each x in [a,a+2p] may be represented by trigonometric series of the form:

a 00

f (x) ti Z + I a_n cos n=,

n 7r x P

b_n sin nTrx]

n=i

where the sign means that the trigonometric series on the right was manufactured by constructing the coefficients a_n and b_n from f in the following manner. To be specific,

let f be real valued. (If f is complex valued, we work separately with its real and imaginary parts.) We write:

"a " n for 1 fa a+2P f (x) cos p n7Tx) dx p (3)

If b_n" for 1 f a+ 2p f (x) sin p n7rx) dx p (4)

where n = 0, 1, 2, ... The series (2) is the Fourier series associated with ,f. Practical working conditions on f which will ensure that its associated Fourier series converges to f in well-defined ways, are the well-known Dirichlet conditions which may be found for example in [38], and which certainly cover many of the hydrodynamic situations encountered in hydrologic optics.

It is possible, by adopting-complex variable concepts, to collapse (2) and the formula_$ (3) and (4) into very compact forms without any essential loss of physical significance. The advantage gained in doing so rests principally in expediting subsequent formal; manipulations of the series and in uncovering the essential 'algebraic ideas behind Fourier (and, generally, harmonic) analysis. When the manipulations are over and numerical work is J'to begin, the following steps can be reversed. To begin,, write:

TI $_y$ *t

for n7rx P

and recall from complex analysis that:

e^{l}y = cosy + i s in y (5) which is Euler's formula. Next, from (3), (4), and (5)

a+ 2P a

To summarize the preceding procedure, we may say that if f is a real (or complex) valued function of one variable defined over [a,a+2p], then its associated Fourier series, in complex form, is obtained by the construction summarized in (6), in which c_n is found from f according to the operation in (7).

It is particularly important to-note that the series in (6), while ostensibly complex valued, is real valued if f is', and this may be readily seen by the working out of a particular example or retracing the steps from (7) to (2).

Common occurrences of the parameter a are o and - iT , while that of p is most often f. Thus, for a = 4, p (6) and (7) become:

00

f [x] e- i nx dx

We now show how functions of several real variables may be methodically analyzed into their Fourier series representations by repeated applications of [6] and (7). We consider

the elevation function ~ which represents the dynamic airwater surface over the closed spatial rectangle [a,a+2p] [b,b+2q] and over a time interval [c,c+2r] where a,b,c and p,q,r are all arbitrary finite real numbers. We begin by fixing the time t in [c,c+2r] and the y coordinate in [b,b+2q]

with the result that ~ (• ,y,t) is a function of a single variable x over the interval [a,a+2p]. Then we have at once from (6) and (7)

00

C (x tY.V t) ti C_m (Y' t) eim7Tx/p (8) $_m$=-C

a+ 2p C $_m$(Y't) = 2p 1 f ~(x~y,t) $_e$-imrrx/p dx (

a

for every y in [b,b+2q] and t in [c,c+2r] and m = 0, ±1, ±2, ... Keeping .t fixed in [c,c+2r], now let y vary in [b,b+2q] so that we can apply (6) and (7) to the Fourier analysis of the function Cm(-,t):

00

Using (14) in (12) and (13) in (15) and reducing the results, we finally arrive at:
.
which is the desired Fourier series representation of over the space-time block ^{r}[a,a+2p] x [b,b+2q] x [c,c+2r],.
Hydrodynamic Basis for Harmonic Analysis of Air-Water Surfaces
The Fourier series representation (16) of a general air ater surface over a space!-time block was derived mechanically and on a strictly mathematical level. We now return to that representation and introduce some appropriate concepts from hydrodynamic theory.' One 'immediate result will be a formal simplification of the representation of ~ (x,y,t) to one which uses only a double sum over the spatial index m and n. This simplification is made possible by introducing the equations of motion of the air-water surface into the analysis and letting them carry some of the burden of describing ~ over the time interval [c,c+2r].
To begin, set c = 0 and let the initial displacement configuration of the air-water surface be given by ~ (x,y, 0) for x and y in [a,a+2p], [b,b+2q], and let initial vertical displacement speeds be given by 3~[x,y,0]/Dt, for x and y over the same intervals. Suppose further that the subsequent displacements during the time interval [0,2r] are described within the rectangle[a, a+ 2p] x [b , b+ 2q] by ~. Then a Fourier series representation of ~ may be obtained as follows. First, let t = 0, so that:
(20)

^{A}mns
This merely serves to suggest that in the case of (18) and (20) we should also expect a special condition linking Amn and B_{mn}. Some experimentation shows that the requisite condition is:
B_{mn} = - i zr Q_{mn} . A_{mn} 1 , (22)
where we have written:
2
(24)
The basis for the condition (22) rests in (64) of Sec. 12.3, since for each pair (m,n) of integers., we have an associated plane wave component of the dynamic surface whose parameter o mn is governed by g and the depth h of the medium. More generally, we could write:
fl_u it mn
T~
f o r
(25)
using (90) of Sec. 12.3. In any event, by imposing the condition (22), we let the natural motions of the waves take up the task of describing C(x,y,t) for t in the range 0 < t < r having

initially prescribed ~(x, y , o) , and a ~(x,y , o) / at . That is, by (19), (21), and (22),
q dx dy (2b)
which fixes the values of A_{mn} given the initial conditions of the wave motion. Then write:

It z (x,Y, t)" for
OD co
izr mx + ~ - A e p q mn
$^{6}mn^{t}$
It follows that:
We now appeal to the theorem in hydrodynamics which states that a fluid undergoing irrotational motion (cf., assumption (ii) below (3n) in Sec. 12.3) has its motion uniquely determined for all t > 9, given its initial displacement and initial displacement speed (see, e.g., Art. 57 of [149] or Sec. 3.77 of [181]). Hence, by the uniqueness of the subsequent flows:
~(x$Ypt) = z(x,Y,t)
for t in [O,r] . That is
(27)
for (x,y,t) in the given space-time block.
The preceding representation of ~(x,y,t) can be returned to real notation by retracing the steps from (2) to (7). The result may be cast into the form:

and where, for m > 1, n > 1,
^{a}mn ^{A}mn + A-m,-n
b mn
^{A}mn _ -m, ,-n
This form may be further reduced by noting that sin 8 = cos. (90-0) which will help change over the sines and leave only cosines. Finally, by renaming a_{00} as '°c " and a_{mn} (after the
general manner in (10) , (11) of Sec. 6.) , as "$c_{2\ j}$^{11}; and b_{mn} as IT c_2j -," where
where for each pair (m,n) we have written:
'tu.+' for
J
"v " for
j
"a." for
J
"k_j" for
($_u$2 + V2)1/2 J J
and where ej is an appropriate phase (0 to 90^0). In this way we can close our analysis of shaving finally returned to the appropriate general version of (99) of Sec. 12.3.
The Periodogram Basis of the Energy Spectrum
In his search for hidden periodicities in natural phenomena, Schuster [277], [278] hit upon a simple but remarkably effective analytical scheme for uncovering the

periodicities which existed in the data of interest. His method (cf. [280]) eventually was developed by Wiener [320], into the method of autocorrelation (or autocovariance) analysis, a powerful tool of harmonic analysis. To see the early form of the method as evolved by Schuster, suppose a function f, defined over some relatively long time interval, [c,c+2T] is to be examined for periodic components. It is tempting to try to reconstruct Schuster's line of reasoning as he sought a means of discovering hidden periodicities. Perhaps it went, in essence, like this (cf. [278]): Suppose, we write

and

r c+$_2$ T

"A_u(T) " for j f [x] cos ux dx

C

$_r$c+2 T

"Bu(T)11 for j f (x) sin ux dx

c

Now if f were periodic, and in fact equal to a cos ux, and 2T = 27T/u, then

rc+2T A_u(T) = a J c

cos^t ux dx

Hence, under such a form of f, A (T) would eventually increase linearly with time T. Stating tRis in another more precise way,

A_u (T) $_1$ c+2T

1'M$_{T\text{-*co}}$ $_T$ = liM$_T$~ $_T$ f [x cos ux dx f c

a

That is, 'the process of finding the limit of A_u(T)/T as T-*oo, would yield the amplitude a of the periodic phenomenon represented by f, if f were exactly of the form a cos ux. Still further, if f were of the 'form a cos vx with u v, then it is clear that

A_u (T) a f c+2T

limT}~ T = limT}~ T cos vx cos ux dx c

D

In a similar way these observations may be shown to hold for the case of B_u(T). This suggests that the operations:

and

c+2T

1 imT~ 1 f

T

c

I c+2T

1 i
mT }oo T [
c

] cos ux dx
] sin ux dx
applied to a function f defined over the time domain would extract from f the amplitude of any sinusoidal component of frequency u that f may have, and discard all other components of different frequency. Thus if we write:

would serve as a quantitative measure of the presence of sinusoidal components of f of frequency u. (Simply adding A_u and B_u might cause cancellation to zero, and thus belie the presence of a component.) The squaring operation is by far the more tractable of the two in analytic work, and therefore is chosen as the requisite measure. The function which assigns to each "u" the value Au + Bu (or some constant factor times this) is the periodogram of f. I t i s essentially the energy spectrum of f, as we shall see below.

When Wiener studied Schuster's periodogram method, he contributed further to the evidence of the efficacy of the method. Suppose that f is data from some physical process recorded over the time interval [-T,T]. To emphasize the fact that f is known only over the interval [-T,T] and that it is set to zero for t > IT I , we write "fT" for this f.

(Hence in the preceding analysis, we would choose c = - T.) The autocorrelation (autocovariance) function OT Of 'fwas then defined by Wiener (following G. I. Taylor- [295]) by writing:

The bar over "f" in (29) denotes the complex conjugate of f. If the data are real valued, then _FT = fT. Observe that the infinite limits of integration mask an essentially finite integration range (namely [-T, T]) for each value x (because of our definition of f). However by being able to put in these infinite limits we open the door to the formal Fourier transform techniques of Sec. 7.14 and are able in particular to gain some deep insight into ~ (x) . To prepare for the application of the Fourier transform, suppose that f has the Fourier series development:

cn

Because of the finite domain over which f is not zero, c in (31) is, to within a multiplicative factor, the Fourier transform of f_T . That is, using the notation of (l0) of

Sec . 7.14 (wi th kernel K (x, w) = e^{-} iwx)

V1 [

Observe next that we can express ~T as -a convolution (Sec. 7-14):

1

rco

~T

We are now ready to apply the Fourier transform operator Y to . Converting ~T into a convolution operation on $T_T R$

and f and using the convolution theorem (6) of Sec. 7.14, T~

we have:

2T in 2T c_n = 2T I c_n 1 (34)

The fact that we can here use the convolution representation for OT, in the argument o f the operator ~, rests on a proof similar to that of (5) of Sec. 7.14. The notation "fT R" means the "reverse of fT", i . e . , (f R) (x) = f_T (- x) .

Com

paring (33) and (34), we see from (3) that applied to any function fT yields the value c_n (in the spectrum of f) multiplied by Z T, the length of the interval over which f is generally not zero. Hence, from (34), it is clear that the corresponding value (for the integer n)-in the spectrum of ~,T is (1/2)1c I^2, (the interval over which OT is not zero is [-2T,2T]). Therefore the Fourier series representation of O_T is00

- I Icnl2 cos u_nx (35) n=o

In this way we have demonstrated that: the spectrum of the autocorrelation function O o f a function f is the energy

spectrum o f f (as defined Blow (100) of Sec. 12.3). This fact, which can be shown to hold for quite general functions f, is known as the Wiener-Khintchine theorem. Its earliest forms may be found in [320], [135].

The relation between the periodogram of f and its energy spectrum is readily seen by referring to (6) and recalling that:

Hence wiener's choice of the autocorrelation function (29) to be the basis for the generalization of Schuster's periodogram is an appropriate one.

It is possible to find out some essential information about the periodic components of a function f by a direct examination of its autocorrelation function prior to an harmonic analysis of f. This is one of the important intuitive features of the autocorrelation function. Some examples will make this clear.

First, we select a sinusoidal f and compute its autocorrelation function. Let f be a cos ux over [-T,T], where

2T = 2f/u, and zero elsewhere on the real line; and then let T-}oo

On the basis of this, it becomes plausible that the autocorrelation function of an extensive periodic function is again periodic and with the same period as the given function.

Note how, in this simple example, the Wiener-Khintchine theorem is illustrated. As a simple exercise, the reader may now evaluate OT for finite T and note in detail the manner in which OT approaches 0 computed above.

Second, we select a function which is very definitely aperiodic. A simple example is that of the real valued function f depicted in Fig. , 12 . 19, such that

fix} = a

0T(X) _ 1 f f(x+t) f (t) dt -T

a^2

-- 2T • (2T- 1xD, for jxj < 2T (38)

and

O_T (x) = 0
for Jxj > 2T (39)
This example shows that for the given f, ~T(x) is a decreasing function and that for large values of x, ~T becomes zero for the aperiodic instance at hand.' Actually this is quite representative of aperiodic functions in general which extend to +oo and -co: ~T(x) is sharply decreasing in value toward zero as jxj increases from o. A rule of thumb is

Fig. 12-:49
FIG. 12.19 Calculating the autocorrelation function of a box-shaped function.
FIG. 12.20 Part (a) depicts the autocorrelation function of a sinusoid with shift x. Part (b) depicts the autocorrelation function of Fig. 12.19.
that: the more sharply decreasing ft with increasing 1XI, the more aperiodic is the f from which ~ is farmed. Graphs of f for the two examples are shown in .1a) and. (b), of Fig. 12.20. A simple graphical interpretation of the calculation of ft in the case of the present aperiodic function is shown in Fig. 12.19. Observe how the graph of f is shifted x units to the left to represent the values f.(x+t) when x is positive. Further examples of autocorrelation functions may be found in the paper by Rice [313] and the book by Lee [152].
Once a visual analysis of ft has-been made on which traces of periodicities of f are evident, .quantitative estimates of the components giving rise to these periodicities are available by simply taking the Fourier transform of ~T,

and this, by (34), yields ~ $1c_n 1^2$, and hence the energy spectrum of f.
Fourier Integral Representations of the Air-Water Surface. Case 1: The Surface is Aperiodic
The concept of the energy spectrum which, as we have just seen, is, the logical outgrowth of the periodogram method of analysis devised for optical problems by Schuster, also lends ,itself to the difficult problem of analytically describing the dynamic air-water surface. The mathematical setting in which the concept attains its full powers of description in the present task is that of two dimensional Fourier integral theory, which we shall now briefly outline. We shall consider two cases: First, the case where the sea surface is aperiodic and whose amplitudes die away rapidly outside some finite (e.g., circular) region.* Second, we consider the cases of periodic and random seas.
The main goal at hand in Case 1 is to find a way of generally describing (i.e., in one fell swoop) the contributions to ~(x,y,t) by wave components of an aperiodic sea surface whose amplitudes belong to either discrete or continuous spectra, or both. Visual observations of the dynamic sea surface result in plots of ocean wave numbers which fall into two main classes as indicated schematically in Fig. 12.21: The first class is that of wave components whose wave numbers k are part of a discrete set. The second class consists of those wave numbers which are part of a continuous set. In other words, in a given sea it may be possible that all the points from a region C in the uv plane have nonzero wave components in that sea in addition to the finite (or infinite). number of points k 'in a discrete set D. Such a

sea surface can be represented as 'follows: Let the amplitude associated with the wave number ki (= (ui,vi)) in D be A(ki) (= A(ui,v-))
and that with k (= (u, v)) in C be A'(k) (=A (u, v)) . A(k) dV (ki is then interpreted as the common amplitude of the wave components cos (k • x - at+£) in the region of area dV(k) about k in the uv plane. Hence our work leading to (28) and our intuitive understanding of A(k) leads us to write:
n

*This is also called the transient case in communication engineering. An interesting parallel between the present cases in hydrodynamic theory and their communication theory counterparts may be generated by comparing the ensuing development with that in [1521.

FIG. 12.21 A given sea surface may have a spectrum which is partly discrete and partly continuous.

where we have written 1'x11 for (x,y), where a and E: are generally functions of k, and where n is finite or infinite. Examples of continuous spectra will be considered in Sec 12.5 For the present we consider theoretical methods by which, at least in principle, the raw data of ~(x,t), can be processed so as to yield estimates of A(ki) and A(k) , where k is in D and k is in C, and hence yield the representation (h). We shall concentrate at first on the aperiodic case, as represented by the continuous amplitude spectrum A(k), as that is the more difficult of the two spectra to handle analytically.* This will take us to (49) below. Then we shall go on to the general combined spectrum problem summarized in (51).

We now consider, for convenience, two copies of Euclidean two space E_2: one which we shall call the 'x--plane,' and the other which we shall call the 'k- plane'.

The x -plane will be the setting for the representations of the air-water surface and k - plane the settings for the wave numbers (and hence the amplitude and energy spectra of ~). If R is any measurable subset of E_2, then:

fR f (x)dV(x)

*Note that the purely aperiodic sea surface whose amplitudes die away. ;at great distances necessarily has no finite discrete spectrum (since individual sine functions never die away). The latter spectra are considered in case 2.

is the integral of a measurable function f over R and which is an alternate notation to:

fR f f (x , y) dx. dy

We shall need the concepts of Lebesgue and Stieltjes integration in order to carry out the present exposition, and any one of several available standard texts on advanced calculus (or [320]) may be consulted for terminology. However, in keeping with the general mathematical level, of, this work our principal emphasis in this discussion will be, on, the physical meanings of the terms and equations,,, and not, on, the formalism.

Recall that a function f on E_2 is square 2ntegrab Ze (briefly: "in L^2 (E_2) ") if

limr4_m I f (x) ^{2} dV (x) f Sr ~Y)

exists and is finite, and where Sr(y) is a circular domain in E_2 of center y and radius r. For example, if for some time t and radius ro, ~(x,t) o for IxI > r_o, and
(x, t) 2 dV (x)
S_r ~Y)
exists and is finite, then mathematically ~ (• , t) is in L^2 (E_2) .9 but more interestingly, the mean square height and hence total energy of the air-water surface at time it is. finite. Hurricanes and other turbulent sea areas outside of which the sea is relatively calm may be represented by such functions.
If and ~(• ,t) are two "seas" at different times, s < t, a measure of the difference of the energies is given by:
fE I ~ (x 3, x) - ~ (x 3, t) 12 dv (X)
2
More generally, the square of the "distance" between two functions f and f_a in $\lim_{n}\}t$

or more generally, if f_a./is a member of a general family {f_a} of functions,, and
1 ima-+b f E g W - f (x) ^{2} dV (x) = o 2
then we will write:
C(x.~ t) = 1.i.M.$_{4;\#t}$ C(xls)
or more generally:
g(x) = 1.i.m. f_a(x) (41) a}b
and say that ~(x,t) (or g(x)) is the Limit in the mean of
~ (x:, s) (or fa(x)). Recall (or verify) from integration theory that i f (41) holds, then
f g (x) dV (x)' = l ima}b f f $_a$ (x) dV (x) (42)
A A
for every measurable set A c E_2.
Now if at some time t, the air-water surface is given by C (• , t) which is in L^2 (E_2) , then let us write-.*

"A46
Then by Plancherel's theorem (cf., e.g., [321]) A(a,t) is in L ^{2} (E $_2$) and
*The manner of placement of the factors ˉ1/27T before (43) and (44) is to some extent arbitrary, and may be done in any way so long as the product of the factors before the integrals is 1/ (4fr2) . Thus we can,e.g., omit 1/2n from before (43) and place 1/(4n2) before (44), etc.

and furthermore: so that the energy content of the air-water surface is given by means of A(• ,t) , according to (45) (cf., (98) of Sec. 12.3; also see (29) and (35) in which x = 0). Relation (45) is known as a Parseval identity.
Now, from a computational point of view the operation in (41)is a bete noire because "l.i.m." denotes an. operation realizable only in the mathematical ivory tower and not the numerical laboratory. Whenever Lebesgue integration theory (or measure theory in general) yields up an object g which is in the form of a limit in the mean, and that object, such as A (k, t) in (43) above, is o f physical interest, it is possible to compute the value A(k,t) almost everywhere by means of the general formula:

for almost every k..- That is, for all k in E , except perhaps for a subset Z of E_2 of at most zero area, (U) will yield a usable value A (k, t) for X outside o f Z. Hence whenever we replace "l i.m." by "lim" in the above manner we can visually clear the way, at least in principle, for a transition from abstract set theory to practical numerical work. The operation (46) or (47) is clearly an averaging process of the function involved, thereby removing complicated wiggles and spikes from the function in question over the set S_r (x) or Sr(k)

As an illustration of the preceding remark, let us reconsider (43). According to (47) we require an initial expression for A(k+k',t) , and this is given by (43) in the form:

that is

The equality above involving the ordinary limit operation is obtained with the help of (42). Equation (48) is the required operational rule for computing A(k,t), and the result holds almost everywhere in E_2. In a similar manner the calculable representation of ; (x, t) takes the form. -

~ (x , t) = l im r} 1 1

The Fourier integral synthesis of ~(x,t) in (49) holds when the indefinite integral of the amplitude function A (• , t) is continuous, so that the preceding theory is adequate to

cope with purely continuous spectra, with no Dirac--delta spikes representing isolated wave components,

Now when the integral of the spectrum function A(•,t) is not continuous, so that the limit operation in (48) may not necessarily be possible, a different tactic must be used in finding A(• ,t) . Evidently, if trouble arises when we go to the limit r}o in (48), it may be profitable to delay this operation until a more judicious stage in the analysis of ~. Thus for each real number s>o, and time t let us write:

"A (k, t; s) " for

In this way we are in effect defining a measure, the amplitude measure, which assigns (for each given k and t) to the circular subset S_s(k) the number A(k,t;s), such that the square of the absolute magnitude is the energy of all wave components in the region S_s(k). In communication theory, this has a one-dimensional analog known as the integrated Fourzer transform or integrated spectrum of ~. At this point our heuristic discussion leading to (101) of Sec. 12.3 may be helpful. Clearly S_s (k) in (50) could be replaced by any arbitrarily shaped subset of the k -plane. However, simple circular neighborhoods will suffice in the present discussion. In order to define A(k,t; s) as we have done above we implicitly required c(• ,t)

to be in L^2 (E_2) , or less stringently, we require ~ (• ,t) to be integrable over regions of finite area, and:

•,t $_{XT}$:

to be in L^2 (E_2) . In. this way ~ (• , t)can describe a broad

range of wild and tumbled seas (still no whitecaps or breakers, unfortunately, since ~ is to be single valued) and essentially all we require is that ~(•,t) be bounded and measurable. The associated A (k, t ; s) may now vary discontinuously with respect to s, and k.

In the special form of+ $(50]^k$ we now recognize, by virtue of the general forms of the definitions (43) and (44), that it is possible to use Plancherel's theorem to conclude:

(x, t) (f $_e$- i (x • k') dV (k I) S_s (0)
1. i.m. 1 r A(k,t; s) $_e$i (x• k) dV(k) r- 2n j . $S_{r\,r}$(o)
For practical work, we can remove "l.i.m." and replace it by "lim" by 'integrating over a small circular region Sq(y): of center y, radius q. This we do, and so arrive at:
Dividing each side by the area of S_s (0) and rewriting the inner integral on the right side of the preceding equation, we have:
(x) =
The right-hand side is the penultimate step in forming a Stieltjes integral with the amplitude measure A (k, t ; s) as the, Stieltjes weighting function with respect to the geometric area measure on the k-plane.
Dividing each side of the preceding equation by the common area q of S_q (0) and S_q (y) , and letting q } 0, we finally arrive at
which holds for almost every Y in E_2. Comparing this representation with (49), it is clear that we have reached our main goal in Case 1 of representing c when ~ is composed of wave trains having both discrete and continuous spectra. Observe how (51) reduces to (49) if A(k,t; •) is absolutely continuous with respect to area measure in the spectral domain. Then
lim } A(k,t,s) = A(k,t) s 0 Tr s 2

for almost every k. In practice we can attain the complex version of (40) using the two--dimensional complex versions of the results of Sec.'21 in [321]. In practice, however, it is customary to use only (49) for the purely aperiodic case. Periodic seas or single samples of random seas are handled somewhat differently as we shall now see. This approach concentrates not on ~ but on its autocovariance--or autocorrelation function.

Fourier Integral Representations of the Air-Water Surface. Case 2: The Surface is Periodic or Random

In the present case the Fourier transform of ~ over all of E_2 generally does not exist. We accordingly introduce for the purposes of Case 2, the two-dimensional time-dependent autocorrelation function of ~ by writing:
~
This function plays an important part in both the theoretical and numerical studies of the dynamic air-water surface. It may be used when the surface ~ has a periodic structure over E_2 or when ~ is simply a bounded measurable function (i.e., is continuing or random).* I t is easy to deduce the following two properties of ~: For every t,
~ (
The first property establishes the fact that 0 is an even function of its spatial variables whenever ~ is real valued (as it is assumed to be throughout this chapter); the second establishes the fact that its magnitude is a nonincreasing function of Ix I as Ix I increases. This shows that W• , t) is a bounded measurable function and satisfies the same general conditions that ~(•,t) does. Indeed, the present analysis rests its case on the mathematically (and physically) observed fact that, while ~ may have no Fourier transform, its autocovariance 0 often does, in the sense defined in the discussion following (50), and we~ assume this to be true in what follows.

Using (52) we can then construct an amplitude measure for ~ (• , t) , in the manner we did for ~ (• , t) in (50). and then go on to find the representation of ~ (• ,t) analogous to (51).

*As in Case 1, the parallel between the present case and the corresponding situation in communication theory should be studied by the serious reader. An overview of the present cases is given in 'Chapter 9 of [152

However, our present interest in 0(`,t) derives from its ability to yield the energy spectrum of C, without having direct knowledge of the spectrum (i.e., Fourier coefficients) of ~,- Furthermore, we are at present interested primarily in continuous spectra of individual samples of random functions ~, since this is the nontrivial part of the present representation problem. (For periodic C we would work in the Fourier series context considered earlier--with finite or infinite numbers of terms:) Finally, our task is to obtain a calculable representation of the amplitude measure of ~ (• , t) . Accordingly, in analogy to (50), let us now assume at the outset that ~ has the same regularity properties* as 4(-,t), and let us write:

"

E(k,t,•) is the energy measure function. To convert this l.i.m. to something we can work with numerically, we can follow the procedure given in (46), or if further theoretical work is necessary we can use the theorem in Lebesgue_ measure theory which states (cf. e.g., [320]) that: if

f (x) = 1. i . m, f x

then there is a subsequence

{ f_n (x) } o f f_n (x) } k

such that the subsequence converges to f almost everywhere. We can construct such a sequence as follows: Write

ri~ $_n$(x,t)" for 1 2 n (x+Y, t) n

% (Y, t)dV(Y) (56)

Trn J E

for each positive integer n and where, in turn, we have written:

*We shall increase the regularity ~ and its transform E (defined below) as we proceed. This increase is dictated by the rather limited mathematical apparatus available to us in this work and by physical considerations applied to ~ and E. The Bochner theory of positive-definite functions (cf., e.g., [320]) is an ideal vehicle for the theory of the correlation function.

(58)

I f "E_n (k, t ; s)" denotes the amplitude measure for $\sim_n$ (x, t) , using the general definition (55), then by the aforementioned theorem,

E_n (k..,¯t; s) = 1 ~ (X ,t) ~ f e¯ ($^{x-k1}$) dV(k') ~ e-l(

Furthermore: there is a subsequence of {on} such that

E(k,t;s) = $\lim_{n\_,.}$ E_n(k,t;s)

almost everywhere on the k-plane. By the Lebesgue bounded convergence theorem, we can move this limit operation inside E_n(k,t;s) , and use (58) to obtain:

E(

we call E (k,t) the unresolved spectral energy density function or unresolved energy spectrum of ~. We are now implicitly assuming that r(k,t;•) is absolutely continuous with respect to area measure, and that its derivative is absolutely integrable.
We find from (60):

(62)
almost everywhere on the k-plane. In this way we arrive at the important representation of the energy spectrum of ~ in terms of the Fourier transform of the autocorrelation function ~ of ~. Observe now E(k,t) is obtained without Fourier analyzing directly. In the case now at hand the Fourier transform of a random (i.e., bounded) function ~ generally does not exist over E2.'._ However, by our present assumptions on E (k, t) , we see that E (- , t) and 0(•,t) can form a Fourier pair. Thus almost everywhere on the x-plane:
(xi t) = E (k 9 t) el (x ok) dV (k) E_2
(63)
The unresolved energy spectrum E (k,t) is an even function of k. This may be seen by first using (53) to deduce:
2
(62a)
where E2 is the right half of the x-plane (i.e., all x = (x,y) such that x > 0). The evenness of E (• , t) now follows from that of the cosine function. (In general: the parity (evenness, oddness) of a function and its Fourier transform agree.)
This property of E (- , t) has a physical interpretation which is important to bring out at this time; E(k,t) is a measure of the combined energy of wave trains moving in both directions k and -k (see (77) below). This interpretation can b e made plausible by studying the one-dimensional case as it is given in (30)-(35), and by pairing (35) with (63). The complex coefficients c_n in (3©) contain directional information in their arguments (phases). This information is unfortunately "folded together" in going over into Icn12 in (35). It is for this reason that we have called E (k,t) the "unresolved" energy spectrum.
We may directly observe this phenomenon of folding together of directional information in E(k,t) and also relate E (k., t) to A (k, t) by repeating for the present setting the derivation of (34). Thus, analogously to (34), we have

The integration over $S_s(k)$ smoothes the l.i.m. functions and serves to make the amplitudes accessible to numerical computation and physically meaningful, as we have already noted several times above. Thus:
An(k,t;s) _ 1 1 r r -i (x.k')
rrs ors JSn (k) E_2
is a working formula for A_n (k, t) , where we write
A_n(k~t ~s)
"A_n (k,t) " for $\lim_{n\}}$~ (65) Ti 5
and which may be compared to (61). Hence for continuous spectra:
almost everywhere on the k-plane. Using (66) in (64), and using (62) applied to E_n and O_n:

E n (k,t) - 1 2 JA n (k_it) 1 ^{2} (67) urn
for every n > 1. Equation (67) is another working formula for the connections between E_n and A_n, and is the analog of (34) which holds in the Fourier series case. Going to the limit, (67) is idealized by:
E (k , t) = 1 im

We continue the present discussion with some pertinent observations on the time dependence of the various functions ~, 0, E, and. A discussed above, and their relations to the unresolved energy spectrum. In our studies of the Fourier series analysis of the sea surface, we found that introducing the condition (22), derived from hydrodynamics ,.into the Fourier representation simplified the analytic form of the amplitude functions. This may be seen by comparing (16) with (27). A similar tactic may be employed in the present case. Thus we may assume
n (k, t) = A n (k, $_{6)}$ e-iat (69)
for all t, where a is given by (23) , or (25), in continuous form. This assumption, therefore,, adopts the linearized model of the air-water surface discussed in Sec. 12.3. As a result, the present version of (44), on recalling (66), becomes:
which holds almost everywhere on the x--plane. It follows from (68) and (69) that E (k, t) = E (k, o) for all t., so that the energy content of the air-water surface is constant during the time the linearized conditions hold. Thus also
O (x, t) = O (x, ©) , by (63). These observations show that the linearized model is adequate only for steady state seas whose waves do not_(or no longer) interact with the wind or with other waves, and in which no energy dissipation processes occur.
The application of the linearized theory of the air-water surface,, therefore, is practically limited both in hydrodynamical and harmonic analyses to steady state and relatively calm seas in a small region. The most the associated harmonic analysis will yield is the unresolved energy spectrum in the form (62). This is quite adequate for many practical investigations (cf. [191]) and is retained as a useful tool for this reason. However, if the energy spectrum is to be further 'resolved, more information must be drawn from either theory or nature herself. In the former case, for example, we are led into the domain of nonlinear wave theory, a subject under continuing development (cf., e.g., [191]).'
In the further appeal to nature, we must Fourier-analyze data taken over E. at more than one instant in time. Thus, we can begin anew and apply Plancherel's and Wiener's theories over the space-time domain E_3 (= E_2 x E,) for the air-water surface. Here E.-is the x-plane and E, is the time domain (i. e:,
This procedure would yield the exact basis for data-reduction schemes yielding energy spectra for the air-water surface without invoking any specific assumptions about the hydrodynamical behavior of the natural hydrosols. Such a procedure would result in completely analogous Fourier integral counterparts to (16) and (17) 6

There are several middle courses that can be taken between the preceding two extreme extensions of the analytic procedures for the energy spectrum, and we now turn to describe such courses.

A Working Representation of the Dynamic Air-Water Surface and its Directional Energy Spectrum

One procedure leading to a middle-road air-water surface representation theory of the kind discussed above would be to postulate, on the basis of our preceding work in hydrodynamic theory (cf. (28)) and harmonic analysis, (cf. (44), (51)) that the air-water surface has a representation of the form:

6Zf $_e$i (k•x-Qt+c) dA(k) E_2

(71)

This is the spectral representation of the dynamic air-water surface which may, if required, be considered as a real stationary stochastic process* in the 'x-plane (cf. Chapter XI [66]-). I t is useful in studying periodic or random seas (cf. Sec. 5.5 of [101]),. Here the integral is of the Fourier Stieltjes type, where c - and a are given functions of u,v, the coordinates of the k-plane. In particular, a is an even function of u,v and can be defined by means of linearized hydrodynamic theory (cf. (64) or (90) of Sec. 12.3 and (23) above). The amplitude measure used above is of the form A(k,s), i.e., the steady state version of A(k,t;s) discussed in detail following definition (50). Because of the Stieltjes form of (71) we can represent ~ (x,t) either in series or integral form or both. If A(k,s) is assumed continuous with respect to the area measure V(•) of the k-plane, then we can write:

for lim } A k,s) (72)

which defines the amplitude spectrum a. In order to expound the essential ideas of the present representation, we shall assume that the amplitude measure A(k,t;-) is indeed continuous

*In this context, we would view ~-(x,t) as but one sample of a large set' IC,(x,t):acA} of surface representations. Here a is an index drawn from a set A of indices, which may be finite or infinite, depending on the type of stochastic process adopted. Concurrently, there would be associated with c_a (x, t) a sample A,(k) of the amplitude measure.

with respect to V(-), and we shall work with one sample of the surface at a time. In this way we avoid the complications of Stieltjes integration and stochastic processes, which one may better tackle once the main issues below have been clarified. Equation (71) may then be written:

~ (x.,-t) = I a(k) cos (X-x.-at+E) dV (k) (73)

E

In the discrete case for A (k, s) , we could write out the representation of ~ as in series form. The autocorrelation function cs is now to be defined by an averaging process over both the k-plane and the time domain (in distinction to (52)) as follows. We write:

_ (y., s) dV (y) ds (74)

Thus (74) allows consideration o f the case where ~;' coul d b e either periodic or random (cf. Case 2 above) . Using (73) in this definition, and writing*

. (76)

*The connection of E(k) with existing treatments in the harmonic analysis of the air-water surface may be made by noting that E (u,v) (= E (k)) is basically the E (u,v) of Languet Longuet Higgins in [155). The rather singular mode of definition of E (k) in (75) is prompted by dimensional considerations and our decision to bypass stochastic theory in the present exposition. (See the discussion after (101) , below.)

The Dirac-delta function in (75) is dimensionless, as a study of the substitution of (73) in (74) will show. The subscript "l" in (75) is a momentary reminder that a is multiplied by a unit-valued function of dimension L-', i.e., inverse length.

E(k) is the resolved energy, spectrum or directional energy spectrum of ~. The reader will find it instructive to derive in detail the form of (7b) when the spectrum is discrete and finite. To begin, suppose (71) is such that:

a . $_e$i (x·kj _oj. $^{t+E}$j)

where the aj are real. I t follows that

n i[k~ • x-107 t+ E~ I . i [k~ • $_Y$~_ a_i s

a

Then perform the integration and limit operations in (74) for the full autocorrelation over space and time:

T

lim - 1 • I f ~ (x+y,-t+s) ~ (x, t) dV (x) dt ; r 2T 7TY,2 $_{-T}$

S (0)

r

and observe that

limn-o.-co

0 if k ~ 0

el k' x dv (X) = = 6 (k)

1 if k = 0

along with:

The bracketed statements are for discrete variables k, the Dirac-delta for continuous k (or a, respectively). The requisite averages are facilitated by-means of the general analytic fact `ihaf : If a, ~ are two complex valued functions, such that

j (6t ot) (6? f

) 0 o

then

Here stands for either a general integral, sum, or linear averaging operation. The result is that, for the present discrete case:

1 n i(k..Y-ojs) (Y is) ~' a. e

a_i cos (k_j ·Y-,$_j$ s)

This equation is the discrete version of (76). The derivation of (76) itself is fundamentally similar to that just outlined.

The relationship between the unresolved energy spectrum E(k) of the linearized theory (we shall omit "t" since E (X, t) = E(k,O) for all t) and the resolved energy

spectrum of the present theory may now be deduced. This relation is important to know since we cannot invert (76) to find E (k) directly from as in the case of (62) and (6-3). This is due to the _term
at in (76) . From (63) and the evenness of E (k)
(x, 0) f F (k) e I (x k) dV (k) E
2
E (-k)]cos (k•x) dV(k)
where E+ is the right half of the k-plane, i. e. , all (u,v) such that u > a. Hence when the present model is applied to the linearized theory, discussed above, which yields E(k), we have (since the two preceding representations of ~ (x, o) hold for all x, and since E(k) is essentially unique):

E (k) = 1 [E (k) + E (-k)] 2
(77)
Actually, on the basis of the meanings of E and E, the relation (76) holds quite generally, and may be used not only in linearized theory but also to find E (k) when estimates of the kind E(k) only are available from real data. Thus suppose that at the same time (t = 0) the temporal derivative of ~(x,0) is estimable. This may be done, for example, by taking two stereo photos of the air-water surface close together in time (cf. [45], [191]), Then from (76)
fE

a a (k,). E (k) sin (k • x) dV (k)
a (k) (E (k) - E(-k)) sin

2

2 a (k) (E (k) - E(-k);) sin

2

(k • x) dV (k)
(k •x) dV (k)
where we have used the fact that the sine is an odd function, and a is an even function of k - (i.e... o f (u, v)) . I t follows that ~' (x, 0) and (1/2) a (k) . (E (k) - E(-k)) are two-dimensional Fourier (sine) transform pairs. Hence:
1 a (k) [E (k) - E(-k)]- - 1 0'(x,,0) sin (k-x} dV (x)
2 4Tr2 fE
Let us write:

for I [E (k) - E(-k)] (78) 2
Then we have:
D (k)
1 j ~' (x.$_1$ 0) sin- (kox) dV(x) 4-n a (k) E2
(79)
We can now state that i f ~' (x,, O) and ~ (x., 0) are estimable from real data, then in principle we can. find E (k) according to the rule:
D(k) . + E (k) = E (k)
16
(80)

Here f3 (k) is the unresolved difference spectrum and r(k) the unresolved (sum] spectrum defined earlier in (61). The time. variable t has been suppressed; all three spectra in (80) pertain to a common instant in time. Equations (62a)(with t = 01 C73) t C79)* and (80) together solve the main air water surface representation problem stated; at the outset of this discussion. For by (62a) with t -= 0, we obtain the unresolved (sum) energy spectrum E (k)', and from (79) "we obtain the unresolved difference spectrum V(k). The directional energy spectrum E(k) then follows from (80)., and the amplitude spectrum (t(k) for the air-water surface is obtained from (75). The representation of the air-water surface function c then follows from (73).

Numerical procedures for realizing the preceding theoretical relations are developed in the one-dimensional case in [241 and in the two-dimensional case in [45].

Geometrical Applications of the Directional Energy Spectrum

It will be shown next how the directional energy spectrum E, and several of its important variants to be introduced below, may be used to determine four important geo metric properties of the air--water surface. These properties are instrumental in the solution of radiative transfer problems associated with the dynamic air-water surface. -The requisite properties leading to the solution are: the mean square elevation of the surface above mean sea level, the mean square slopes of the surface in each of two given perpendicular directions in the mean sea level surface, and the mean of the product of these two slopes. We now discuss, in turn, each of these-properties. The derivations will hold for either periodic ~ or random ~, where ~ is real valued on E_2.

First, the mean square elevation of-the air-water surface is defined by writing
f -T f S_r (0,)
(81)
Observe that if the air-water surface is statistically stationary (i.e., if the integral of 42 over E_2- is independent of t) then the additional averaging integration over t is unnecessary. It is at once clear from (74) and (76) that
= 0000)
E (k) dV(k_)(82)
If the spectrum of ~ is discrete, then 7 is a (finite or infinite) sum of squares of component amplitudes of ~, as the example following (76) 'illustrates,

Second$_s$ the mean square slope of the air-water surface, as it is sectioned by planes parallel to the xz-plane, is defined by writing:
For example, if ~ is given by (73) with a continuous. spectrum and we write out x in the farm "Cx,y)", when necessary for clarity, then:
[84)
Using this in (83) along with (75), we have:

(85)
which is the general form of the representation using the directional energy spectrum,
Third, defining the mean square slopes in planes parallel to the yz-plane by writing:
1 1 fT dV x dt ,
a
it D~ fo r l~im$_r$ ~ T..~ ~ 2 ay ~
y err -T f Sr~O~
(86)
It follows that:

(I r E (k) v^2 dV(k) \$^a$y/ = J E
2
(87)
where again we have used the rectangular coordinate representations x = (x,y) and k = (u,v).
Equations (85} and (87) use special weighted integrals of E (k), the general form of which is defined by writing:

Pq
for F E (k)up vq
j IE 2 dV (k) (88)
It turns out that the preceding three mean square quantities are representable as:
2
= ^{m}oo = 00,90)
(89)
(a~)2 _
^{m} ax 20 (47 mat

(90)
(91)
We call mp$_q$ the pgth moment of E. It is found that, in natural hydrosols moll $m_{1\ o}$, and the mean of ~ are for most cases zero. However, m,,, i.e., the mean of (a ~/ ax) - (a ~/ ay) is often nonzero. Clearly:
(i~ (2-C) = m _ ^{32}1001 ax ay ~~ ax ay
(ot)
which is the fourth and final geometric feature of the airwater surface needed for our hydrologic optics studies. The specific integral representation of m_i, is:

a . a ~ = tE (k) uv dV(k) ax aY E
2
(93)
We will now free ourselves from the restriction of describing the energy spectrum E in one particular coordinate system over the k-plane. The first result will be a practical rule for expressing the mean square slope of the air-water surface along any direction in terms of the three basic means m_{ot}, m20, m,,. Let the coordinate frame in the be rotated an angle ~ radians as shown in Fig. 12.22. k with coordinates (u,v) will now have coordinates
in the new frame, where:
k-plane A point (u' v')
and reciprocally:
u' =
u cos ~ + v sin
(94)

v' = -
u sin g + v cos

u = u' cos - v' sin

(95)

v = u' sin + v' cos

FIG. 12.22 Diagram for constructing the equations for a rotation transformation of angle ~ about the origin.
Now, the mean square slope of the air-water surface in any direction ~ may be obtained by passing a vertical plane through the u' axis of a coordinate frame rotated ~ radians and then computing m_{2e} in the new frame. To emphasize the relative orientation of this frame, we shall denote m_{9_o} in the rotated frame by "m_2 M t".

The "2" denotes the mean square aspect of the moment. We compute m_2(~) as follows: Strictly by the definitions, and (85), if E' is the energy spectrum with respect to variables measured in the rotated frame, then:
m_2 M = f E' (k) $(u')^2$ dv (k') E
E' (U': v') $(u')^2$ du' dv' (96)
JE 2

where we have used the representation k -l = (u' ,v') , and the fact that the element of area in the new system is du' dvt in contrast to du dv (= dV (k)) in the old system..

To perform this conversion in a systematic way, observe that the conversion of variables from one frame to another is governed
by the transformation (94). The change of area elements then requires the Jacobian of (94) which in this case is:

au' au'

au av cos

a (u' v' }

au,v) av' av'

- sin

'au av

E (k

Therefore, the mean square slope m_2 (~) along an arbitrary .direction 0 in some frame-of reference i s calculable i f in that frame the three numbers m2o, m_{o^*}, and mil are known. It should be noted that this connection is quite general; however to effect it, the representation ~ of the surface is to have integrable second derivatives.

As a final topic in the discussion of various geometric applications of the directional energy spectrum, we consider certain types of experimentally determined energy spectra which are concerned only with the frequencies (or period) of the water waves and not with their directional properties. This suggests that we establish a polar coordinate system in the k-plane, as depicted in Fig. 12.23. The transformation from the u v coordinates to the k,~ coordinates is

wow

u

FIG. 12.23 Changing from cartesian to polar spectrum coordinates.

Let us write

"S(kfor E(u(k, ~ v(ks ~J a u v $_a$ (100) The Jacobian for (98) is:

au

- k cos ^{2} ~ + k s in^2 0 = k

Therefore we have the connection:

S (k) = S'(k , 0) = E (u . v) k = E (k) k (101)

when (u,v) and (k.0) are but two representations of one and the same point k in the k-plane, with their coordinates related by (98). The function S is the polar form of the directional energy spectrum E.

Some words about the dimensions of the energy spectra E and S are in order, as it will become increasingly important to keep track of the dimensions of the various derivates of E and S below, preparatory to the study of experimental data in the following section. The basic point of departure in the dimensional analysis can be (74). A dimensional analysis of the autocorrelation function ~ shows that it has dimension L^2. The measure V in (76) in the two dimensional case is an "area" measure in the k-plane. The dimension of k is L^{-1} so that the dimension of V in this case. is L' 2. Since the cosine function is dimensionless, it follows from (76)that the dimension X of E is the solution of the equation

$L^2 = X L^{2}$

that is, dim (E (k)) = L^4 . An alternate analysis may be based on (71) and (75) by noting via (73) that dim [A (k)] = L, and dim (O.(k)) = L^3. Then since dim (6'(k-k')) = 1, the result follows from (75). According to (101), the dimension of S (k) is then L4 L- 1= W

Returning to the main line of discussion we next write:

"T k1

r2Tr

for j S[k, ~ d~ . (l02) a

Tk is the spatial frequency spectrum associated with the wave numbers k. From this definition and (101) we see that dim(Tk) = L^3. Furthermore,

0

which may be established by observing that:

$_4$2

It is also occasionally convenient to have available a frequency spectrum associated with the temporal frequency a occurring in the theories of this -and the preceding section. To lay the ground work for the introduction of the temporal frequency, we must decide on a workable connection between a and k. From (64) of Sec. 12.3 we have, on the basis of the linear theory of hydrodynamics:

0

= I f tanh (kh)

(104)

which in infinitely deep media (h = oo) becomes:

Q2 =.gk

(105)

When .surface tension effects are to be taken into account spectrum studies, we could use (90) of Sec. 12.3:

in

where we have written

1f QT 1

for

g (Pw " Pa) ^{P}a + Pw

7

and

for

T

Pa + Pw

We shall work, for illustrative purposes, with (105); hence forth, 'unless specifically stated otherwise, u and k are to he functionally related by (105). In many practical computations, we may set p_w = 1, pa = 0 ; s o that = g, Y = T, (cgs system).

The definition of the directional spectrum based on temporal frequency a and direction ~ is now obtained by writing

a

It F(atfor E(u(a,0 ,v(6 u v $_{a'v}$ (107)
From (98) and (105) we have the transformation from the' uv coordinate frame to the ao coordinate frame in the k--plane
u(^{Cj}~ .90
$_a$2
cos . _ k cos
g2

v(a,.~) = v = 6 g sin 0 = k sin 0
Hence

Du au

as a~
au av

ad a0
From this and [107; we have
26k $_k$3 ,f2
F (k) = F(a, ~)= E (u,v) g = ~E (k) 2 g . (108)

Furthermore from (101) and (108) we have:
S k) = E (u,v)_ -g - F (k)
k 2ak
s o that
26 S (k , ~) = F (a s ~) (109) g
From this relation¯the dimension of F (k) is readily found:
dim (F (k) }= dim, a) • dim S k) dim (g)
L2 T
L $T^{r\ 2}$
Corresponding to Tk, in (102) , we can write:

Z ?T
"To~f for f F(os) +~ (110) 0
for the (temporal) frequency spectrum and observe that
r
From (110.) we have dim-(To) = L^2 T.
oceanographers often find it convenient to work with temporal .periods ¯r rather than. temporal frequencies a. Since:
o = -2n /T (112)
we can write r
"T T^{11} for To da - (113)
where the derivative day/dT is now a one-dimensional version of the Jacob ian. T..$_r$
is the (temporal) period spectrum.

T T
so that dim (TT) = dim (a^2) • dim (T_O) dim (T..) -- L T^{-1}.
T7-2 * L^2 T; that is
directly from (111) and (113):
f 00 0
T_T dT =
f 00 0
The importance of the various frequency and period spectra, S (k,~) , Tk , T_o, and T_T for hydrologic optics rests in the means (!) they make available for converting data on wave spectra of the surfaces of natural hydrosols into useful information about the mean square elevations and slopes o f those surfaces. As an illustration, consider S (k, ~). and Tk. From (89) and (103), we have

^{m}oo
f 00 0
T_k dk
Furthermore from (85) and (90)
m_{20} = fE E (k) u^2 dV(k) = fE E(u,v) u^2 du dv

fE
2
S(k,~) u^2 (k,~) d~ dk
Therefore by (98)
m
20
f 00 0
27r
S (k,) k^2 $\cos^t$ d~ dk
f_o

In a similar manner we find:

f0D f 27 S (k, ~) k^2 sin ~ cos d~ dk 0 0
As a simple instructive exercise, the reader should now find,
in a similar manner, the preceding four moments in terms of F (Q,$) . (See, e.g., (42)-(46) of Sec. 12.9.)
A considerable simplification in computations of the moments moo, ^{M}92) M20 can be. effected whenever the directional spectrum S is isotropic, i.e.,
S (k, ~) W IT T_k (120) ~r
for every ~, 0 S ~ < 2-ff. Under the isotropy condition on S, (117)-(119) Become:

m 20
Go
(124)

for every ~, 0 < ~ ~ 2w, under the isotropy condition (120).
In general, isotropy or not, we have, from (117) and (118):

k^2 T_k dk (125)

In this way we see that from knowledge of S (k, ~), and Tk the four important moments m_{oot} m $_2$, m_2 ,-and m~~ are determinable. From these moments ana. the 33tatistlcal distributions governing ~, a ~/ ax, and a ~/ay, which we, shall study in Sec. 12.9, we can deduce in turn the basic functional relations governing radiative transfer phenomena at the air-water surface (Sec. 12.12). In this connection, (121)-(125) and their following variants will be found helpful (Sec. 12.8)

First, we have quite generally;

T_o

_ 2a T g k

(126)

which follows from (109) and (110). This, together with (114), provides connections among the three versions of the one-dimensional spectrum. Furthermore, under the isotropy condition:

(127)

we obtain from (121) or (122) ;

m = m

2o 02 $_{29}$2