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Invariant Imbedding Relation for the Principles Of Invariance*

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1. Introduction - The principle of invariant imbedding, recently stated by Bellman and Kalaba,[1] is a rule of action which may be followed in the mathematical formulation of any of a wide class of problems concerned with transfer phenomena in general media. The statement of the principle generalizes the methodology originally developed by Ambarzumian[2,3] and extended by Chandrasekhar[4] in their studies of the transfer of radiation through scattering and absorbing media. In this note we exhibit an explicit analytic embodiment of the principle for radiative transfer and neutron transport processes. This symbolic statement of the principle - which we shall call the invariant imbedding relation - yields in particular the general symmetric forms of the principles of invariance[5] for these processes. The semi-group features which generally are associated with an invariantly imbedded process are also implicit in the invariant imbedding relation.

The present discussion will be limited to the steady state case on a class of inhomogeneous one-parameter carrier spaces[6] (e.g., plane-parallel, cylindrical, spherical, and general one-parameter space filling families of surfaces). Additional features of the invariant imbedding relation such as those

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associated with time dependence, polarization, and more general carrier spaces, will be reserved for subsequent discussions.

2. Invariant Imbedding Relation - Let $\Phi = X \times \Xi$ be a one parameter carrier space in E_3 i.e., $X = \{X_x : x \in [a,b]\}$, where $[a,b]$ is a closed interval in the extended real number system, and Ξ is the unit sphere in E_3. For each $y \in [a,b]$ there is a pair $(N_+(y), N_-(y))$ o (real or vector valued) functions on $\Phi_y = X_y \times \Xi$. Let $\mathcal{N}$ denote the collection of all ordered pairs of functions $(N_+(z), N_-(x))$, $[x,z] \subset [a,$ with subcollections $\mathcal{N}_+$ and $\mathcal{N}_-$ defined as $\{N_+(z);\ z \in [a,b]\}$ and $\{N_-(x) : x \in [a,b]\}$, respectively. Then for each $y \in [x,z] \subset [a$ there is a map $\mathcal{M}(x,y,z)$ of $\mathcal{N}$ into $\mathcal{N}$ such that

$$(N_+(y), N_-(y)) = (N_+(z), N_-(x))\, \mathcal{M}(x,y,z) \tag{1}$$

where

$$\mathcal{M}(x,y,z) = \begin{pmatrix} \mathcal{T}(z,y,x) & \mathcal{R}(z,y,x) \\ \mathcal{R}(x,y,z) & \mathcal{T}(x,y,z) \end{pmatrix},$$

in which $\mathcal{R}(x,y,z)$ is the complete reflectance operator with domains $\mathcal{N}_+$ $\mathcal{N}_-$ and ranges $\mathcal{N}_-$, $\mathcal{N}_+$ respectively; $\mathcal{T}(x,y,z)$ the complete transmittance operator with domains $\mathcal{N}_+$, $\mathcal{N}_-$ and ranges $\mathcal{N}_+$, $\mathcal{N}_-$ respectively. In addition, $\mathcal{T}(x,z,z) = T(x,z)$ is t standard transmittance operator and $\mathcal{T}(x,x,z) = I$, the identity operato $\mathcal{R}(x,x,z) = R(x,z)$ is the standard relectance operator and $\mathcal{R}(x,z,z$ the zero operator.

Expression (1) is the invariant imbedding relation for the radiative transfer and neutron transport contexts. The pair of functions: $N_y = (N_+(y), N_-(y))$ is the abstract version of the classical intensity (or radiance) function on surface X_y, y units of depth "below" the upper boundary of the medium. $N_+(y)$ is a function which physically describes the "outward" directed flux ... $N_-(y)$ describes the "inward" directed flux at each point of X_y, (Figure 1). $N(y)$ need not be constant over X_y.

Starting with the S and T functions of Chandrasekhar, one may obtain concrete examples of the various operators and of the relation (1) in the classical case of a finitely deep plane-parallel medium.

Figure 1.

3. Principle of Invariant Imbedding - In the present context, the invariant process mentioned in the general statement of the principle is represented by the collection $\{ \mathfrak{M}(x,y,z) : y \in [x,z] \subset [a,b] \}$ of transformations and the state vectors are the elements of the collection $\mathfrak{N}$.

4. Semi-Group Properties of the Imbedding Relation - The semi-group relations are obtained by setting $N_+(b) = O$ (the zero function on $X_b \times \Xi_+$). By (1), for all $y \in [a,b]$,

$$(N_+(y), N_-(y)) = (N_+(b), N_-(a))\,\mathfrak{M}(a,y,b) = (N_-(a)\,\mathcal{R}(a,y,b),\ N_-(a)\,\mathcal{T}(a,y,b)),$$

and for $b \in [y,b]$,

$$(N_+(b),\ N_-(b)) = (N_+(b), N_-(y))\,\mathfrak{M}(y,b,b) = (O,\ N_-(y)\,\mathcal{T}(y,b,b)),$$

whence

$$N_-(b) = N_-(a)\,\mathcal{T}(a,b,b) = N_-(a)\,\mathcal{T}(a,y,b)\,\mathcal{T}(y,b,b),$$

that

$$\mathcal{T}(a,b,b) = \mathcal{T}(a,y,b)\,\mathcal{T}(y,b,b) \quad , \quad y \in [a,b] \qquad (2)$$

a similar way it may be shown that

$$\mathcal{R}(a,z,b) = \mathcal{T}(a,y,b)\,\mathcal{R}(y,z,b), \quad y \in [a,z] \qquad (3)$$

.he complementary set of semi-group relations are obtained by setting $N_-(a)=0$.

:. <u>Principles of Invariance</u> - The four general principles of invariance associated .ith X are obtained by first writing out in full the matricial relation (1) and then suitably choosing the spread of the interval $[x, z]$ and the location of the parameter y in $[x,z]$. Thus, from (1):

I. Let $x=y$, use component $N_+(y)$:

$$N_+(y) = N_+(z)\,T(z,y) + N_-(y)\,R(y,z).$$

II. Let $z=y$, use component $N_-(y)$:

$$N_-(y) = N_-(x)\,T(x,y) + N_+(y)\,R(y,x).$$

III. Use I twice: Let $y=a$, $z=b$; then $y=a$, z arbitrary:

$$N_+(a) = N_+(b)\cdot T(b,a) + N_-(a)\,R(a,b) = N_+(z)\,T(z,a) + N_-(a)\,R(a,z).$$

IV. Use II twice: Let $y=b$, $x=a$; then $y=b$, x arbitrary:

$$N_-(b) = N_-(a)\,T(a,b) + N_+(b)\,R(b,a) = N_-(x)\,T(x,b) + N_+(b)\,R(b,x).$$

Three observations may be made here: (i) the four principles of invariance stated above are in <u>undecomposed</u> operator form. The advantage gained by retaining this form is the resultant general symmetry of the statements. (ii) Principles III and IV are special cases of I and II respectively, and the latter in turn are obtained by the appropriate permutation of $+$, $-$, and x, z, in a single basic statement derived from (1). (iii) The invariant imbedding relation and the principles of invariance refer (as do their classical counterparts) to an arbitrary source-free subset of X defined by the interval $[x, z]$. This is in keeping with the original purpose [2-5] of defining and characterizing the inherent reflecting and transmitting properties of source-free subsets of a space. Thus the problem of formulating the general principles of invariance is, by definition, independent of the boundary value problem associated with internally distributed sources. It should be observed, however, if sources are present in X, the relation (1) and its corollaries I-IV continue to hold for all those subsets (defined by $[x, z]$) which are free of sources.

The classical forms of the principles of invariance are obtained by: (i) decomposing the functions $N(y)$, $y \in [a, b]$ into the sum of their reduced and diffuse components: $N(y) = N^{\circ}(y) + N^{*}(y)$; (ii) imposing the boundary conditions: $N_{-}(a) = N^{\circ}(a)$ with Dirac-delta structure over Ξ_{-}, and $N_{+}(b) = 0$; (iii) assuming X to be a plane-parallel homogeneous slab so that $R(x,y) = R(y,x) = R(|x-y|)$, and $T(x,y) = T(y,x) = T(|x-y|)$.

The decomposition of $N(y)$ carries through to the general case, and the classical boundary conditions may be imposed on $N_{-}(a)$ and $N_{+}(b)$. However, the general inhomogeneity and curvilinear structure of X first of all give rise to the polarity of the R and T operators: $R(x,y) \neq R(y,x)$, $T(x,y) \neq T(y,x)$; and secondly, remove some of the original invariant features of the R and T operators such as those associated with the physical operations of adding or subtracting finite layers of the media. The general functional relations satisfied by the R and T operators will be considered at a later time.

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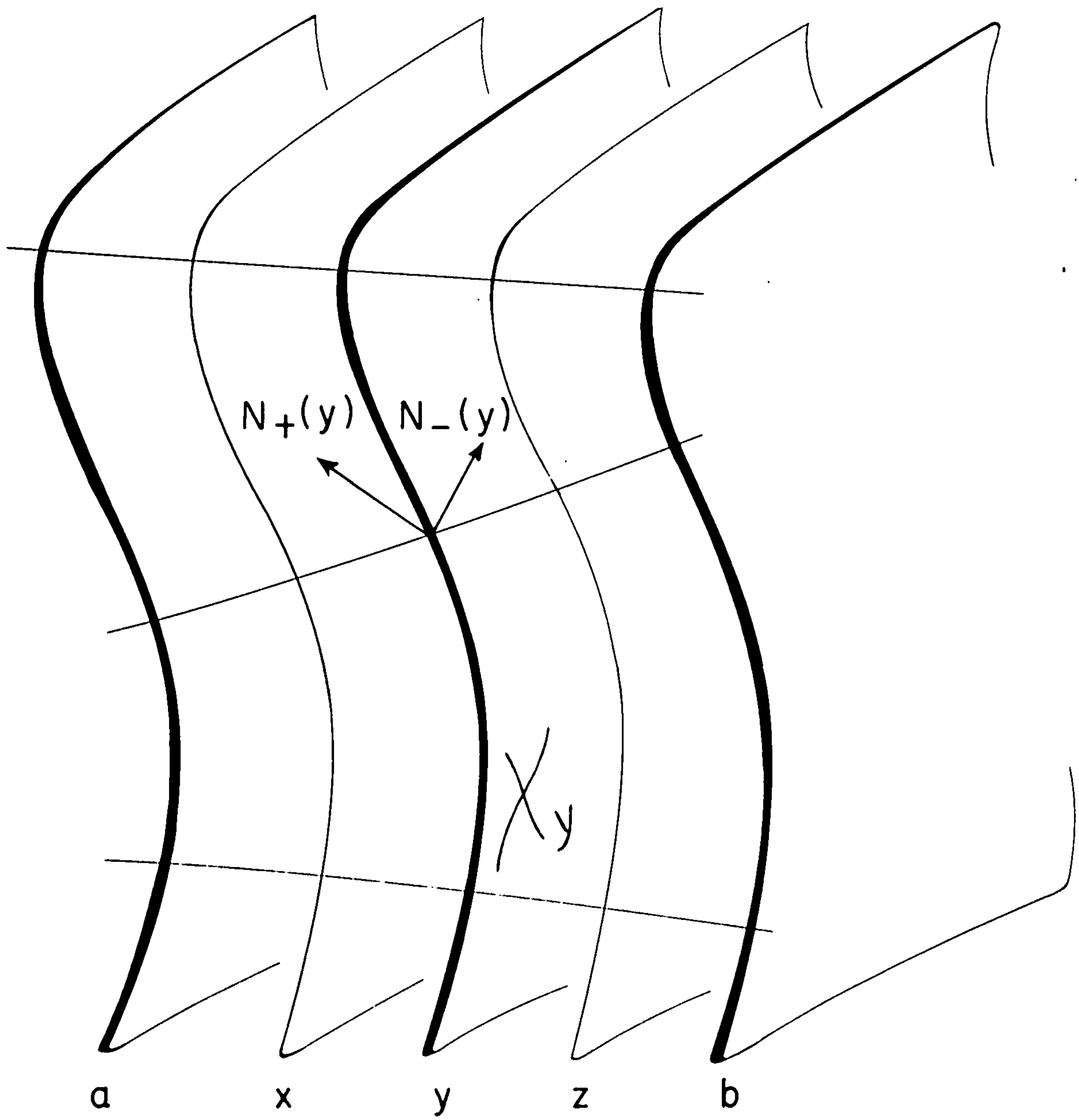

Figure 1
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