Visibility Laboratory
University of California
Scripps Institution of Oceanography
San Diego 52, California

TIME-DEPENDENT PRINCIPLES OF INVARIANCE

Rudolph W. Preisendorfer

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Approved:

Seibert Q. Duntley

Seibert Q. Duntley, Director
Visibility Laboratory

Approved for Distribution:

Roger Revelle

Roger Revelle, Director
Scripps Institution of Oceanography

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Time-Dependent Principles of Invariance*

by Rudolph W. Preisendorfer

University of California, La Jolla, California

1. <u>Introduction</u> - Radiative transfer phenomena, whether in the transient or stationary states, may be classified according to the relative importan of the roles played by the three general processes of conduction, convect and radiation. <u>Strict radiative transfer</u> refers to the transfer by radiat processes alone. In 1926 work on time-dependent strict radiative transfer processes had been carried out by Milne and later extended by Chandrasekhar In these studies the time-dependent radiation field is considered to be governed principally by the various decay rates of excited atoms. This is in contrast to the time dependent process initiated and sustained by the mechanism of simple scattering and the speed of propagation of radiant energy through the medium (the multiple scattering process). In another early study, simultaneous consideration of time-dependent convection and radiation processes was made by Rosseland[2]. Aside from these studies the time dependent case of strict radiative transfer theory has been relatively unexplored. This is particularly true for the time-dependent multiple scattering problem.

Most of the tools presently available for the study of time-dependent multiple scattering phenomena are imports from neutron transport theory. In this field the approach to the transient case has been successf

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effected by means of the linearized Boltzmann transport equation. — the exact counterpart to the equation of transfer in radiative transfer theory. The approach may be analyzed into two alternate procedures: the direct and indirect. The direct procedure, as exemplified in the papers of Lehner and Wing,[3] is based on the transport equation, in which the time-dependence is explicitly retained throughout. In the indirect procedure, the time-dependent problem is immediately shunted to a stationary problem by means of the Laplace transform, is solved in that context, and then is switched back to a non-stationary problem by an inverse Laplace transform. Examples may be found in the works of Grosjean,[4] Case, de Hoffman and Placzek,[5] and Davison.[6] We note that this procedure is possible only if the attenuation and scattering functions are time-independent. The mathematical questions of the existence of solutions sought in the time-dependent case by these methods can be settled for example by applications of the results obtained by Kurth.[7]

There remains, however, to explore a potentially useful approach to the time-dependent multiple scattering processes — an approach based on the principle of invariance formulations which have been instrumental in the exact solution of a large class of stationary multiple scattering problems in radiative transfer theory. In this note, we exhibit the formulation of the requisite statements of the four time-dependent principles of invariance and show that they are sufficient to derive the functional relations governing the time-dependent R and T functions and their corresponding integral operators. The derivations follow closely the pattern used in some earlier studies of the stationary case.[8] In this way we add to the evidence that the classical

procedures initiated by Chandrasekhar can be extended to more general setting
For brevity the present discussion is limited to plane-parallel media. Howe
the media will be assumed finite, non-separable, and such that the attenuati
and scattering functions are generally dependent on time.

2. <u>Local Forms of the Time Dependent Principles of Invariance</u>. - With the exception of the explicit incorporation of time parameters, the geometrical setting and the notation used in the present investigation is the same as that used in the study of the steady-state case.[8] In particular the time-dependent equation of transfer will be written

$$\mu \frac{\partial N(y,t,\mu,\phi)}{\partial y} + \frac{1}{v} \frac{\partial N(y,t,\mu,\phi)}{\partial t} = -\alpha(y,t) N(y,t,\mu,\phi) + N_*(y,t,\mu,\phi), \quad (1)$$

where

$$N_*(y,t,\mu,\phi) = \int_{\Xi} \sigma(y,t;\mu,\phi;\mu',\phi') N(y,t,\mu',\phi') \, d\mu' d\phi'.$$

As before we use the notions of outward and inward radiance distributions:

$$N_+(y,t) \;, \quad N_-(y,t).$$

The local reflectance and transmittance operators are defined as

$$\rho(y,t) = \frac{1}{\mu} \int_{\Xi_+} \sigma_-(y,t;\mu,\phi;\mu',\phi') [\;\;] \, d\mu' d\phi'$$

$$\tau(y,t) = \frac{1}{\mu} \int_{\Xi_+} \sigma_+(y,t;\mu,\phi;\mu',\phi') [\;\;] \, d\mu' d\phi' - \frac{1}{\mu} \alpha(y,t) ,$$

so that the local forms of the time-dependent principles of invariance may written:

$$\frac{D_- N_+(y,t)}{Dy} = -\frac{\partial N_+(y,t)}{\partial y} + \frac{1}{\mu v} \frac{\partial N_+(y,t)}{\partial t} = N_+(y,t)\tau(y,t) + N_-(y,t)\rho(y,t) \quad (2$$

$$\frac{D_+ N_-(y,t)}{Dy} = \frac{\partial N_-(y,t)}{\partial y} + \frac{1}{\mu v} \frac{\partial N_-(y,t)}{\partial t} = N_-(y,t)\tau(y,t) + N_+(y,t)\rho(y,t)$$

3. <u>Time-Dependent R and T Operators, and Principles of Invariance</u>. - For every pair (x,z) of geometric depths and ordered pair (t',t) of times, t' $\leq$ t, we define two non negative real valued functions $R(x,z,t',t; \;\; ;$

$T(x,z;t',t;\ ;)$, on $\Xi_+ \times \Xi_+$ and their associated integral operat

$$R(x,z,t',t) = \frac{1}{\mu}\int_{\Xi_+} R(x,z,t',t;\mu,\phi;\mu',\phi')[\ ]\,d\mu' d\phi',$$

$$T^*(x,z,t',t) = \frac{1}{\mu}\int_{\Xi_+} T(x,z,t',t;\mu,\phi;\mu',\phi')[\ ]\,d\mu' d\phi'.$$

These are clearly analogous to the operators $R(x,z)$ and $T^*(x,$
considered in the steady-state case. In addition we define

$$T^0(x,z,t',t) = \int_{\Xi_+} T_r(x)\,\delta(\mu-\mu')\delta(\phi-\phi')\,\delta(t-\frac{r}{v}-t')[\ ]\,d\mu'$$

where $T_r(x)$, $r = |z-x|/\mu'$, is defined as in the stationary case.
Set

$$T(x,z,t',t) = T^0(x,z,t',t) + T^*(x,z,t',t).$$

In addition to the usual analytic conditions imposed on the R an T functions in the stationary context, we require at present three additional properties: (i) If $N_-(x,\cdot)$ is arbitrary and $N_+(z,\cdot) =$
then

$$N_+(x,t) = \int_{-\infty}^{t} N_-(x,t')R(x,z,t',t)\,dt' = N_-(x,\bar{t'})R(x,z,\bar{t'},t$$

$$N_-(z,t) = \int_{-\infty}^{t-\frac{|z-x|}{v}} N_-(x,t')T(x,z,t',t)\,dt' = N_-(x,\bar{t}')T(x,z,\bar{t'},$$

The right-hand equalities define a convenient integration convention.
(ii) In particular, if $N_-(x,\cdot)$ is constant over $(-\infty,t)$ t finite, and α and σ are independent of t, then

$$N_+(x) = N_-(x,\bar{t'})\ R(x,z,\bar{t}',t) = N_-(x)\ R(x,z),$$

$$N_-(z) = N_-(x,\bar{t'})T(x,z,\bar{t'},t) = N_-(x)\ T(x,z),$$

where R(x,z) and T(x,z) are the steady-state operators considered earlier.
(iii) Finally,

$$R(x, z, t, t) = 0 \qquad \text{for all } t,$$

and

$$T(x, z, t', t) = 0 \qquad \text{for} \qquad t - t' \leq |z - x|/v$$

These adopted conditions are sufficient to allow the formulation of both the time-dependent principles of invariance and the necessary functional relations governing the time-dependent R and T operators which reduce, under steady-state conditions, to their respective steady-state counterparts.

The two main time-dependent principles of invariance may then be written:

$$\begin{aligned} &\text{I} \quad N_+(y,t) = N_+(z,\overline{t'})\, T(z,y,\overline{t'},t) + N_-(y,\overline{t'})\, R(y,z,\overline{t'},t) \\ &\text{II} \quad N_-(y,t) = N_-(x,\overline{t'})\, T(x,y,\overline{t'},t) + N_+(y,\overline{t'})\, R(y,x,\overline{t'},t), \\ &\qquad a \leq x \leq y \leq z \leq b. \end{aligned} \tag{3}$$

The structural similarity of the sets (2) and (3) can be noted. The set (3) yields, after appropriate choices of the spread of [x,z] and the location of y within it, the operator forms of the four principles of invariance for the present context. The general resultant forms, being structured similarly to the steady-state set considered earlier, need not be given here. Instead, for brevity, we assume that the usual boundary conditions are already in force namely: $N_-(a,\cdot)$ is arbitrary, and $N_+(b,\cdot) = 0$ (the zero function). The appropriate principles then follow directly from (3):

$$\text{I} \quad N_+(y,t) = N_-(y,\overline{t'})\, R(y,b,\overline{t'},t)$$

$$\text{II} \quad N_-(y,t) = N_-(a,\overline{t'})\, T(a,y,\overline{t'},t) + N_+(y,\overline{t'})\, R(y,a,\overline{t'},t)$$

$$\text{III} \quad N_+(a,t) = N_-(a,\overline{t'})\, R(a,b,\overline{t'},t) = N_+(y,\overline{t'})\, T(y,a,\overline{t'},t) + N_-(a,\overline{t'})\, R($$

$$\text{IV} \quad N_-(b,t) = N_-(a,\overline{t'})\, T(a,b,\overline{t'},t) = N_-(y,\overline{t'})\, T(y,b,\overline{t'},t)$$

4. Functional Relations for the Time-Dependent R and T Operators – In what follows, let $N_-(a,\cdot)$ be of arbitrary angular structure, but of Dirac-delta time dependence: $N_-(a,t) = N_-(a,t_0)\,\delta(t-t_0)$. This procedure results in relatively simple relations for the R and T operators (cf. with the adoption of a Dirac-delta angular structure for $N_-(a)$ in the steady-state formulation which results in the relatively simpler relations for the R and T functions). The subsequent operations involving the Dirac delta-functions, their products, and derivatives are governed by the usual conventions.[9]

We begin by applying the operator D_-/Dy to I:

$$\frac{D_- N_+(y,t)}{Dy} = \frac{D_- N_-(y,\overline{t'})}{Dy} R(y,b,\overline{t'},t) + N_-(y,\overline{t'}) \frac{D_- R(y,b,\overline{t'},t)}{Dy} . \qquad (4)$$

To this we apply in turn the operation $\lim_{y \to a}$. The left side of (4) yields, by means of (2),

$$N_+(a,t)\,\tau(a,t) + N_-(a,t)\,\rho(a,t) = N_-(a,t_0)\left[R(a,b,t_0,t)\,\tau(a,t) + \delta(t-t_0)\,\rho(a\right.$$

the alternate expression comes from the left-hand side of III and the adopted form of $N_-(a,\cdot)$. Further, by (2),

$$\frac{D_- N_-(y,t')}{Dy} = -\frac{\partial N_-(y,t')}{\partial y} = \frac{1}{\mu v}\frac{\partial N_-(y,t')}{\partial t'} - N_-(y,t')\tau(y,t') - N_+(y,t')\rho$$

so that

$$\lim_{y\to a}\frac{D_- N_-(y,t')}{Dy} = N_-(a,t_0)\left[\frac{1}{\mu v}\delta'(t'-t_0) - \delta(t'-t_0)\tau(a,t') - R(a,b,t_0,\right.$$

where $\delta'(t'-t_0)$ is the symbolic derivative of the Dirac-delta function with respect to t'. Finally,

$$\lim_{y\to a}\frac{D_- R(y,b,t',t)}{Dy} = -\frac{\partial R(a,b,t',t)}{\partial a} + \frac{1}{\mu v}\frac{\partial R(a,b,t',t)}{\partial t}\ .$$

Using these results and the fact that $N_-(a,\cdot)$ is of arbitrary angular structure, we have:

$$\text{I}' \quad -\frac{\partial R(a,b,t_0,t)}{\partial a} + \frac{1}{\mu v}\left[-\frac{\partial R(a,b,t_0,t)}{\partial t_0} + \frac{\partial R(a,b,t_0,t)}{\partial t}\right]$$

$$= \delta(t-t_0)\rho(a,t) + \tau(a,t_0)R(a,b,t_0,t) + R(a,b,t_0,t)\tau(a,t)$$

$$+ R(a,b,t_0,\overline{t'})\,\rho(a,\overline{t'})\,R(a,b,\overline{t'},t)\ .$$

Now starting with II, applying in turn the operations D_+/Dy and $\lim_{y\to b}$, and then making use of (2) and the left side of IV we have in a similar way:

$$\text{II}' \quad \frac{\partial T(a,b,t_0,t)}{\partial b} + \frac{1}{\mu v}\frac{\partial T(a,b,t_0,t)}{\partial t} =$$

$$= T(a,b,t_0,t)\tau(b,t) + T(a,b,t_0,\overline{t'})\,\rho(b,\overline{t'})\,R(a,b,\overline{t'},t)$$

The third relation follows from the right-hand equality of III by applying the procedure used for I'. The result is:

$$\text{III}' \quad \frac{\partial R(a,b,t_0,t)}{\partial b} = T(a,b,t_0,\overline{t'})\,\rho(b,\overline{t'})\,T(a,b,\overline{t'},t).$$

The fourth relation follows from the right-hand equality of IV by applying the procedure used for II'. The result is:

$$\text{IV}' \quad -\frac{\partial T(a,b,t_0,t)}{\partial a} - \frac{1}{\mu v}\frac{\partial T(a,b,t_0,t)}{\partial t_0} =$$

$$= \tau(a,t_0)\,T(a,b,t_0,t) + R(a,b,t_0,\overline{t'})\,\rho(a,\overline{t'})\,T(a,b,\overline{t'},t).$$

We conclude by observing that, (i) the time-dependent R and T operators will be homogeneous with respect to time (i.e., will depend on time differences t-t') if and only if the attenuation and scattering functi are independent of t. (ii) Furthermore, for non-separable spaces, the time-dependent R and T operators exhibit polarity: $R(a,b,t',t) \neq R(b,a,t',t)$, $T(a,b,t',t) \neq$ (iii) The solution procedures for the time-dependent relations are, in principle, the same as those followed in the stationary case.

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