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Determination of the Non-zero Asymptote of an Exponential Decay Function*

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It frequently becomes necessary in physical investigations to deal with exponential decay functional forms where the asymptote is displaced from zero. The method of least squares as normally applied cannot be used, and other methods are complicated by the presence of the non-zero asymptote. However, if the data is based on equal progressive increments of the independent variable, the asymptote may be readily found by the following method. The method may also be applied to data involving unequal increments provided that the data can be reduced to effectively equal incremental form. Given a set of data points and knowing or assuming that they should fall on a curve of the type: $y = ae^{bx} + c$

Where b is less than 0

Divide data into three groups, each of which centers about x_1, x_2, x_3

so that $x_2 - x_1 = x_3 - x_2$ and $2x_2 = x_1 + x_3$

Then take the geometric mean of each group and consider this mean the

$y_1(x_1), y_2(x_2), y_3(x_3)$ respectively.

$y_i = ae^{bx_i}$ or $e^{bx_i} = \frac{y_i - c}{a}$

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$$\left(\frac{y_2-c}{a}\right)^{\frac{x_1}{x_2}} = \frac{y_1-c}{a} \qquad \text{or} \qquad a^{\frac{x_2-x_1}{x_2}} = \frac{y_1-c}{(y_2-c)^{\frac{x_1}{x_2}}}$$

$$\left(\frac{y_2-c}{a}\right)^{\frac{x_3}{x_2}} = \frac{y_3-c}{a} \qquad \text{or} \qquad a^{\frac{x_2-x_3}{x_2}} = \frac{y_3-c}{(y_2-c)^{\frac{x_3}{x_2}}}$$

Since $x_2 - x_1 = x_3 - x_2$: $a^{\frac{x_2-x_1}{2}} = a^{\frac{x_3-x_2}{x_2}} = \dfrac{1}{a^{\frac{x_2-x_3}{x_2}}}$

and $\dfrac{(y_2-c)^{\frac{x_3}{x_2}}}{y_3-c} = \dfrac{y_1-c}{(y_2-c)^{\frac{x_1}{x_2}}}$; or $(y_2-c)^{\frac{x_3+x_1}{x_2}} = (y_1-c)(y_3-c)$

since $x_3 + x_1 = 2x_2$: $(y_2-c)^2 = (y_1-c)(y_3-c)$

$$y_2^2 - 2cy_2 + c^2 = y_1y_3 - cy_1 - cy_2 + c^2$$

$$c(y_1+y_3-2y_2) = y_1y_3 - y_2^2$$

$$\boxed{c = \frac{y_1y_3 - y_2^2}{y_1+y_3-2y_2}}$$

EXAMPLE: In this case a series of attenuation constants varying with depth has been determined. It appeared that the constants diminished exponentially with depth. Applying the method to the following data:

j	x	k
1	0	0.505
2	40	0.463
3	80	0.455
4	120	0.444
5	160	0.440
6	200	0.437

$$k_1 = (0.505 \times 0.463)^{1/2} = 0.483{,}544$$
$$k_2 = (0.455 \times 0.444)^{1/2} = 0.449{,}466$$
$$k_3 = (0.440 \times 0.437)^{1/2} = 0.438{,}497$$

$$k_\infty = \frac{k_1 k_3 - k_2^2}{k_1 + k_3 - 2k_2} =$$

$$\frac{0.483{,}544 \times 0.438{,}497 - 0.449{,}466^2}{0.483{,}544 + 0.438{,}497 - 2 \times 0.449{,}466} =$$

$$\frac{0.010{,}013}{0.023{,}109} =$$

$$0.433$$

Notes:

1. This method may be used wherever three or more points are known, satisfying the condition that the resulting $x_2 - x_1 = x_3 - x_2$

2. Knowing the three meanpoints y_1, y_2 and y_3 and the asymptote, the equation to fit these points may be found by subtracting the asymptote from the values of y_i and applying any of various methods, including the method of least squares.

3. The principal source of error is the subtractive process in the numerator and particularly in the denominator of the formula. Hence it is advisable to carry as many figures in the geometric means as is practicable.