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THE PLANETARY HYDROSPHERE PROBLEM I. GENERAL PRINCIPLES

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The Planetary Hydrosphere Problem. I. General Principles.*

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ABSTRACT

The problem of the penetration of natural light into the sea is analyzed by means of the general principles of radiative transfer theory into a set of boundary problems for the slab geometry. The formulations are intended to be solved by an automatic computor.

1. INTRODUCTION

The problem of the penetration of natural light into the sea is of fundamental interest to geophysicists, marine biologists, naval scientists and personnel, and mathematicians alike. Of course each group's interest centers on widely different consequences of the quantity and quality of the light reaching a given level below the surface of a natural body of water. To the geophysicist, knowledge of the character of the light yields detailed information about the hydrosol as regards its light-scattering and light-absorbing properties. Marine biologists attempt to draw connections among the abundance of food, the abundance of various forms of marine animals, and the levels of light at various depths in the ocean. Naval scientists and personnel are concerned with the detection and classification of submarine objects, a task vitally dependent upon knowledge of the submarine light field. Mathematicians (but far too few) have wrestled with the challenging mathematical problems associated with the precise phenomenological description of the complex character of the light field in a medium that scatters and absorbs light.

The problem of the penetration of natural light into the sea is but one member of a large set of closely related problems. Some of these related problems are: the penetration of natural light into the atmosphere, the passage of light through the incandescent layers of a star, the translucency of pigments in paints. All these together are but part of the general problem of the transfer of radiant energy through media that emit, absorb, and scatter radiant energy, the study of which is the domain of radiative transfer theory.

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The purpose of this paper is to define and formulate, by means of the general principles of radiative transfer theory, and in hitherto unattained detail and completeness, the mathematics of the problem of the penetration of natural light into the sea. The formulation particularly emphasizes the role of the reflecting boundaries: the air-water boundary at the surface of the sea, and the presence of a reflecting bottom for the case of optically finitely deep regions of the sea. The ultimate goal of the present studies is the eventual construction of useful mathematical tables describing the state of the natural light field in a carefully chosen collection of simulated physical situations.

As a first step toward this goal the problem is defined and formulated in its general radiative transfer context. In this way an overall mathematical and physical perspective of the problem is attained. The reduction of the solution procedure to a numerical level requires further discussion (notably of the S and T functions). These details will be presented in a subsequent paper.

2. THE GENERAL TWO-BOUNDARY PROBLEM FOR THE SLAB GEOMETRY

(a) The Equation of Transfer for Radiance.

Both the mathematical and empirical description of the transfer of radiant energy in optical media are carried out by means of the notion of radiance (or specific intensity) which describes at each point of a medium the "intensity" of a given wavelength of radiation in a given direction at a given time. Taking the notion of radiant energy as the basic undefined radiometric concept in this discussion, radiance may be more precisely defined as follows. First we define radiant flux as the time rate of change (the flow) of radiant energy, and agree to fix for the entire discussion the wavelength λ of the radiant energy.

Radiance: Let $(\underset{\sim}{x},\underset{\sim}{\xi})$ be a point in an optical medium M, where $\underset{\sim}{x}$ is a location vector, and $\underset{\sim}{\xi}$ is a unit direction vector. Let $A_{\underset{\sim}{\xi}}$ be a unit area at $\underset{\sim}{x}$ normal to $\underset{\sim}{\xi}$. The magnitude of the radiant flux at time t across $A_{\underset{\sim}{\xi}}$ in the direction ξ and confined to a unit solid angle about $\underset{\sim}{\xi}$ is called the radiance of the radiant flux at $(\underset{\sim}{x},\underset{\sim}{\xi},t)$ and is denoted: $N(\underset{\sim}{x},\underset{\sim}{\xi},t)$.

The mathematical description of radiance rests on the following conventions. First, let X denote some well-defined subset of euclidean 3-space E_3 (which has volume measure V). X supplies the geometric setting of the transfer of radiant energy; it is called the location space. Elements of X are called location vectors. An important example of X occurs in the definition of the two-boundary problem below. Further, let Ξ denote the subset of E_3 consisting of all unit vectors ξ. Ξ is the direction space and elements of Ξ are direction vectors. Ξ has a measure Ω (solid angle measure). The cartesian product $X \times \Xi = \Phi$ is the phase space. Usually radiative processes (even in the steady state) are studied over a certain time interval $T = (t_1, t_2)$ (e.g., $(-\infty, +\infty)$, $(0, +\infty)$, $(0, t)$, etc.). Thus $\Phi \times T$ will denote the space-time setting of the radiative transfer process. The length of T will be arbitrary but fixed throughout the following discussion.

Mathematically, radiance is described by means of a non negative real valued function $N(\ ,\ ,\ )$ with domain $\Phi \times T$. At a point (x, ξ, t) in $\Phi \times T$, the value $N(x, \xi, t)$ of $N(\ ,\ ,\ )$ is the radiance at that point. The restriction $N(x,\ , t)$ of $N(\ ,\ ,\ )$ to a point x in X and a time t in T is called the radiance distribution at x at time t.

We observe that for a complete discussion of the mathematical properties of $N(\ ,\ ,\ )$ (and the functions α, σ, a, n considered below) more needs to be said about measurability, integrability, continuity, differentiability, etc. With regard to the main purposes of this paper, however, these analytical features are subordinate and will be assumed to be in force in order to allow all formal operations to be legitimate.

The functions α and σ, the volume attenuation function and the volume scattering function, respectively, summarize the point by point responses of X to radiant flux with regard to the mechanisms of absorption and scattering. The roles the play in the theory is made clear by an inspection of the equation of transfer below. For the present it suffices to define α as a non negative real valued function with domain $X \times T$; and σ as a non negative real valued function with domain $X \times \Xi \times \Xi \times T$. The index of refraction function n plays a relatively small but indispensible role. It is understood here to be a non negative real valued function with domain $X \times T$. Mathematically, the notion of an optical medium is completely described by means of the quintuple $M = (\Phi \times T, N, \alpha, \sigma, n)$, and the radiative transfer process on $\Phi \times T$ is described by the equation of transfer:

$$(n^2(\underset{\sim}{x},t)/v(\underset{\sim}{x},t))\; D(N(\underset{\sim}{x},\underset{\sim}{\xi},t)/n^2(\underset{\sim}{x},t))/Dt \; =$$

$$= -\alpha(\underset{\sim}{x},t)N(\underset{\sim}{x},\underset{\sim}{\xi},t) + \int_{\Xi} \sigma(\underset{\sim}{x},\underset{\sim}{\xi},\underset{\sim}{\xi}',t)N(\underset{\sim}{x},\underset{\sim}{\xi}',t)\, d\Omega(\underset{\sim}{\xi}') + N_\eta(\underset{\sim}{x},\underset{\sim}{\xi},t). \qquad \text{(A)}$$

Here $v(\underset{\sim}{x},t) = c/n(\underset{\sim}{x},t)$, c is the speed of light in vacuum. $N_\eta(\ ,\ ,\ )$ is a non negative real valued function on $\Phi \times T$, the emission function, which accounts for the possibility of sources of radiant energy within X. $N_\eta(\underset{\sim}{x},\underset{\sim}{\xi},t)$ has the dimensions of radiance per unit length in the direction $\underset{\sim}{\xi}$. $D()/Dt$ is the lagrangian derivative operator on $\Phi \times T$.

As it stands above, the equation of transfer yields a description, on a phenomenological level, of the transfer of radiant energy within an optical medium. The physical quantity associated with each symbol is directly observable by means of an appropriately designed instrument.

(b) The Optical Medium for the Slab Geometry.

(i) Using a cartesian coordinate system in the euclidean three-space (see Fig.1) with coordinates $(x,y,z) = \underset{\sim}{x}$, let $X = (\underset{\sim}{x} : 0 \leq z \leq z_1 \leq \infty)$. If $z_1 < \infty$, X is called a finite slab. If $z_1 = \infty$, X is called a semi-infinite slab. $B_0 = (\underset{\sim}{x} : z = 0)$ is the upper boundary of X, $B_1 = (\underset{\sim}{x} : z = z_1)$ is the lower boundary of X. With each point of the boundaries B_0 and B_1 will be associated the non negative real valued reflectance functions r_0 and r_1 (defined in greater detail in (iv)).

Fig. 1

(ii) The steady state condition will hold, i.e., N, α, σ, and n of M will be constant functions on T. The T-component of their domains will henceforth be omitted from the notation.

(iii) Let, ad hoc, $\underset{\sim}{k} = (0,0,-1)$. We define $\Xi_+ = (\underset{\sim}{\xi} : \underset{\sim}{\xi}\cdot\underset{\sim}{k} \geq 0)$ as the totality of outward directions. Further, $\Xi_- = (\underset{\sim}{\xi} : \underset{\sim}{\xi}\cdot\underset{\sim}{k} < 0)$ is the totality of inward directions. Occaisionally it will be necessary to consider the special class $\Xi_0 = (\underset{\sim}{\xi} : \underset{\sim}{\xi}\cdot\underset{\sim}{k} = 0)$ of singular directions. At all points $\underset{\sim}{x}$ of X such that $0 \leq z < z_1$, the functions $N(\underset{\sim}{x},+)$ and $N(\underset{\sim}{x},-)$, which denote restrictions of $N(\underset{\sim}{x},\ )$ to Ξ_+ and Ξ_-, respectively, are the outward and the inward radiance distributions at $\underset{\sim}{x}$. For $\underset{\sim}{x}$ such that $z = z_1$, the outward radiance distributions are the are the restrictions $N(\underset{\sim}{x},+\ )$ of $N(\underset{\sim}{x},\ )$ to $\Xi_+ - \Xi_0$, and the <u>inward radiance distributions</u> are the restrictions $N(\underset{\sim}{x},-\ )$ of $N(\underset{\sim}{x},\ )$ to $\Xi_- + \Xi_0$. This convention arises in an attempt to treat in a symmetric way the radiant energy entering and leaving a medium with a slab geometry. This convention is but one of many which may be used to overcome (but never wholly satisfactorily) the inherent asymmetries of coordinate systems for a slab geometry. To continue, let $\mu = \underset{\sim}{\xi}\cdot\underset{\sim}{k}$, and let φ be the angle that the projection of $\underset{\sim}{\xi}$ on the xy plane makes with the positive x-axis. A direction vector $\underset{\sim}{\xi}$ is therefore completely defined by the pair (μ,φ), $-1 \leq \mu \leq 1$, $0 \leq \varphi < 2\pi$, and conversely.

(iv) With each point of B_0 and B_1 associate a non negative real valued reflectance function r_0 and r_1 on $\Xi_+ \times \Xi_+$ such that

$$(1/4\pi\mu)r_0(\mu,\varphi;\mu',\varphi')N(x,+\mu',\varphi') = N(\underset{\sim}{x},-\mu,\varphi)$$

and

$$(1/4\pi\mu)r_1(\mu,\varphi;\mu',\varphi')N(x,-\mu',\varphi') = N(\underset{\sim}{x},+\mu,\varphi),$$

where $\underset{\sim}{x} = (x,y,0)$ in the first equation, and $\underset{\sim}{x} = (x,y,z_1)$ in the second equation. Thus $(1/4\pi\mu)r_0$ assigns an inward radiance distribution to an outward radiance distribution at each point of B_0, and $(1/4\pi\mu)r_1$ assigns an outward radiance distribution to an inward radiance distribution at each point of B_1. The radiance distribution convention holds here as well as in all discussions in the sequel. This convention has the advantage of obviating the need for endless chains of inequalities in the subsequent formulations. Special examples of r_0 and r_1 will be encountered in section 8. The functions r_0 and r_1 are completely general insofar as they characterize reflectance processes occurring at a surface between two optical media. An idea of the generality of r_0 and r_1 will be gained when section 7 is discussed.

(v) n is a constant function on X.

(vi) X is <u>emission-free</u>, i.e., $N_\eta(\ ,\ ) = 0$ (the zero function) on Φ.

(vii) X is <u>stratified</u>, i.e., α and σ depend only on z of $\underset{\sim}{x} = (x,y,z)$. Formally, $\alpha(\underset{\sim}{x}) = \alpha(\underset{\sim}{x}')$ if $z = z'$, where $\underset{\sim}{x} = (x,y,z)$, $\underset{\sim}{x}' = (x,y,z')$, and $\sigma(\underset{\sim}{x},\underset{\sim}{\xi},\underset{\sim}{\xi}') = \sigma(\underset{\sim}{x}',\underset{\sim}{\xi},\underset{\sim}{\xi}')$ if $z = z'$.

(viii) X is <u>separable</u>, i.e., the function σ/α is a constant function on X. In addition, X is <u>scattering-isotropic</u>, i.e., the value $\sigma(\underset{\sim}{x},\underset{\sim}{\xi},\underset{\sim}{\xi}')$ depends at $\underset{\sim}{x}$ only upon the value $\underset{\sim}{\xi}\cdot\underset{\sim}{\xi}'$. Formally, $\sigma(\underset{\sim}{x},\underset{\sim}{\xi},\underset{\sim}{\xi}') = \sigma(\underset{\sim}{x},\underset{\sim}{\xi}''',\underset{\sim}{\xi}'')$ if $\underset{\sim}{\xi}\cdot\underset{\sim}{\xi}' = \underset{\sim}{\xi}'''\cdot\underset{\sim}{\xi}''$.

The result of these conventions may be summarized notationally as follows: $N(z,\mu,\varphi)$, $\alpha(z)$, $\sigma(z;\mu,\varphi;\mu',\varphi')$, n, which shows on what domains the functions are now defined. z is the geometric depth below B_0.

(c) The Equation of Transfer for the Slab Geometry.

(i) The phase function p(;) on $\Xi \times \Xi$ is defined as $(1/4\pi)p(\; ; \;) = \sigma(\; , \; , \;)/\alpha(\;)$, i.e., at each depth z, and for every pair of directions (μ,φ), (μ',φ') we have $(1/4\pi)p(\mu,\varphi;\mu',\varphi') = \sigma(z;\mu,\varphi;\mu',\varphi')/\alpha(z)$. Define $\bar{\omega}_0 = (1/4\pi)\int_{-1}^{+1}\int_0^{2\pi} p(\mu,\varphi;\mu',\varphi')\, d\mu' d\varphi' = (1/4\pi)\int_{\Xi} p(\mu,\varphi;\mu',\varphi')\, d\mu' d\varphi' = \int_{\Xi} \sigma(z;\mu,\varphi;\mu',\varphi')\, d\mu' d\varphi'/\alpha(z) = s(z)/\alpha(z)$. (From other parts of the theory it is known that $\alpha = s + a$, where a is the non negative real valued volume absorption function with general domain $X \times T$--now appropriately restricted. s is the (volume) total scattering function.) Hence $\bar{\omega}_0 \leq 1$. This latter quantity is the albedo for single scattering.

(ii) Define $\tau = \int_0^z \alpha(z)\, dz$, $0 \leq z \leq z_1$. τ is the optical depth, corresponding to the geometric depth z below B_0.

With the conventions established above, the equation of transfer for the slab geometry (as defined in (b)) may be written

$$\mu\, dN(\tau,\mu,\varphi)/d\tau = N(\tau,\mu,\varphi) - (1/4\pi)\int_{\Xi} p(\mu,\varphi;\mu',\varphi')N(\tau,\mu',\varphi')\, d\mu' d\varphi' \qquad (A')$$

N(, ,) is now defined on $Z_\tau \times \Xi$, Z_τ is the image, under the transformation defined in (ii), of the closed interval $(0, z_1)$. Observe that (A') explicitly contains only one of the several characteristics of the response of X to radiant flux, namely the phase function p. (The general equation contains five) Thus in the present discussion of the general two-boundary problem the physical characteristics of X are summarized explicitly by a single function, namely p. Despite the relative simplicity of (A') as compared to (A), the solution of (A') still presents formidable difficulties both abstractly and numerically. The single notable published discussion[1] to date of (A') on an abstract level is for

[1] J. Lehner and G.M. Wing, "Solution of the Linearized Boltzmann Transport Equation for the Slab Geometry," Duke Math. J. 23, 125-142 (1956).

the transient case with p = constant function on $\Xi \times \Xi$. Methods designed to aid in the numerical solution of (A') are contained in the standard treatise[2] (Chapter VI). These methods are specifically for the important Standard Problem (the no-boundary problem of the present classification scheme). The presence of boundaries presents some additional difficulties of formulation and solution. The formulation of the equations governing the boundary problems will now be considered.

(d) Definition of the Two-Boundary Problem (The General Planetary Hydrosphere Problem).

Given: (i) An optical medium $M = (\Phi \times T, N, \alpha, \sigma, n)$ as defined in (b).

(ii) Boundary B_0 of X with reflectance function r_0.

(iii) Boundary B_1 of X with reflectance function r_1.

(iv) At all points $(x,y,0)$ of E_3 there exists an incident radiance distribution $N^0(0,-\mu,\varphi)$, $0 < \mu \leq 1$; and at all points (x,y,z_1) of E_3 there exists an incident radiance distribution $N^0(\tau_1,+\mu,\varphi)$, $0 < \mu \leq 1$.

Required:

(v.p) The radiance distribution at optical depths $\tau = 0$, $\tau = \tau_1$.

(v.c) The radiance distribution at every optical depth τ, $0 \leq \tau \leq \tau_1$.

3. CLASSIFICATION OF PROBLEMS

In the attempt to formulate the equations governing the radiance distributions in the slab geometry, there arises quite naturally a set of integral equations for the inward and outward radiance distributions on the boundaries. Further analysis shows that the radiance distributions at levels τ, $0 < \tau < \tau_1$ can be found with the help of the boundary radiance distributions. Thus the problem of finding a radiance distribution $N(\tau, \,,\,)$, $0 \leq \tau \leq \tau_1$ may be analyzed into two steps: the partial problem in which $N(0, \,,\,)$ and $N(\tau_1, \,,\,)$ are

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[2] S. Chandrasekhar, Radiative Transfer, (Clarendon Press, Oxford, 1950)

found, and the complete problem in which the remainder of the radiance distributions $N(\tau, \ , \ )$, $0 < \tau < \tau_1$, are found. This procedure will be used in the study of the no-, one-, and two-boundary problems, i.e., the problems in which the possible combinations of the choices $r_o = 0$, $r_o \neq 0$, $r_1 = 0$, $r_1 \neq 0$ are exhaustively considered. The most general problem to consider is therefore the complete two-boundary problem which is defined by (i)-(iv) and (v.c) above. It will be shown, however, that the basic problems are the standard (partial no-boundary) and the partial one-boundary problems in the sense that all others are reducible to these in the slab geometry context, a fact which is of help in the formulation of the present problems and their reduction to numerical solution procedures.

4. THE STANDARD PROBLEM (PARTIAL NO-BOUNDARY)

(a) Definition of the Standard Problem.

Using (d) of section 2, the Standard Problem is defined by giving: (i) and (iv), setting $r_o = 0$ and $r_1 = 0$ in (ii) and (iii); and requiring: (v.p) only. In the general classification scheme, the Standard Problem is the partial no-boundary problem.

(b) The Principle of Relative Scattering Order (p.o.r.s.o.).

In the slab geometry context, the principle may be stated as follows:

(i) A radiance function $N(\ , \ , \ )$ may be arbitrarily assigned an (axiomatized notion of) scattering order $j = 0$, the result being designated by $N^o(\ , \ , \ )$.

(ii) A radiance distribution $N^j(\ , \ , \ )$ has scattering order $j \geq 0$ (j an integer) if and only if every radiance distribution $N^j(z, \ , \ )$ has scattering order j.

(iii) If a radiance distribution $N^j(z, \ , \ )$ has a scattering order $j \geq 0$, the operator $(1/4\pi)\int_{\Xi} ( )p \, d\Omega$ assigns to $N^j(z, \ , \ )$ a radiance distribution $N^{j+1}(z, \ , \ )$ of scattering order $j+1$.

(iv) If the radiance distributions $N^j(z,+ \ , \ )$, $N^j(z,- \ , \ )$ have a scattering order $j \geq 0$, the operator $(1/4\pi\mu)\int_{\Xi_{\pm}} ( )r \, d\Omega$ assigns to them, respectively, the radiance distributions $N^{j+*}(z,- \ , \ )$, and $N^{j+*}(z,+ \ , \ )$ of scattering order $j+*$, where $* = (1) + (2) + \cdots$, and r stands for r_0 or r_1 defined earlier.

(v) Any radiance distribution may be analyzed into the sum $N^0(z, ,) + N^*(z, ,)$ of two radiance distributions of scattering order $j = 0$, and $j = *$.

The symbol $* = (1) + (2) + \cdots$, is to denote that N^* consists (relative to N^0) of components of scattering order $j = 1$, <u>and</u> $j = 2$, etc., and may be symbolically expressed as $N^* = N^1 + N^2 + \cdots$. It is possible that N^* could be so generated that any or all of its components N^j are identically zero. This relatively minor but yet indispensible principle has not received its full share of attention in mathematical investigations of radiative transfer theory. A rigorous formulation of the problems of radiative transfer require this principle in no less a way than the apparently more important principles of invariance. In fact the principles of invariance and the p.o.r.s.o. are not wholly independent principles: relations exist between the two which indicate eventual subsumption of one by the other. As a conjecture, the p.o.r.s.o. would seem the more basic of the two. We will only note here that the p.o.r.s.o. has been successfully used in the simultaneous derivation and solution of the integral form of the equation of transfer on a formal level[3].

(c) The Decomposition of (A'). The S and T Functions.

With (a) of this section now in force, use (i) of p.o.r.s.o. to assign zero scattering order to $N^0(0,-\mu,\varphi)$ and $N^0(\tau_1,+\mu,\varphi)$, and again (v) to write $N(\tau, ,) = N^0(\tau, ,) + N^*(\tau, ,)$. Then (A') may be written

$$\mu \, d(N^0(\tau,\mu,\varphi) + N^*(\tau,\mu,\varphi))/d\tau =$$

$$= N^0(\tau,\mu,\varphi) + N^*(\tau,\mu,\varphi) - (1/4\pi)\int_{\Xi} (N^0(\tau,\mu',\varphi') + N^*(\tau,\mu',\varphi'))p(\mu,\varphi;\mu',\varphi')d\mu' d\varphi'$$

Using (ii) and (iii) of the p.o.r.s.o., it may be shown that

$$\mu \, dN^0(\tau,\mu,\varphi)/d\tau = N^0(\tau,\mu,\varphi)$$

whence

[3] R.W. Preisendorfer, <u>A Preliminary Investigation of the Transient Radiant Flux Problem</u>, unpublished lecture notes (Visibility Laboratory, Scripps Institution of Oceanography, La Jolla, California, February 1954).

$$N^o(\tau,-\mu,\varphi) = N^o(0,-\mu,\varphi)e^{-\tau/\mu}, \quad 0 < \mu \leqq 1,$$
$$N^o(\tau,+\mu,\varphi) = N^o(\tau_1,+\mu,\varphi)e^{-(\tau_1-\tau)/\mu}, \quad 0 < \mu \leqq 1.$$

(The remaining statement(iv) is used later in the formulation of the boundary problems.)

With these observations the equation of transfer becomes

$$\mu\, dN^*(\tau,\mu,\varphi)/d\tau = N^*(\tau,\mu,\varphi) - (1/4\pi)\int_{\Xi} N^*(\tau,\mu',\varphi')p(\mu,\varphi;\mu',\varphi')\, d\mu' d\varphi'$$
$$- (1/4\pi)\int_{\Xi} N^o(\tau,\mu',\varphi')p(\mu,\varphi;\mu',\varphi')\, d\mu' d\varphi'. \qquad (A^*)$$

The solution of this equation is subject to the boundary conditions (which follow from (i) and (v) of the p.o.r.s.o.):

$$N^*(0,-\mu,\varphi) = 0$$
$$N^*(\tau_1,+\mu,\varphi) = 0 \qquad (0 < \mu \leqq 1).$$

(A*) and the above boundary conditions yield two non negative real valued functions $S(\tau_1;\ ;\ )$ and $T(\tau_1;\ ;\ )$ on $\Xi_+ \times \Xi_+$ with the properties

$$N^*(0,+\mu,\varphi) = (1/4\pi\mu)\int_{\Xi_+} S(\tau_1;\mu,\varphi;\mu',\varphi')N^o(0,-\mu',\varphi')\, d\mu' d\varphi' +$$
$$+ (1/4\pi\mu)\int_{\Xi_+} T(\tau_1;\mu,\varphi;\mu',\varphi')N^o(\tau_1,+\mu',\varphi')\, d\mu' d\varphi'$$

$$N^*(\tau_1,-\mu,\varphi) = (1/4\pi\mu)\int_{\Xi_+} S(\tau_1;\mu,\varphi;\mu',\varphi')N^o(\tau_1,+\mu',\varphi')\, d\mu' d\varphi' +$$
$$+ (1/4\pi\mu)\int_{\Xi_+} T(\tau_1;\mu,\varphi;\mu',\varphi')N^o(0,-\mu',\varphi')\, d\mu' d\varphi'.$$

The functions S and T satisfy the Helmholtz principle of reciprocity, and are indispensible ingredients in the statements of the principles of invariance. These principles in turn give rise to integral equations satisfied by S and T. The resulting equations may in turn be used to study the properties of the S and T functions. For further details about S and T, see Chap. VII of reference 2. The present discussions must eventually return to the question of obtaining S and T numerically, for as will be shown below, the solutions of all the boundary problems incorporate in an essential way (but are not necessarily reducible to) the solutions of the Standard Problem.

(d) The Integral Operators r_0^*, r_1^*, S^*, T^*,

It should be observed that the functions p, r_0, r_1, S and T are all members of a family of transformations which assign, in the senses made clear above, a given radiance distribution to another. The physical significance of some of the mappings are made clear in Fig. 2.

Fig. 2

A completely symmetric formulation of the family can be made by first adjoining the surface transmittance functions t_0 and t_1 on $\Xi_+ \times \Xi_+$ defined analogously to r_0 and r_1. Finally, p itself can be analyzed into two functions s and t on $\Xi_+ \times \Xi_+$ which are the lamina counterparts to S and T. The following table summarizes these functions:

TABLE I

	Reflectance	Transmittance
Surface	r_0, r_1	t_0, t_1
Lamina	s	t
Slab	S	T

The integral operations yielding $N^*(0,+\mu,\varphi)$ and $N^*(\tau_1,-\mu,\varphi)$ ________ will be used repeatedly. The discussions in this and following sections will be greatly simplified and made essentially algebraic if the following integral operators are defined.

Let

$$Y^* = (1/4\pi\mu)\int_{\Xi_+} (\quad) \, Y \, d\Omega$$

denote the integral operator Y^* where Y can be any member of the reflectance-transmittance family in Table I. For example, if Y is taken to be S, the definition written out in full reads:

$$S^*(\tau_1;\mu,\varphi;\,,\,) = (1/4\pi\mu)\int_{\Xi_+}(\ )S(\tau_1;\mu,\varphi;\ ,\ )\,d\Omega,$$

and $S^*(\tau_1;\mu,\varphi)$ operates on $N(\tau_1;+,\,)$, say, to yield $N(\tau_1,-\mu,\varphi)$, that is, the value of the radiance distribution $N(\tau_1,-,\ )$ at (μ,φ). Symbolically,

$$S^*(\tau_1;\mu,\varphi)N(\tau_1,+\mu',\varphi') = N(\tau_1,-\mu,\varphi).$$

Further abbreviation can be adopted profitably by dropping reference to (μ,φ) in $S^*(\tau_1;\mu,\varphi)$, and therefore simply writing $S^*(\tau_1)$, the pair (μ,φ) being understood. With these conventions, $N^*(0,+\mu,\varphi)$ and $N^*(\tau_1,-\mu,\varphi)$ can be written

$$N^*(0,+\mu,\varphi) = T^*(\tau_1)N^0(\tau_1,+\mu',\varphi') + S^*(\tau_1)N^0(0,-\mu',\varphi')$$
$$N^*(\tau_1,-\mu,\varphi) = S^*(\tau_1)N^0(\tau_1,+\mu',\varphi') + T^*(\tau_1)N^0(0,-\mu',\varphi').$$

The τ_1 following S^* and T^* indicate the optical depth of the slab to which they refer.

One further definition will be helpful:

$$T(\tau) = T^*(\tau) + e^{-\tau/\mu}I_\mu,$$

where

$$I_\mu = \int_{\Xi_+}(\ )\,\delta(\mu-\mu')\,\delta(\varphi-\varphi')\,d\mu' d\varphi',$$

so that

$$I_\mu N(\tau,\mu',\varphi') = \int_{\Xi_+} N(\tau,\mu',\varphi')\,\delta(\mu-\mu')\,\delta(\varphi-\varphi')\,d\mu'\,d\varphi' = N(\tau,\mu,\varphi).$$

Here δ is the Dirac delta function.

Observe that $S^*(0) = 0$, and $T^*(0) = 0$ (the zero operators), and that $T(0) = I_\mu$. Finally, it should be noted that the integral operators defined above do not, in general, commute.

(e) The Solution of the Standard Problem

By the p.o.r.s.o.,

$$N(0,+\mu,\varphi) = N^{o}(0,+\mu,\varphi) + N^{*}(0,+\mu,\varphi),$$

$$N^{*}(0,-\mu,\varphi) = 0,$$

$$N^{o}(0,+\mu,\varphi) = N^{o}(\tau_1,+\mu,\varphi)e^{-\tau_1/\mu},$$

so that:

$$N_o(0,+\mu,\varphi) = T(\tau_1)N^{o}(\tau_1,+\mu',\varphi') + S^{*}(\tau_1)N^{o}(0,-\mu',\varphi') \quad \text{(i)}$$

$$N_o(0,-\mu,\varphi) = N^{o}(0,-\mu,\varphi) \quad \text{(ii)}$$

$$N_o(\tau_1,-\mu,\varphi) = S^{*}(\tau_1)N^{o}(\tau_1,+\mu',\varphi') + T^{*}(\tau_1)N^{o}(0,-\mu',\varphi') \quad \text{(iii)}$$

$$N_o(\tau_1,+\mu,\varphi) = N^{o}(\tau_1,+\mu,\varphi). \quad \text{(iv)}$$

The subscripts "o" are to denote a solution of the no-boundary problem. Clearly, if S and T are known, the partial no-boundary problem is immediately solvable by performing on $N^{o}(0,-\mu,\varphi)$ and $N^{o}(\tau_1,+\mu,\varphi)$ the operations indicated in equations (i)-(iv) above.

(f) Formulation of the (complete) Standard Problem.

The principles of invariance may be adapted to the formulation of the complete no-boundary problem (and all the other "complete" problems) in the following way. At any level τ, $0 < \tau < \tau_1$,

$$N_o(\tau,+\mu,\varphi) = S^{*}(\tau_1-\tau)N_o(\tau,-\mu',\varphi') + T(\tau_1-\tau)N_o(\tau_1,+\mu',\varphi') \quad \text{(v)}$$

$$N_o(\tau,-\mu,\varphi) = S^{*}(\tau)N_o(\tau,+\mu',\varphi') + T(\tau)N_o(0,-\mu',\varphi'). \quad \text{(vi)}$$

Using equation (vi) in (v), and making use of the solutions (i)-(iv) of the partial problem,

$$N_o(\tau,+\mu,\varphi) = S^*(\tau_1-\tau)S^*(\tau)N_o(\tau,+\mu'',\varphi'') + S^*(\tau_1-\tau)T(\tau)N^o(0,-\mu'',\varphi'') + T(\tau_1-\tau)N^o(\tau_1,\mu',\varphi') \quad \text{(vii)}$$

$$N_o(\tau,-\mu,\varphi) = S^*(\tau)S^*(\tau_1-\tau)N_o(\tau,-\mu'',\varphi'') + S^*(\tau)T(\tau_1-\tau)N^o(\tau_1,\mu'',\varphi'') + T(\tau)N^o(0,-\mu',\varphi') \quad \text{(viii)}$$

The operator compositions $S^*(\tau_1-\tau)S^*(\tau)$ and $S^*(\tau)S^*(\tau_1-\tau)$ are, in general, distinct operators. This pair of equations is basically a pair of integral equations of the second order of the Fredholm type, second kind. As will be seen, this type of structure appears in all the formulations of the partial and complete boundary problems considered in this paper. Therefore, if this type can be solved numerically by some routine, rapid procedure, the general two-boundary problem can therefore be solved and the results usable on an engineering level.

5. THE TWO-BOUNDARY PROBLEM

(a) Formulation of the Partial Problem.

Let $N_2(0, \,,\,)$ and $N_2(\tau_1, \,,\,)$ designate the required radiance distributions on B_o and B_1, respectively. The the extended forms of the principles of invariance yield the following set of relations between the inward and outward radiance distributions on B_o and B_1:

$$N_2(0,+\mu,\varphi) = T(\tau_1)N_2(\tau_1,+\mu',\varphi') + S^*(\tau_1)N_2(0,-\mu',\varphi') \quad \text{(i)}$$

$$N_2(0,-\mu,\varphi) = r_o^*N_2(0,+\mu',\varphi') + N^o(0,-\mu,\varphi) \quad \text{(ii)}$$

$$N_2(\tau_1,-\mu,\varphi) = T(\tau_1)N_2(0,-\mu',\varphi') + S^*(\tau_1)N_2(\tau_1,+\mu',\varphi') \quad \text{(iii)}$$

$$N_2(\tau_1,+\mu,\varphi) = r_1^*N_2(\tau_1,-\mu',\varphi') + N^o(\tau_1,+\mu,\varphi) \quad \text{(iv)}$$

These equations do not follow directly from the <u>classical</u> forms of the principles of invariance. It will be observed that the classical forms of the principles of invariance deal only with the radiances of the type N^*, and the above radiances are of the type $N^o + N^*$. In addition, the presence of the boundaries B_o and B_1 require additional statements ((ii) and (iv)) to be made in order that the general two-boundary problem be completely formulated. Implicit use has been made of clause (iv) of the p.o.r.s.o. in the formulation of (i)-(iv) above in order to insure a complete tally of the interreflected radiant flux at B_o and B_1. It is clear that the p.o.r.s.o. and the extended forms of the principles of invariance (or some equivalent set of principles) are indispensible for any systematic and consistent approach to the general two-boundary problem. The present formulations are consistent with the formulation of the two-boundary problem in which the radiometric quantity irradiance (rather than the radiance) was of central interest[4]. In fact it was the detailed study of of this latter problem which resulted in the inductive process leading to the forms (i)-(iv) above.

As before, the two radiance distributions $N_2(0,+\ ,\ )$ and $N_2(\tau_1,-\ ,\ )$ are singled out as the basic radiance distributions of interest in a partial problem. The set (i)-(iv) can be recast so as to form a pair of simultaneous integral equations in these radiance distributions. This is done by combining (i), (ii), and (iii), (iv) as follows:

$$N_2(0,+\mu,\varphi) = T(\tau_1)(r_1^* N_2(\tau_1,-\mu'',\varphi'') + N^o(\tau_1,+\mu',\varphi')) + $$
$$+ S^*(\tau_1)(r_o^* N_2(0,+\mu'',\varphi'') + N^o(0,-\mu',\varphi'))$$

$$= S^*(\tau_1) r_o^* N_2(0,+\mu'',\varphi'') + S^*(\tau_1) N^o(0,-\mu',\varphi') + $$
$$+ T(\tau_1) r_1^* N_2(\tau_1,-\mu'',\varphi'') + T(\tau_1) N^o(\tau_1,+\mu',\varphi'),$$

so that

$$(S^*(\tau_1) r_o^* - I_\mu) N_2(0,+\mu'',\varphi'') + T(\tau_1) r_1^* N_2(\tau_1,-\mu'',\varphi'') + N_o(0,+\mu,\varphi) = 0 \quad \text{(v)}$$

similarly

$$T(\tau_1) r_o^* N_2(0,+\mu'',\varphi'') + (S^*(\tau_1) r_1^* - I_\mu) N_2(\tau_1,-\mu'',\varphi'') + N_o(\tau_1,-\mu,\varphi) = 0, \quad \text{(vi)}$$

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[4] R.W. Preisendorfer, "A Derivation of h(x) (Scalar Irradiance)," unpublished manuscript, (Visibility Laboratory, Scripps Institution of Oceanography, La Jolla, California, September 1953).

which is the required pair of integral equations. Observe the role played by the solutions of the partial no-boundary problem.

(b) The Formulation of the Complete Problem.

The extended forms of the principles of invariance once again yield the formulation of the complete problem (compare with(v) and (vi) of section 4). At any level τ, $0 < \tau < \tau_1$,

$$N_2(\tau,+\mu,\varphi) = S^*(\tau_1-\tau)N_2(\tau,-\mu',\varphi') + T(\tau_1-\tau)N_2(\tau_1,+\mu',\varphi') \qquad \text{(vii)}$$

$$N_2(\tau,-\mu,\varphi) = S^*(\tau)N_2(\tau,+\mu',\varphi') + T(\tau)N_2(0,-\mu',\varphi') \qquad \text{(viii)}$$

whence

$$\begin{aligned} N_2(\tau,+\mu,\varphi) = {} & S^*(\tau_1-\tau)S^*(\tau)N_2(\tau,+\mu'',\varphi'') + \\ & + T(\tau_1-\tau)N_2(\tau_1,+\mu',\varphi') + \\ & + S^*(\tau_1-\tau)T(\tau)N_2(0,-\mu'',\varphi'') \end{aligned} \qquad \text{(ix)}$$

$$\begin{aligned} N_2(\tau,-\mu,\varphi) = {} & S^*(\tau)S^*(\tau_1-\tau)N_2(\tau,-\mu'',\varphi'') + \\ & + S^*(\tau)T(\tau_1-\tau)N_2(\tau_1,+\mu'',\varphi'') + \\ & + T(\tau)N_2(0,-\mu',\varphi') \end{aligned} \qquad \text{(x)}$$

It is tn be observed that the solution of the complete problem is contingent upon the solution of the partial problem (the solutions of the partial problem supply the non-homogeneous terms in (ix) and (x)). The solution of the partial two-boundary problem is in turn dependent upon a successful unraveling of the required radiance distributions from (v) and (vi). The non commutativity of the integral operators makes even a formal solution of (v) and (vi) difficult. It turns out, however, that the partial two-boundary problem may be solved if an <u>explicit</u> solution of the partial one boundary can be found. This fact will be demonstrated in section 7 after the definition and formulation of the one-boundary problem, which will now be done.

6. THE ONE-BOUNDARY PROBLEM

(a) Formulation of the Partial Problem of the First Kind.

Using (d) of section 2, a partial one-boundary problem of the first kind is defined by giving: (i), (iii), (iv), setting $r_o = 0$; and requiring only (v.p). The integral equations for this problem follow immediately from (v) and (vi) of the preceding section in which $r_o^* = 0$ (the zero operator):

$$N_1(0,+\mu,\varphi) = T(\tau_1)r_1^*N_1(\tau_1,-\mu'',\varphi'') + N_o(0,+\mu,\varphi) \quad \text{(i)}$$

$$N_1(\tau_1,-\mu,\varphi) = S^*(\tau_1)r_1^*N_1(\tau_1,-\mu'',\varphi'') + N_o(\tau_1,-\mu,\varphi) \quad \text{(ii)}$$

Equation (ii) represents the basic problem encountered in the solution of all boundary problems of the present study. The mathematical structure of (ii) is that of a second order linear Fredholm integral equation of the second kind. If (ii) can be solved with relative ease, the one-boundary problem can therefore be solved in its entirety; and in view of the remarks of the preceding section, the solution of (ii) in all detail will lead to the solution of the two-boundary problem.

(b) Formulation of the Partial Problem of the Second Kind.

Using (d) of section 2, a partial one-boundary problem of the second kind is defined by giving: (i), (ii), (iv), setting $r_1 = 0$; requiring only (v.p). The integral equations for this problem follow immediately from (v) and (vi) of the preceding section in which $r_1^* = 0$:

$$N_1(0,+\mu,\varphi) = S^*(\tau_1)r_o^*N_1(0,+\mu'',\varphi'') + N_o(0,+\mu,\varphi) \quad \text{(iii)}$$

$$N_1(\tau_1,-\mu,\varphi) = T(\tau_1)r_o^*N_1(0,+\mu'',\varphi'') + N_o0,-\mu,\varphi) \quad \text{(iv)}$$

It is easy to see that the mathematical structures of the problems of the first and second kind are identical. Mathematically there is only one partial one-boundary problem. However, a distinction between the two is physically advantageous. Thw artifice of introducing two kinds of one-boundary problems yields some useful analyses of other boundary problems, especially for the case of the two-boundary problem.

(c) The Complete One-Boundary Problem.

The problem is defined as in (a) or (b) above (depending upon what kind of problem is being considered) with now (v.c) required. The formulation of the complete problem for the case of one boundary goes exactly as the complete two-boundary problem, with now the subscripts 2 replaced by 1.

7. THE UBIQUITY OF THE PARTIAL ONE-BOUNDARY PROBLEM

(a) The Complete No-Boundary Problem Formulated as a Collection of Partial One-Boundary Problems.

Writing out equation (ii) of section 6 in full:

$$N_1(\tau_1,-\mu,\varphi) = S^*(\tau_1)r_1^*N_1(\tau_1,-\mu'',\varphi'') + T(\tau_1)N^0(0,-\mu',\varphi') + S^*(\tau_1)N^0(\tau_1,+\mu',\varphi'),$$

and comparing terms with those in (viii) of section 4, the following correspondences can be made:

Complete No-Boundary Problem	Partial One-Boundary Problem
τ	τ_1
$N_0(\tau,-\mu,\varphi)$	$N_1(\tau_1,-\mu,\varphi)$
$S^*(\tau)$	$S^*(\tau_1)$
$S^*(\tau_1-\tau)$	r_1^*
$T(\tau_1-\tau)N^0(\tau_1,\mu'',\varphi'')$	$N^0(\tau_1,+\mu',\varphi')$
$N^0(0,-\mu',\varphi')$	$N^0(0,-\mu',\varphi')$

Hence (viii) of section 4 is <u>mathematically equivalent to a collection of partial one-boundary problems of the first kind</u>. (The collection arises from the fact that (viii) of section 4 must be solved for all τ, $0 < \tau < \tau_1$.) Fig. 3.

This situation is illustrated in Fig. 3.(a)

In a similar manner, equation (vii) of section 4 of the complete no-boundary problem may be shown to be mathematically equivalent to the equations of a collection of partial one-boundary problems of the second kind. The correspondences in this instance (between (vii) of section 4 and (iii) of section 6) are:

Complete No-Boundary Problem	Partial One-Boundary Problem
$\tau_1-\tau$	τ_1
$N_0(\tau_1,+\mu,\varphi)$	$N_1(0,+\mu,\varphi)$
$S^*(\tau_1-\tau)$	$S^*(\tau_1)$
$S^*(\tau)$	r_0^*
$T(\tau)N^0(0,-\mu'',\varphi'')$	$N^0(0,-\mu',\varphi')$
$N^0(\tau_1,+\mu',\varphi')$	$N^0(\tau_1,+\mu',\varphi')$

This situation is illustrated in Fig. 3.(b).

(b) The Complete One-Boundary Problem Formulated as a Collection of Partial One-Boundary Problems.

In view of the previous observations on the mathematical equivalence of the formulations of all complete problems, the demonstnation of the present assertion (and that of (d) below) has essentially been made in (a).

(c) The Partial Two-Boundary Problem Formulated as a Pair of One-Boundary Problems of the First Kind.

Because of the relative intractibility of the pair of equations representing the partial two-boundary problem, the present assertion is perhaps the most important of those given in the present section, for it leads to an alternate, more tractable formulation of the partial two-boundary problem. The proof depends upon the formal inversibility of the operator $(I_\mu - S^*(\tau_1)r_1^*)$. The details of the proof may be arranged as follows:

From equation (ii) of section 6,

$$(I_\mu - S^*(\tau_1)r_1^*)N_1(\tau_1,-\mu'',\varphi'') = N_o(\tau_1,-\mu,\varphi).$$

Formally inverting the operator:

$$N_1(\tau_1,-\mu'',\varphi'') = (I_\mu - S^*(\tau_1)r_1^*)^{-1}N_o(\tau_1,-\mu,\varphi).$$

Using this in (i) of section 6,

$$N_1(0,+\mu,\varphi) = T(\tau_1)r_1^*(I_\mu - S^*(\tau_1)r_1^*)^{-1}N_o(\tau_1,-\mu,\varphi) + N_o(0,+\mu,\varphi).$$

Now using the explicit expressions for the solutions of the partial no-boundary problem, the above expression may be written

$$\begin{aligned} N_1(0,+\mu,\varphi) = {} & T(\tau_1)r_1^*(I_\mu - S^*(\tau_1)r_1^*)^{-1}(T(\tau_1)N^o(0,-\mu',\varphi') + S^*(\tau_1)N^o(\tau_1,+\mu',\varphi')) \\ & + T(\tau_1)N^o(\tau_1,+\mu',\varphi') + S^*(\tau_1)N^o(0,-\mu',\varphi') \qquad \text{(i)} \end{aligned}$$

In this way the solution $N_1(0,+\mu,\varphi)$ of the partial one-boundary problem of the first kind is formally expressed in terms of the incident radiance distributions $N^o(0,+\ ,\ )$ and $N^o(\tau_1,-\ ,\ )$. In view of (i) above, we may say that $N_1(0,+\ ,\ )$ is the one-boundary response of the first kind to the incident radiance distributions $N^o(0,+\ ,\ )$ and $N^o(\tau_1,-\ ,\ )$. A similar statement can be made about the radiance distribution $N_1(\tau_1,-\ ,\ )$.

To continue with the proof, we note that on the other hand (vi) of section 5 may be rearranged so that

$$\begin{aligned} & (I_\mu - S^*(\tau_1)r_1^*)^{-1}T(\tau_1)r_o^*N_2(0,+\mu'',\varphi'') - N_2(\tau_1,-\mu'',\varphi'') + \text{[illegible]} \\ & \qquad + (I_\mu - S^*(\tau_1)r_1^*)^{-1}N_o(0,+\mu,\varphi) = 0 \qquad \text{(ii)} \end{aligned}$$

and from (v) of section 5,

$$T(\tau_1)r_1^*N_2(\tau_1,-\mu'',\varphi'') = -N_o(0,+\mu,\varphi) - (S^*(\tau_1)r_o^* - I_\mu)N_2(0,+\mu'',\varphi''). \qquad \text{(iii)}$$

But from (ii) and (iii) above in that order,

$$T(\tau_1)r_1^*N_2(\tau_1,-\mu'',\varphi'') = T(\tau_1)r_1^*(I_\mu - S^*(\tau_1)r_1^*)^{-1}T(\tau_1)r_o^*N_2(0,+\mu'',\varphi'') +$$
$$+ T(\tau_1)r_1^*(I_\mu - S^*(\tau_1)r_1^*)^{-1}N_o(0,+\mu,\varphi)$$
$$= -N_o(0,+\mu,\varphi) - (S^*(\tau_1)r_o^* - I_\mu)N_2(0,+\mu'',\varphi'').$$

Solving for $N_2(0,+\mu,\varphi)$ (and making use of (ii) section 5):

$$N_2(0,+\mu,\varphi) = T(\tau_1)r_1^*(I_\mu - S^*(\tau_1)r_1^*)^{-1}(T(\tau_1)N_2(0,-\mu'',\varphi'') + S^*(\tau_1)N^o(\tau_1,+\mu'',\varphi''))+$$
$$+ T(\tau_1)N^o(\tau_1,+\mu',\varphi') + S^*(\tau_1)N_2(0,-\mu',\varphi') \qquad \text{(iv)}$$

Comparison of (i) and (iv) yields the following conclusion:

I. The boundary radiance distribution $N_2(0,+\ ,\ )$ in a two-boundary problem is the one-boundary response of the first kind to the incident radiance distributions $N_2(0,-\ ,\ )$ and $N^o(\tau_1,+\ ,\ )$.

A corresponding statement about $N_2(\tau_1,-\mu,\varphi)$ may be obtained as follows:

Starting once again from (ii) of section 6 and using (iii) of section 4,

$$N_1(\tau_1,-\mu'',\varphi'') = (I_\mu - S^*(\tau_1)r_1^*)^{-1}N_o(\tau_1,-\mu,\varphi) =$$
$$= (I_\mu - S^*(\tau_1)r_1^*)^{-1}(T(\tau_1)N^o(0,-\mu',\varphi') + S^*(\tau_1)N^o(\tau_1,+\mu',\varphi')).$$

But from (vi) of section 5,

$$N_2(\tau_1,-\mu'',\varphi'') = (I_\mu - S^*(\tau_1)r_1^*)^{-1}(T(\tau_1)r_o^*N_2(0,+\mu'',\varphi'') + N_o(\tau_1,-\mu,\varphi)), \qquad \text{(v)}$$

which, with the aid of (ii) of section 5, and (iii) of section 4,

$$= (I_\mu - S^*(\tau_1)r_1^*)^{-1}(T(\tau_1)N_2(0,-\mu',\varphi') + S^*(\tau_1)N^o(\tau_1,+\mu',\varphi')) \qquad \text{(vi)}$$

Comparison of (v) and (vi) yields the conclusion:

II. The boundary radiance distribution $N_2(\tau_1,-\ ,\ )$ in a two-boundary problem is the one-boundary response of the first kind to the incident radiance distributions $N_2(0,-\ ,\ )$ and $N^o(\tau_1,+\ ,\ )$.

These conclusions point up the role played by the one-boundary problems in the class of boundary problems in the slab geometry context. The Standard Problem of course rests at the base of all these discussions, but with the introduction of boundaries, the center of gravity of interest shifts from the Standard Problem to somewhere between the Standard Problem and the one-boundary problem. The operators S^* and T^* (and therefore $T(\tau)$) completely determine the solutions of the StandardProblem. In the case of the partial one-boundary problem, the operators

$$T(\tau_1)r_1^*(I_\mu - S^*(\tau_1)r_1^*)^{-1}T(\tau_1) + S^*(\tau_1)$$

$$T(\tau_1)r_1^*(I_\mu - S^*(\tau_1)r_1^*)^{-1}S^*(\tau_1) + T(\tau_1)$$

along with

$$(I_\mu - S^*(\tau_1)r_1^*)^{-1}T(\tau_1)$$

$$(I_\mu - S^*(\tau_1)r_1^*)^{-1}S^*(\tau_1)$$

play an analogous role. Specific examples of some of these operators will occur in section 8.

Observe that the one-boundary problem of the first kind has been used in the above discussion. However, in view of the complete symmetry between the problems of both kinds, what has been concluded here about problems of the first kind can also be concluded about the second kind. Historically, however, the one-boundary problem of the first kind is the natural mathematical setting for the classical planetary atmosphere problem (Chap. X, reference 2), and is also the natural springboard from which to leap into the planetary hydrosphere problem.

(d) The Complete Two-Boundary Problem as a Collection of Partial One-Boundary Problems.

See the remarks in (b).

8. THE (CLASSICAL) PLANETARY HYDROSPHERE PROBLEM

(a) Definition and Formulation.

The (classical) planetary hydrosphere problem is defined as a two-boundary problem in which:

(i) r_0 is the Fresnel reflectance function, i.e.,

$(1/4\pi\mu\mu')r_0(\mu,\varphi;\mu',\varphi) = (r(\mu')/\mu')\,\delta(\mu+\mu')\,\delta(\varphi-\varphi')$, where

$$r(\mu') = ((2m^2+1)/(m^2-1)^2)\,(\,(\cos(\cos^{-1}\mu''-\cos^{-1}\mu') - a)^2 + {}$$
$$+(\sec(\cos^{-1}\mu''-\cos^{-1}\mu') - a)^2\,),\quad 0 \leq \mu' \leq 1,$$

and $a = (m^2+1)/2m$. μ' and μ'' are related indirectly by Snell's Law. If $\theta' = \cos^{-1}\mu'$, and $\theta'' = \cos^{-1}\mu''$, and the relative index of refraction throughout X is $m = 4/3$ (relative to points of euclidean three-space in the complement of X) then $(4/3)\sin\theta' = \sin\theta''$. θ' is the angle of incidence of light coming from X to B_0. Total reflection occurs when $\theta'' = \pi/2$, i.e., when $\theta' = \sin^{-1}(3/4)$ (hence when $0 \leq \mu' \leq \sqrt{7}/4$).

(ii) r_1 is the Lambert reflectance function, i.e.,

$(1/4\pi\mu\mu')r_1(\mu,\varphi;\mu',\varphi') = r_1/\pi$, r_1 is a constant, the albedo of B_1.

(iii) $N^0(0,-\mu,\varphi) = N^0\,\delta(\mu-\mu_0)\,\delta(\varphi-\varphi_0)$, $0 < \mu_0 \leq 1$, $N^0 \neq 0$ is a fixed real number.

(iv) $N^0(\tau_1,+\ ,\ ) = 0$.

In view of the conclusions I and II of the preceding section, we consider at the outset of the formulations, the associated one-boundary problem of the first kind. The associated integral equations are therefore:

$$N(0,+\mu,\varphi) = Tr_1^*N(\tau_1,-\mu'',\varphi'') + N_0(0,+\mu,\varphi)$$
$$N(0,-\mu,\varphi) = N^0(0,-\mu,\varphi)$$
$$N(\tau_1,-\mu,\varphi) = S^*r_1^*N(\tau_1,-\mu'',\varphi'') + N_0(\tau_1,-\mu,\varphi)$$
$$N(\tau_1,+\mu,\varphi) = r_1^*N(\tau_1,-\mu',\varphi'),$$

where the solutions of the partial no-boundary problem now take the forms:

$$N_o(0,+\mu,\varphi) = (1/4\pi\mu)N^o S(\mu,\varphi;\mu_o,\varphi_o)$$

$$N_o(\tau_1,-\mu,\varphi) = (N^o/4\pi\mu)T(\mu,\varphi;\mu_o,\varphi_o) + $$
$$+ N^o e^{-\tau_1/\mu}\,\delta(\mu - \mu_o)\,\delta(\varphi - \varphi_o).$$

References to τ_1 in the integral operators and in the functions S and T have been dropped for brevity. For the work in the sequel, it will be convenient to adopt (as in reference 2) the following notation:

$$t(\mu) = (1/4\pi)\int_0^1\int_0^{2\pi} T(\mu,\varphi;\mu',\varphi')\,d\mu' d\varphi'$$

$$s(\mu) = (1/4\pi)\int_0^1\int_0^{2\pi} S(\mu,\varphi;\mu',\varphi')\,d\mu' d\varphi'$$

$$\bar{t} = 2\int_0^1 t(\mu)\,d\mu$$

$$\bar{s} = 2\int_0^1 s(\mu)\,d\mu$$

$$\gamma_1(\mu) = e^{-\tau_1/\mu} + t(\mu)/\mu$$

Evaluating the effect of the Lambert reflectance integral operator, we have:

$$r_1^* N(\tau_1,-\mu'',\varphi'') = (r_1/\pi)\int_0^1\int_0^{2\pi} N(\tau_1,-\mu'',\varphi'')\mu''\,d\mu'' d\varphi''$$
$$= N(\tau_1,+\mu,\varphi),$$

wherein the last equality follows from the usual relation between boundafy radiance distributions and the special relation (iv) in the definition of the planetary hydrosphere problem. Let the integral (which represents an irradiance) be designated by $H(\tau_1,-)$, so that

$$r_1 H(\tau_1,-) = \pi N(\tau_1,+\mu,\varphi).$$

Since the outward radiance distribution at $\tau = \tau_1$ is evidentally a constant function, we abbreviate the preceding relation to

$$r_1 H(\tau_1,-) = \pi N(\tau_1,+),$$

where r_1 is the albedo of B_1.

An examination of the integral equations shows that the effect of the integral operators $T(\tau_1)$ and $S^*(\tau_1)$ on $N(\tau_1,+)$ must be evaluated:

$$T^*N(\tau_1,+) = (N(\tau_1,+)/4\pi\mu)\int_0^1\int_0^{2\pi}T(\mu,\varphi;\mu',\varphi')\,d\mu'd\varphi' = N(\tau_1,+)t(\mu)/\mu.$$

$$e^{-\tau_1/\mu}I_\mu N(\tau_1,+) = e^{-\tau_1/\mu}N(\tau_1,+).$$

$$S^*N(\tau_1,+) = (N(\tau_1,+)/4\pi\mu)\int_0^1\int_0^{2\pi}S(\mu,\varphi;\mu',\varphi')\,d\mu'd\varphi' = N(\tau_1,+)s(\mu)/\mu.$$

Hence the integral equations reduce to the linear algebraic equations

$$N(0,+\mu,\varphi) = \gamma_1(\mu)N(\tau_1,+) + (N^0/4\pi\mu)S(\mu,\varphi;\mu_0,\varphi_0) \qquad \text{(i)}$$

$$N(0,-\mu,\varphi) = N^0\,\delta(\mu-\mu_0)\,\delta(\varphi-\varphi_0) \qquad \text{(ii)}$$

$$N(\tau_1,-\mu,\varphi) = N(\tau_1,+)s(\mu)/\mu + (N^0/4\pi\mu)T(\mu,\varphi;\mu_0,\varphi_0) + N^0e^{-\tau_1/\mu}\,\delta(\mu-\mu_0)\delta(\varphi-\varphi_0) \qquad \text{(iii)}$$

$$N(\tau_1,+\mu,\varphi) = N(\tau_1,+) \qquad \text{(iv)}$$

The partial one-boundary problem associated with the planetary hydrosphere problem will be solved if $N(\tau_1,+)$ can be determined. From the definition of $H(\tau_1,-)$ we observe that (iii) above yields an alternate method of its evaluation. Performing the indicated operation on (iii),

$$\begin{aligned} H(\tau_1,-) &= \pi\bar{s}N(\tau_1,+) + N^0t(\mu_0)/\mu_0 + N^0\mu_0e^{-\tau_1/\mu_0} \\ &= \pi\bar{s}N(\tau_1,+) + \gamma_1(\mu_0)\mu_0N^0. \end{aligned}$$

But $(\pi/r_1)N(\tau_1,+) = H(\tau_1,-)$, so that

$$N(\tau_1,+) = N^0\mu_0r_1\gamma_1(\mu_0)/\pi(1-r_1\bar{s}) \qquad \text{(v)}$$

Using (v) in the set (i)-(iv), the solution of the associated partial one-boundary problem may be written

$$N(0,+\mu,\varphi) = \frac{N^o}{4\pi\mu}\left[S(\mu,\varphi;\mu_o,\varphi_o) + \frac{4r_1}{(1-r_1\bar{s})}\mu\mu_o\gamma_1(\mu)\gamma_1(\mu_o)\right] \qquad \text{(vi)}$$

$$N(0,-\mu,\varphi) = N^o\,\delta(\mu-\mu_o)\,\delta(\varphi-\varphi_o) \qquad \text{(vii)}$$

$$N(\tau_1,-\mu,\varphi) = \frac{N^o}{4\pi\mu}\left[T(\mu,\varphi;\mu_o,\varphi_o) + \frac{4r_1}{(1-r_1\bar{s})}\mu\mu_o\frac{s(\mu)}{\mu}\gamma_1(\mu_o)\right] + N^o e^{-\tau_1/\mu}\,\delta(\mu-\mu_o)\,\delta(\varphi-\varphi_o) \qquad \text{(viii)}$$

$$N(\tau_1,+\mu,\varphi) = N^o\mu_o r_1\gamma_1(\mu_o)/\pi(1-r_1\bar{s}) \qquad \text{(ix)}$$

Equations (vi)-(ix) above give explicit solutions to the partial one-boundary problem, and furnish correspondingly explicit examples of the operators associated with a partial one-boundary problem discussed in (c) of section 7. Of course the operators presented above are direct counterpartx to S and T of the Standard Problem. If these operators are designated by S_1 and T_1, the set (vi)-(ix) may be written ((vii) and (ix) are unchanged):

$$N(0,+\mu,\varphi) = (N^o/4\pi\mu)S_1(\mu,\varphi;\mu_o,\varphi_o) \qquad \text{(vi)}_1$$

$$N(\tau_1,-\mu,\varphi) = (N^o/4\pi\mu)T_1(\mu,\varphi;\mu_o,\varphi_o) + N^o e^{-\tau_1/\mu}\,\delta(\mu-\mu_o)\,\delta(\varphi-\varphi_o) \qquad \text{(viii)}_1$$

Now under the definition of the planetary hydrosphere problem, the application of conclusions I and II of section 7 takes the following form:

$$N(0,+\mu,\varphi) = (1/4\pi\mu)\int_0^1\int_0^{2\pi} S_1(\mu,\varphi;\mu',\varphi')N(0,-\mu',\varphi')\,d\mu'd\varphi' \qquad \text{(x)}$$

$$N(0,-\mu,\varphi) = r_o^* N(0,+\mu',\varphi') + N^o(0,-\mu,\varphi) \qquad \text{(xi)}$$

$$N(\tau_1,-\mu,\varphi) = (1/4\pi\mu)\int_0^1\int_0^{2\pi} T_1(\mu,\varphi;\mu',\varphi')N(0,-\mu',\varphi')\,d\mu'd\varphi' + N(0,-\mu,\varphi)e^{-\tau_1/\mu} \qquad \text{(xii)}$$

$$N(\tau_1,+\mu,\varphi) = r_1^* N(\tau_1,-\mu',\varphi') \qquad \text{(xiii)}$$

In the interests of brevity, the subscripts denoting boundary problem solutions have been dropped. It is clear, however, that the set (vi)-(ix) refers to a one-boundary solution, and the set (x)-(xiii) refers to a two-boundary solution. Using the adopted form of r_o, (xi) can readily be evaluated.

$$N(0,-\mu,\varphi) = r(\mu)N(0,+\mu,\varphi) + N^o\,\delta(\mu-\mu_o)\,\delta(\varphi-\varphi_o) \qquad \text{(xi)}_1$$

Equations (x) and $(xi)_1$ may be combined to yield

$$N(0,+\mu,\varphi) = (1/4\pi\mu)\int_0^1\int_0^{2\pi} S_1(\mu,\varphi;\mu',\varphi')r(\mu')N(0,+\mu',\varphi')\, d\mu' d\varphi' + (N^0/4\pi\mu)S_1(\mu,\varphi;\mu_0,\varphi_0). \quad \text{(xiv)}$$

In view of equations (xi), (xii), and (xiii), the solution of the partial planetary hydrosphere problem rests on (xiv).

(b) Solutions in Terms of Dimensionless Parameters; The Parameters of the Classical Planetary Hydrosphere Problem.

The one-boundary response principles I and II lead to the set (x)-(xiii) of equations which in turn lead to the basic integral equation (xiv) of the partial problem associated with the classical planetary hydrosphere problem. As observed, solution of (xiv) is tantamount to the solution of the entire partial problem. In order that the solution of (xiv) cover as wide a range of cases as possible, the terms should be made dimensionless. The only terms of (xiv) as yet not dimensionless are the radiance terms. By defining

$$\bar{N}(\tau, \ , \ ) = N(\tau, \ , \ )/N^0,$$

(xiv) becomes

$$\bar{N}(0,+\mu,\varphi) = (1/4\pi\mu)\int_0^1\int_0^{2\pi} S_1(\mu,\varphi;\mu',\varphi')r(\mu')\bar{N}(0,+\mu',\varphi')\, d\mu' d\varphi' + (1/4\pi\mu)S_1(\mu,\varphi;\mu_0,\varphi_0). \quad ((xiv)_1$$

Equation $(xiv)_1$ is possible because the response of the optical medium (the output of M) is linear with respect to the input.

The number of parameters required for a complete specification of the classical planetary hydrosphere problem is now clear. Since r_0 is essentially fixed, the parameters are r_1 (the albedo of B_1), the attenuation function α, and the phase function p. Hence the number is three. (In actuality, p is so complex that an approximate representation in terms of, say, Legendre polynomials is necessary. Experience has shown that a five parameter representation of p is adequate for all practical purposes. There is a dearth of experience, however as to how sensitive the light field is to changes in the structure of p. This knowledge will be a natural by-product of the subsequent investigations.)

(c) The Semi-Infinite Case.

If the optical depth of X is infinite (which implies, but not conversely, that the geometric depth is infinite) the basic integral equation (xiv) is essentially unaffected. Hence in the semi-infinite case, the mathematical problem is virtually unchanged from the finite case. Of course, in the semi-infinite case, $T(\tau_1) = 0$, so that computation of $N(\tau_1, \;, \;)$ is obviated.

9. RELATION OF THE GENERAL TWO-BOUNDARY PROBLEM TO THE INTERREFLECTION PROBLEMS

(a) Interreflection problems have been considered[5] in which the volume attenuation and volume scattering functions are identically zero on X. The present two-boundary problem is a generalization of this interreflection problem for the slab geometry context. The integral equations governing the interreflection process in an optical medium with no attenuation action on the light field follow immediately from the set (i)-(iv) of section 5 by observing that now $S^*(\tau_1) = 0$, $T(\tau_1) = 1$ (the zero and unity operators, respectively). Hence for this case,

$$N_2(0,+\mu,\varphi) = N_2(\tau_1,+\mu,\varphi) \quad \text{(i)}$$

$$N_2(0,-\mu,\varphi) = r_0^* N_2(0,+\mu',\varphi') + N^0(0,-\mu,\varphi) \quad \text{(ii)}$$

$$N_2(\tau_1,-\mu,\varphi) = N_2(0,-\mu,\varphi) \quad \text{(iii)}$$

$$N_2(\tau_1,+\mu,\varphi) = r_1^* N_2(\tau_1,-\mu',\varphi') + N^0(\tau_1,+\mu,\varphi) \quad \text{(iv)}$$

The formulation of the associated complete problem (from (ix) and (x) of section 5) reduces to the trivial statements

[5] F. Moon, ''On Interreflections,'' J. Opt. Soc. Am. 30, 195-205(1940).

$$N_2(\tau,+\mu,\varphi) = N_2(\tau_1,+\mu,\varphi) \quad \text{(v)}$$

$$N_2(\tau,-\mu,\varphi) = N_2(0,-\mu,\varphi) \quad \text{(vi)}$$

(b) The results of the present work are in formal agreement with those obtained for the special case of irradiance. This problem was considered in reference 4 in which the slab geometry was used for the study of the depth dependence of irradiance in the presence of attenuation of the light field and reflecting boundaries.

(c) Further generalizations of the problem considered in the present study are desirable. Two generalizations of immediate interest are: (1) the inclusion of the possibility of sources in X, so that $N_\eta \neq 0$. This case, but with no attenuation, is discussed in[6]. (2) Generalization of the geometry of X from the slab to other settings: spheres, cylinders, arbitrary convex and concave media. The first obstacle to remove in these generalizations is the extension of the principles of invariance to geometries other than the slab geometry.

10. SUMMARY

The purpose of this study is two-fold: (1) To define and formulate the general two-boundary problem in radiative transfer theory for the slab geometry (along with the special one- and no-boundary problems). The prototype of this problem is the classical planetary hydrosphere problem of geophysical optics. (2) To delineate all useful interconnections among the formulations and solutions of the no-, one- and two-boundary problems. Beyond (1) and (2), the ultimate purpose is to obtain numerical solutions of the classical planetary hydrosphere problem. It is planned that the computation of the numerical solutions be facilitated by means of some large scale automatic computor.

The principle results of the investigation are the realization of purposes (1) and (2). In particular it is found that the mathematical

[6] H. Bateman and C.L. Pekeris, "Transmission of Light from a Point Source in a Medium Bounded by Diffusely Reflecting Parallel Plane Surfaces," J. Opt. Soc. Am. 35, 651-657(1945).

core of the formulation of both the partial and complete problems lies in the integral equation of the form

$$N(\tau,x,y) = N_0(\tau,x,y) + (1/4\pi x)\int_0^1\int_0^{2\pi} K(\tau';x,y;x',y')N(\tau,x',y')\,dx'dy', \quad \text{(i)}$$

where $N_0(\tau, \ , \ )$ is a known function, $K(\tau'; \ ; \ )$ is a given kernel and $N(\tau, \ , \ )$ is the function sought. τ and τ' are parameters fixed for each equation, but which are drawn from any point in a closed interval of real numbers.

For the one-boundary problems, K is of the form: (partial problems)

$$K(\tau_1,r;\mu,\varphi;\mu'',\varphi'') =$$
$$= (1/4\pi)\int_0^1\int_0^{2\pi} S(\tau_1;\mu,\varphi;\mu',\varphi')r(\mu',\varphi';\mu'',\varphi'')\,\frac{d\mu'}{\mu'}\,d\varphi',$$

where r is the boundary reflectance function for either B_0 or B_1, and S is the solution operator of the no-boundary problem (the Standard Problem), and the parameter τ' is the pair (τ_1,r).

For the complete problems (of any number of boundaries), K is of the form,

$$K(\tau_1-\tau,\tau;\mu,\varphi;\mu'',\varphi'') =$$
$$= (1/4\pi)\int_0^1\int_0^{2\pi} S(\tau_1-\tau;\mu,\varphi;\mu',\varphi')S(\tau;\mu',\varphi';\mu'',\varphi'')\,\frac{d\mu'}{\mu'}\,d\varphi',$$

where now the parameter τ' is the pair $(\tau_1-\tau,\tau)$. Another kernel is obtained by permuting the pair of optical depths. A natural reciprocity relation holds between kernels of this type:

$$K(\tau,\tau_1-\tau;\mu,\varphi;\mu',\varphi') = K(\tau_1-\tau,\tau;\mu',\varphi';\mu,\varphi).$$

For the classical planetary hydrosphere partial problem, K has the form shown in $(xiv)_1$ of section 9.

Thus (i) above is the representative form for all the problems considered in this study.

Captions

Figure 1

Illustrating the slab geometry.

Figure 2

The response of either a surface, lamina or slab to an incoming radiance distribution (symbolized by $N(\mu',\varphi')$) is an outgoing radiance distribution (symbolized by $N(\mu,\varphi)$). In any case there is a function A on $\Xi_+ \times \Xi_+$ with the property that $N(\mu,\varphi) = (1/4\pi\mu)A(\mu,\varphi;\mu',\varphi')N(\mu',\varphi')$, for each pair (μ,φ), (μ',φ') of incident and reflected directions. For a surface, A is of the form r, for a lamina, A is of the form p, and for a slab, A is of the form S. Similarly, there is a function B on $\Xi_+ \times \Xi_+$ which plays the general role of a transmittance function for surfaces, laminae and slabs.

Figure 3

Illustrating the relation between complete problems and partial one-boundary problems. The general reflectance functions r_1 and r_0 of the one-boundary problems subsume the properties of S.

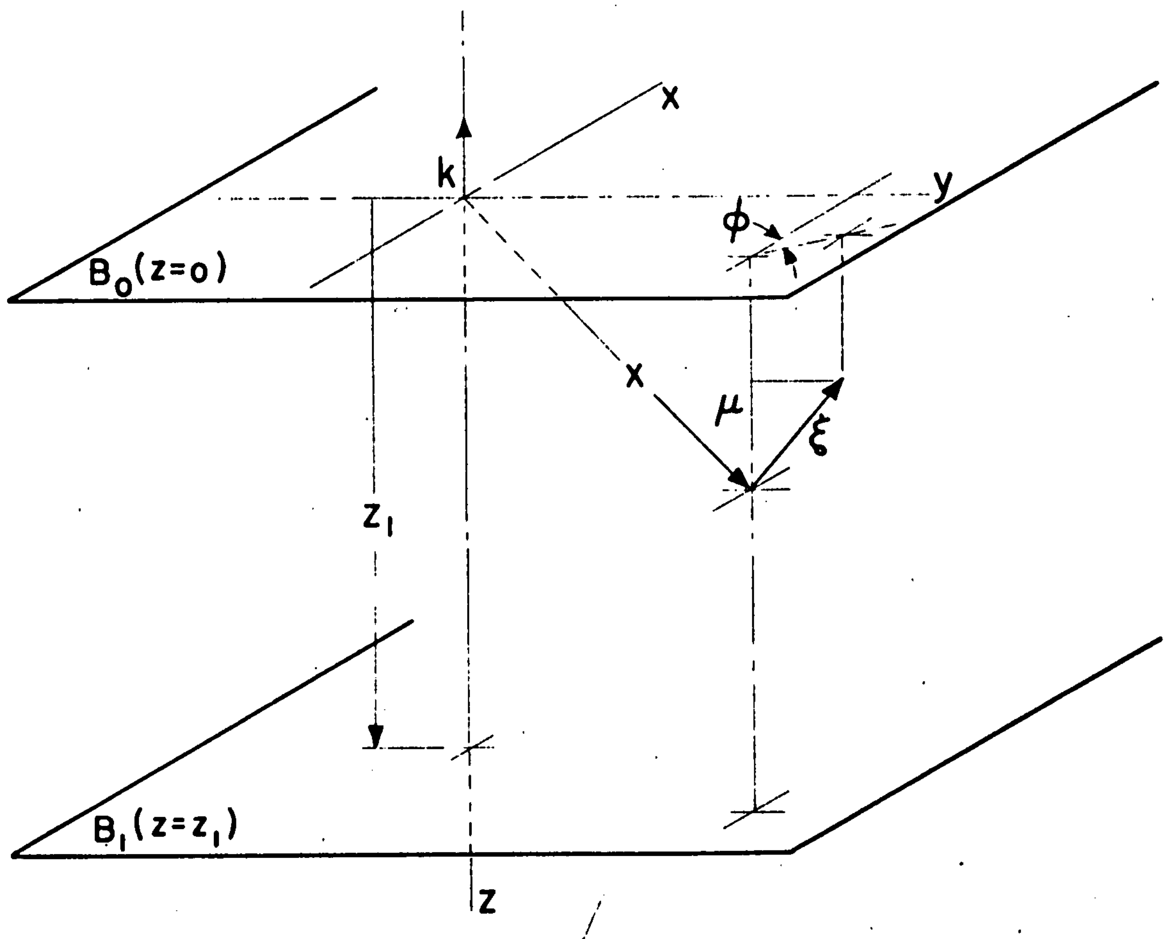

Figure 1. Rudolph W. Preisendorfer

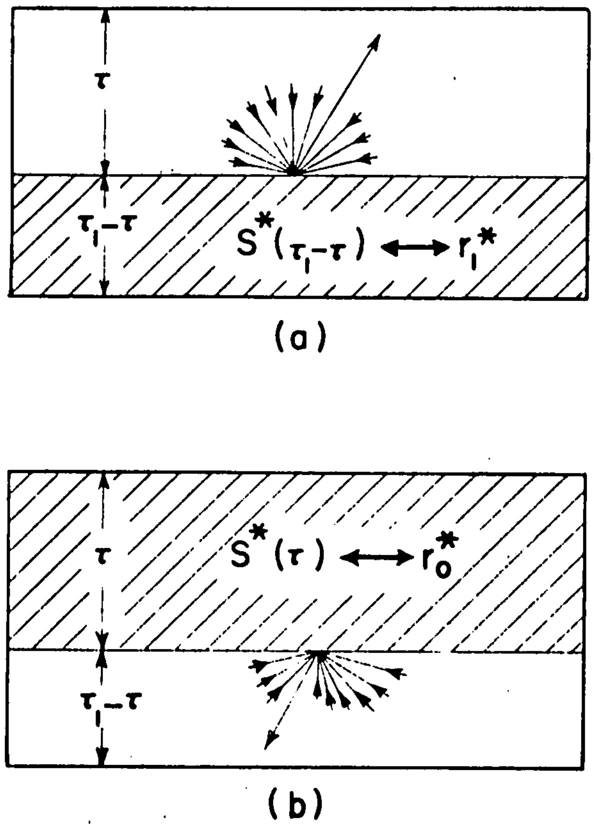

Figure 3. Rudolph W. Preisendorfer