Visibility Laboratory
University of California
Scripps Institution of Oceanography
San Diego 52, California

MODEL FOR RADIANCE DISTRIBUTIONS IN NATURAL HYDROSOLS

R. W. Preisendorfer

1 August 1957
Index Number NS 714-100

Bureau of Ships
Contract NObs-72092

SIO REFERENCE 58-42

Approved:

Seibert Q. Duntley

Seibert Q. Duntley, Director
Visibility Laboratory

Approved for Distribution:

Roger Revelle

Roger Revelle, Director
Scripps Institution of Oceanography

A Model for Radiance Distributions in Natural Hydrosols+

by Rudolph W. Preisendorfer

Scripps Institution of Oceanography, University of California, La Jolla, California

ABSTRACT

A simple model for radiance distributions in optically deep natural hydrosols is derived from the equation of transfer for radiance using a special form of the path function derived from the classical two-flow Schuster differential equations for irradiance. The model yields explicit expressions for the apparent radiance of submerged objects and the contrast transmittance for arbitrary paths of sight in the hydrosol. The model reproduces the major features of experimentally determined radiance distributions under a variety of external lighting conditions.

INTRODUCTION

The main problem of hydrological optics, and indeed of any branch of applied radiative transfer theory, is to determine by means of experiments and theory the values of the radiance distribution and optical functions at each point of the optical medium under study. Once these are known the important subclass of problems associated with the visibility of objects in natural hydrosols can be solved.

+Contribution from the Scripps Institution of Oceanography, New Series No.________.
This paper represents results of research which has been supported by the Bureau of Ships, U. S. Navy.

The relatively simple model of the radiance distribution presented below has for the past several years been useful in calculations of the apparent radiance and apparent contrast of objects submerged in optically deep natural hydrosols (oceans, harbors, rivers and lakes). The model allows an estimate of the radiance in a given direction (μ, ϕ) at a given depth z below the surface of the hydrosol, given the incident radiance distribution just below the surface and the attenuating coefficients α and σ (defined below).

The author was introduced to this subject in 1950 by Professor S. Q. Duntley at the Massachusetts Institute of Technology, who had two years earlier formulated the essential structure of the model described in this paper by setting up an equation of transfer for radiance for a vertical path which expressed his experimental finding that luminous density decreases exponentially with depth within uniform hydrosols. It was the author's assignment during a summer employment in Duntley's laboratory to study solutions of this equation in an endeavor to ascertain if the model led to expressions for the apparent radiance of submerged objects and for the reduction of apparent contrast along inclined paths of sight through uniform hydrosols which were in agreement with the exponential character of these phenomena which Duntley's 1948 experiments had disclosed. Interested readers should consult the recently declassified account of those experiments[1a] as well as a Visibility Laboratory report[1b] which summarizes the results of the combined theoretical and experimental program and includes the essential content of an unpublished laboratory note-book manuscript prepared by the author in 1950 and entitled "Luminance Attenuation along Oblique Lines of Sight in Natural Hydrosols". The present paper, which carries the theoretical model to completion and relates it to the basic constructs of radiative transfer theory, is a natural outgrowth of continued studies of hydrological optics by the author and his colleagues in the Visibility Laboratory throughout the intervening years.

note-book manuscript prepared by the author in 1950 and entitled "Luminance Attenuation along Oblique Lines of Sight in Natural Hydrosols". The present paper, which carries the theoretical model to completion and relates it to the basic constructs of radiative transfer theory, is a natural outgrowth of continued studies of hydrological optics by the author and his colleagues in the Visibility Laboratory throughout the intervening years.

Recent experimental determinations of the radiance distributions in the Lake Pend Oreille experiments by J. E. Tyler have confirmed the fact that the present model reproduces the major features of the radiance distributions at all depths from the surface down to 65 meters (the maximum depth attained in the experiments). These experimental findings will soon be published by Tyler.

In its broad outlines the formula resembles the classical Koschmieder expression for the apparent radiance of objects seen along horizontal paths of sight in the atmosphere[2]. The essential difference between the two stems from the assumed form of the path function along the path of sight. In the Koschmieder model the path function is assumed to be constant; in the present model the path function has an exponential structure which is derived from the solutions of the classical two-flow Schuster equations for irradiance.

[1a] Minutes and Proceedings of the Armed Forces—NRC Vision Committee, 23rd Meeting, 4-5 March 1949, pp. 123-126;

[1b] The Visibility of Submerged Objects, Final Report (Visiblity Laboratory, Massachusetts Institute of Technology, Cambridge, Massachusetts, August, 1952).

[2] H. Koschmieder, "Theorie der horizantalen Sichtweite," Beitr. Phys. Freien Atm. **12**, 33-53; **12**, 171-181 (1924).

The resemblance of the two apparent radiance functions is not accidental. It arises from the result of transforming the integro-differential equation for radiance to a linear first order differential equation for the radiance by assuming the path function to behave in a given manner along the path of sight. This has been done implicitly, for example, in the paper by Duntley on meteorological optics[3]. In fact, as is shown below, by assuming the path function to be known but of general character, the most general expression for the apparent radiance of a distant object can be obtained from a formal integration of the equation of transfer. This fact is the basis for the postulation of the general analytical form of apparent radiance of objects in the atmosphere, as given in an earlier paper[4].

In a certain sense the present formula is a first iterative solution of the equation of transfer, for it may be used to generate a new path function which in turn gives rise to a second iterative solution, and so on <u>ad infinitum</u>[5].

---

[3] S. Q. Duntley, "The Reduction of Apparent Contrast by the Atmosphere," J.Opt.Soc. Am 38, 179-191 (1948)

[4] S. Q. Duntley, A. R. Boileau, R. W. Preisendorfer, "Image Transmission by the Troposphere I," J. Opt. Soc. Am. 47, 499-506 (1957)

[5] Under suitable conditions, it may be shown that the resulting sequence of radiance functions converges to a radiance function which is the solution of the equation of transfer.

The fine-grained detail which the present model exhibits (uncommon for a first iteration) stems from the fact that the zeroth iteration is itself obtained from the solutions of the two-flow Schuster equations, which are actually modified forms of the equation of transfer whose solution is sought.

THE GENERAL APPARENT RADIANCE EXPRESSION

For the present purposes the equation of transfer for radiance may be written:

$$dN(z,\theta,\phi)/dr = -\alpha(z)N(z,\theta,\phi) + N_*(z,\theta,\phi)\,, \tag{1}$$

where N_* is the path function having the form:

$$N_*(z,\theta,\phi) = \int_0^{\pi}\int_0^{2\pi} \sigma(z;\theta,\phi;\theta',\phi')\,N(z,\theta',\phi')\sin\theta'\,d\theta'\,d\phi'\,. \tag{2}$$

Equation (1) may be written more briefly as:

$$dN/dr = -\alpha\cdot N + N_* \tag{1'}$$

or

$$dN/dr = \alpha(-N + N_q) \tag{1''}$$

where N_q is the equilibrium radiance (in the astrophysical context it is known as the source function). The function α is the volume attenuation function, σ is the volume scattering function. If, in addition, we designate by a the volume absorption function, and define

$$s(z) = \int_0^{\pi}\int_0^{2\pi} \sigma(z;\theta',\phi';\theta,\phi)\sin\theta'\,d\theta'\,d\phi' \tag{3}$$

as the total (volume) scattering function, then:

$$\alpha(z) = a(z) + s(z). \tag{4}$$

The form of (1) summarizes the present assumptions about the hydrosols under discussion: The hydrosols are in the steady state, with constant index of refraction, and are stratified, i.e., α, N, N_* depend only on z. The slab geometry is used and is depicted in Figure 1. Of central importance in what follows is the notion of a path in the hydrosol. A path is symbolically summarized by the quadruple (z_t, θ, ϕ, r). z_t is the depth of the initial (or target) point, θ and ϕ give direction of the path, and r is its geometrical length (or range). The terminal (or observation) point has depth $z = z_t - r\cos\theta$. Because the hydrosol is stratified, the inclusion of the x and y coordinates of the target and observation points is superfluous and has therefore been omitted. The radiance used in this paper is specific radiance, i.e., the radiant flux (of fixed wavelength λ) at depth z travelling in the direction (θ, ϕ) (Figure 1). Specific radiance is most useful in theoretical discussions and is the precise geophysical counterpart of the astrophysical specific intensity. Specific radiance has a natural experimental counterpart[6] in the field radiance which is defined operationally as follows: If $N(z, \theta, \phi)$ is the specific radiance at depth z in the direction (θ, ϕ), it is necessarily measured by pointing, at level z, a radiance meter (a Gershun tube[7]) in the direction $\pi - \theta$, $\phi + \pi$.

[6]Field radiance has been used implicitly, for example, in Reference 4.

[7]A. Gershun, "The Light Field," J. Math. Phys. 18, 51-151 (1939).

A formal solution of (1') is obtained by assuming N_* known so that (1') is in effect a linear first order differential equation of N along a path. The integrating factor is:

$$1/T_r(z,\theta,\phi) = exp\left\{\int_0^r \alpha(z')\,dr'\right\} = e^{\bar{r}}, \tag{5}$$

the integration being taken along a path (z_t , θ, ϕ, r) of indefinite length r. Physically, T_r(z, θ, ϕ) is the <u>beam transmittance</u> for the path, and $\bar{r}$ is the <u>optical range</u> of the path. (If $\theta = \pi$ and $z_t = 0$, $\bar{r}$ is the optical depth of the terminal point, usually denoted in general radiative transfer theory by τ). By means of the integrating factor, (1') becomes:

$$d(e^{\bar{r}}N)/dr = e^{\bar{r}}N_*, \tag{6}$$

which is immediately integrable over the path (z_t, θ, ϕ, r), the result being

$$e^{\bar{r}}N\Big|_0^r = \int_0^r e^{\bar{r}'}N_*\,dr'. \tag{7}$$

Let N evaluated at the target point of the path be designated by $N_o(z_t, \theta, \phi)$, and let N evaluated at the observation point of the path be designated by $N_r(z, \theta, \phi)$. Then (7) becomes, in abbreviated notation,

$$e^{\bar{r}}N_r - N_o = \int_0^r e^{\bar{r}'}N_*\,dr', \tag{8}$$

or
$$N_r = N_r^o + N_r^* \tag{9}$$

where
$$N_r^o = T_r N_o \tag{10}$$

and
$$N_r^* = \int_0^r T_{r-r'} N_* \, dr'. \tag{11}$$

The physical names attached to these quantities are as follows: N_o is the <u>inherent radiance</u>, N_r is the <u>apparent radiance</u> associated with (z_t, θ, ϕ, r). The expression (9) for N_r holds in quite general situations[8], and expresses the fact that the apparent radiance associated with a path of sight may be written as the sum of a <u>transmitted inherent radiance</u> N_r^o and a <u>path radiance</u> N_r^*, the former consisting of radiant flux transmitted from the initial point without having suffered scattering or absorption, the latter consisting of radiant flux having been first scattered into the direction (θ, ϕ) at each point along (z_t, θ, ϕ, r) and then transmitted to the observation point. Mathematically, of course, (9) arises from the formal solution of the equation of transfer. Whenever the explicit dependence of the various values of the radiance function on depth z and direction (θ, ϕ) must be shown, (9), (10), and (11) are written out in full:

$$N_r(z,\theta,\phi) = N_r^o(z,\theta,\phi) + N_r^*(z,\theta,\phi), \tag{9'}$$

---

[8] For meteorological optics context see Reference 4.

$$N_r^{\circ}(z,\theta,\phi) = T_r(z,\theta,\phi)\, N_o(z_t,\theta,\phi)\,, \tag{10'}$$

$$N_r^{*}(z,\theta,\phi) = \int_0^r T_{r-r'}(z',\theta,\phi)\, N_*(z',\theta,\phi)\, dr'\,, \tag{11'}$$

and

$$z' = z_t - r\cos\theta\,, \tag{12}$$

otherwise the abbreviated notation will be used.

THE MODEL FOR THE RADIANCE DISTRIBUTION

For horizontal paths of sight ($z_t, \pi/2, \phi, r$) in natural hydrosols (and aerosols), it is known that the values of the path function N_* and the values of α are essentially constant along ($z_t, \pi/2, \phi, r$) for appreciable ranges r. In this case the beam transmittance is of the form:

$$T_r(z,\pi/2,\phi) = e^{-\alpha r}\,, \tag{13}$$

and the path radiance has the form:

$$N_r^{*}(z,\pi/2,\phi) = N_q(z,\pi/2,\phi)\left(1 - e^{-\alpha r}\right), \tag{14}$$

where $$N_q(z, \pi/2, \phi) = N_*(z, \pi/2, \phi)/\alpha \tag{15}$$

is the equilibrium radiance associated with (z_t, $\pi/2$, ϕ, r), and in this case, is constant along the path. In this special case, the formula for N_r takes on a particularly simple form:

$$N_r(z, \pi/2, \phi) = N_o(z, \pi/2, \phi)e^{-\alpha r} + N_q(z, \pi/2, \phi)(1 - e^{-\alpha r}), \tag{16}$$

or,

$$N_r = N_o e^{-\alpha r} + N_q(1 - e^{-\alpha r}), \tag{16'}$$

which is the classical expression for apparent radiance given by Koschmieder.

Now let α and σ be independent of z, then in the appendix it is shown that an approximate form for $N_*(z, \theta, \phi)$ can be derived by means of the solutions of the two-flow Schuster equations for irradiance. The expression derived is:

$$N_*(z, \theta, \phi) = N_*(0, \theta, \phi)\, e^{-kz} \quad \left\} \begin{array}{l} z \geq 0 \\ 0 \leq \theta \leq \pi \\ 0 \leq \phi < 2\pi, \end{array} \right. \tag{17}$$

where the forms of k and $N_*(0, \theta, \phi)$ are defined during the derivation of (17). N_* in (17) arises from an assumed "zeroth" approximation to N. N_* in turn is used to find a "first" approximation to N in the following way:

Let (z_t, θ, ϕ, r) be a given path in the hydrosol (i.e., $z_t \geq r\cos\theta$, $\theta \leq \pi/2$). Then under the above conditions on N_* and α, the path radiance associated with (z_t, θ, ϕ, r) is:

$$N_r^*(z,\theta,\phi) = N_*(0,\theta,\phi)\int_0^r e^{-\alpha(r-r')}e^{-kz'}\,dr'$$

$$= \left[N_*(z,\theta,\phi)/(\alpha + k\cos\theta)\right]\left[1 - e^{-(\alpha + k\cos\theta)r}\right]. \tag{18}$$

Hence the apparent radiance has the form:

$$N_r(z,\theta,\phi) = N_0(z_t,\theta,\phi)e^{-\alpha r} + \left[N_*(z,\theta,\phi)/(\alpha + k\cos\theta)\right]\left[1 - e^{-(\alpha + k\cos\theta)r}\right]. \tag{19}$$

SOME PROPERTIES OF THE MODEL

1. The Koschmieder Case. It should be observed first of all that (19) reduces to (16) for $\theta = \pi/2$, and if in addition, $r = \infty$, we have

$$N_\infty(z,\pi/2,\phi) = N_q(z,\pi/2,\phi). \tag{20}$$

2. The Radiance Distribution. For paths of sight of the form $(z_t, \theta, \phi, \infty)$ in optically infinitely deep hydrosols with $\theta \leq \pi/2$, (19) reduces to:

$$N_\infty(z,\theta,\phi) = N_*(z,\theta,\phi)/(\alpha + k\cos\theta) \left.\begin{matrix} z \geq 0 \\ \theta \leq \pi/2. \end{matrix}\right\} \tag{21}$$

For paths of sight of the form $(0, \theta, \phi, r)$ with $\theta > \pi/2$, (19) reduces to:

$$N_r(z,\theta,\phi) = N_0(0,\theta,\phi)\, e^{-\alpha r} + N_r^*(z,\theta,\phi) \left.\vphantom{\begin{matrix}a\\b\\c\end{matrix}}\right\} \begin{matrix} z \geq 0 \\ r = -z\sec\theta \\ \theta > \pi/2 . \end{matrix} \tag{22}$$

(21) and (22) together constitute the present model for the radiance distribution values $N(z, \theta, \phi)$, $0 \leq \theta \leq \pi$, $0 \leq \phi < 2\pi$ at each depth z in the natural hydrosol, given α, σ (and hence k) and incident radiance distribution $N_0(0, \theta, \phi)$, $\theta > \pi/2$, $0 \leq \phi < 2\pi$.

3. <u>The Asymptotic Radiance Distribution</u>. Let z, the depth of the terminal point of the path (z_t, θ, ϕ, r) be large so that in (19) we have, essentially,

$$\left.\begin{matrix} 1 - e^{-(\alpha + k\cos\theta) r} = 1 \\ N_0(0,\theta,\phi)\, e^{-\alpha r} = 0 \end{matrix}\right\} \begin{matrix} \theta > \pi/2 \\ r = -z\sec\theta . \end{matrix} \tag{23}$$

Then, in view of (21) and (22), we have for all $0 \leq \theta \leq \pi$, $0 \leq \phi < 2\pi$ (dropping the subscript r)

$$N(z,\theta,\phi) = N_*(z,\theta,\phi) / (\alpha + k\cos\theta) . \tag{24}$$

If in particular σ is independent of $(\theta, \phi; \theta', \phi')$ (i.e., scattering is isotropic) then by (2) and (3),

$$N_*(z,\theta,\phi) = (s/4\pi)\, h(z) , \tag{25}$$

where $h(z)$ is the scalar irradiance at depth z :

$$h(z) = \int_0^{\pi}\int_0^{2\pi} N(z,\theta',\phi')\sin\theta'\,d\theta'\,d\phi'. \tag{26}$$

By (10) , evidentally, $h(z) = h(0)\,e^{-kz}$, (27)

so that (24) may be written

$$N(z,\theta,\phi) = c\,e^{-kz}/(\alpha + k\cos\theta), \tag{28}$$

where $c = s\,h(0)/4\pi$. This indicates that for large depths z in a plane-parallel medium which scatters isotropically (and which has an incident radiance distribution at the upper boundary and no additional sources) the radiance distribution $N(z, \;, \;)$ when plotted has the form of an ellipsoid of revolution with eccentricity k/α, $0 \leq k/\alpha < 1$. It turns out that (28) is actually a formal solution of the equation of transfer under the above conditionsand for a suitable k. This may be seen by direct substitution of (28) into the equation of transfer.[9]

---

[9] In this connection see, e.g., S. Glasstone and C. Edlund, The Elements of Nuclear Reactor Theory (D. Van Nostrand Co., New York, 1952), P. 395; and S. Chandrasekhar, Radiative Transfer (Clarendon Press, Oxford, 1950), P.18.

4. The Non-Linear Features of N_r. The special structure of (19) namely the additive combination of two terms involving different exponential functions implies that, for each path $(0, \theta, \phi, r)$ with fixed $\theta > \pi/2$ and ϕ, a semi-log plot of $N_r(z, \theta, \phi)$ with respect to, say r, would in general be non-linear. It turns out that, under suitable conditions, (given below) N_r increases, reaches a maximum, and then decreases indefinitely. The value r_m at which a maximum is attained may be found from (19) by the usual calculus rules. The predicted value of r_m is:

$$r_m(\theta, \phi) = z_m(\theta,\phi)\sec\theta = -(\alpha + k\cos\theta)^{-1} \ln\left\{-\alpha k^{-1}\sec\theta\left[1-(\alpha+k\cos\theta)\frac{N_0(0,\theta,\phi)}{N_*(0,\theta,\phi)}\right]\right\}. \tag{29}$$

where $z_m(\theta, \phi)$ is the depth corresponding to $r_m(\theta, \phi)$. Clearly, the requirement that $r_m(\theta, \phi) > 0$ (i.e., that $r_m(\theta, \phi)$ exist as a positive real number) is that the argument of the logarithm in (29) be greater than unity. This yields, after some manipulation, the equivalent requirement that

$$N_*(0,\theta,\phi)/\alpha > N_0(0,\theta,\phi), \quad \theta > \pi/2, \tag{30}$$

i.e., that

$$N_q(0,\theta,\phi) > N_0(0,\theta,\phi). \tag{31}$$

Now this is simply the usual requirement that the equilibrium radiance N_q at a point on a path exceed the actual radiance N at the point in order that should increase locally along the path. That is, from the equation of transfer in the form (1"),

$$dN/dr > 0 \quad \text{if} \quad N_q > N ,$$

$$dN/dr < 0 \quad \text{if} \quad N_q < N . \tag{32}$$

We observe parenthetically that, on the other hand, for paths with $\theta \leq \pi/2$, it is clear from (21) that N_r plotted against depth z will have a linear semi-log plot with slope $-k$.

Condition (31) occurs frequently. For example, it occurs when the sky is clear with a bright sun and when $N_o(0, \theta, \phi)$ is associated with a point in the sky not on the sun's disk. On the other hand, if the sky is heavily overcast, condition (31) may not be realized and consequently r_m would not exist. The experiments referred to in the introduction have verified the predicted values of r_m under sunny sky conditions, and have verified the conclusion that r_m may not exist for overcast skies; in such cases (31) did not hold. The predicted linearity of $N_\infty(z, \theta, \phi)$, $\theta \leq \pi/2$, $z \geq 0$ with the slope $-k$ was also essentially verified.

5. <u>Contrast and Contrast Transmittance</u>. Let (z_t, θ, ϕ, r) be a path in the hydrosol, and let a target of inherent radiance ${}_tN_o(z_t, \theta, \phi)$ be placed at the initial point of the path. Further, designate by ${}_bN_o(z_t, \theta, \phi)$ the inherent radiance of the background of the target. Let ${}_tN_r(z, \theta, \phi)$

and $_bN_r(z, \theta, \phi)$ be the apparent radiances of target and background at the observation point of the path. Then the <u>inherent</u> and <u>apparent contrasts of the target with respect to its background</u> are defined, respectively, as:

$$C_o(z_t, \theta, \phi) = \left[{}_tN_o(z_t, \theta, \phi) - {}_bN_o(z_t, \theta, \phi)\right] / {}_bN_o(z_t, \theta, \phi) \quad (33)$$

$$C_r(z, \theta, \phi) = \left[{}_tN_r(z, \theta, \phi) - {}_bN_r(z, \theta, \phi)\right] / {}_bN_r(z, \theta, \phi) \,. \quad (34)$$

These definitions are general in the sense that their formulation requires no restrictive assumptions on the nature of the radiance function in the optical medium. The <u>contrast transmittance</u> of the path (z_t, θ, ϕ, r) is defined as

$$C_r(z, \theta, \phi) / C_o(z_t, \theta, \phi) , \quad (35)$$

and by means of (9), (33) and (34) may be expressed as

$$C_r(z, \theta, \phi) / C_o(z_t, \theta, \phi) = T_r(z, \theta, \phi)\, {}_bN_o(z_t, \theta, \phi) / {}_bN_r(z, \theta, \phi) . \quad (36)$$

Using the present model of the radiance distribution, $_bN_o$ and $_bN_r$ are given by (21) and (22) under the assumption that the presence of the target at the initial point does not perturb the radiance of the background. Thus (36) becomes:

$$C_r(z,\theta,\phi)/C_o(z_t,\theta,\phi) = \begin{cases} e^{-(\alpha+k\cos\theta)r}, & \theta \leq \pi/2 \\ e^{-\alpha r} \cdot \dfrac{N_{-z_t\sec\theta}(z_t,\theta,\phi)}{N_{-z\sec\theta}(z,\theta,\phi)}, & \theta > \pi/2 . \end{cases} \qquad (37)$$

If the asymptotic radiance distribution conditions (23) hold for both the depths z and z_t for each $\theta > \pi/2$, then (37) reduces to:

$$C_r(z,\theta,\phi)/C_o(z_t,\theta,\phi) = e^{-(\alpha+k\cos\theta)r} \left\{ \begin{array}{l} r = (z_t - z)\sec\theta \\ 0 \leq \theta \leq \pi \\ 0 \leq \phi < 2\pi . \end{array} \right. \qquad (38)$$

APPENDIX

The expression for $N_*(z, \theta, \phi)$ given in (17) is derived from the solution of the two-flow Schuster equation applied to an optically infinitely deep hydrasol as follows:

First:
$$N_*(z,\theta,\phi) = \int_0^{\pi/2}\int_0^{2\pi} \sigma(\theta,\phi;\theta',\phi')\, N(z,+) \sin\theta' d\theta' d\phi'$$
$$+ \int_{\pi/2}^{\pi}\int_0^{2\pi} \sigma(\theta,\phi;\theta',\phi')\, N(z,-) \sin\theta' d\theta' d\phi',$$

where, in accordance with the classical hypothesis of the Schuster theory, $N(z, \theta, \phi) = N(z, +) =$ constant for $\theta \leq \pi/2$, $N(z, \theta, \phi) = N(z, -) =$ constant for $\theta > \pi/2$ at each depth $z \geq 0$. Define

$$s_+(\theta,\phi) = \int_0^{\pi/2}\int_0^{2\pi} \sigma(\theta,\phi;\theta',\phi') \sin\theta' d\theta' d\phi'$$
$$s_-(\theta,\phi) = \int_{\pi/2}^{\pi}\int_0^{2\pi} \sigma(\theta,\phi;\theta',\phi') \sin\theta' d\theta' d\phi'.$$

Then

$$N_*(z,\theta,\phi) = s_+(\theta,\phi) N(z,+) + s_-(\theta,\phi) N(z,-), \quad z \geq 0, \quad \begin{array}{l} 0 \leq \theta \leq \pi \\ 0 \leq \phi < 2\pi. \end{array}$$

Let $H(z, +)$ and $H(z,-)$ be the irradiances induced by $N(z,+)$ and $N(z,-)$ respectively. Hence $H(z,\pm) = \pi N(z, \pm)$.

Solving the Schuster equations for the pair $(H(z,+), H(z,-))$, we have:

$$H(z,\pm) = H(0,\pm) e^{-kz},$$

where $k = 2[a(a+b)]^{1/2}$, a is the volume absorption coefficient and b is the back scattering coefficient which occurs in the Schuster theory. Hence:

$$N_*(z,\theta,\phi) = [s_+(\theta,\phi) N(0,+) + s_-(\theta,\phi) N(0,-)] e^{-kz}$$
$$= N_*(0,\theta,\phi) e^{-kz}, \quad z \geq 0, \quad \begin{array}{l} 0 \leq \theta \leq \pi \\ 0 \leq \phi < 2\pi. \end{array}$$

CAPTIONS

Figure 1. Illustrating the slab geometry and the distinction between specific radiance and field radiance.

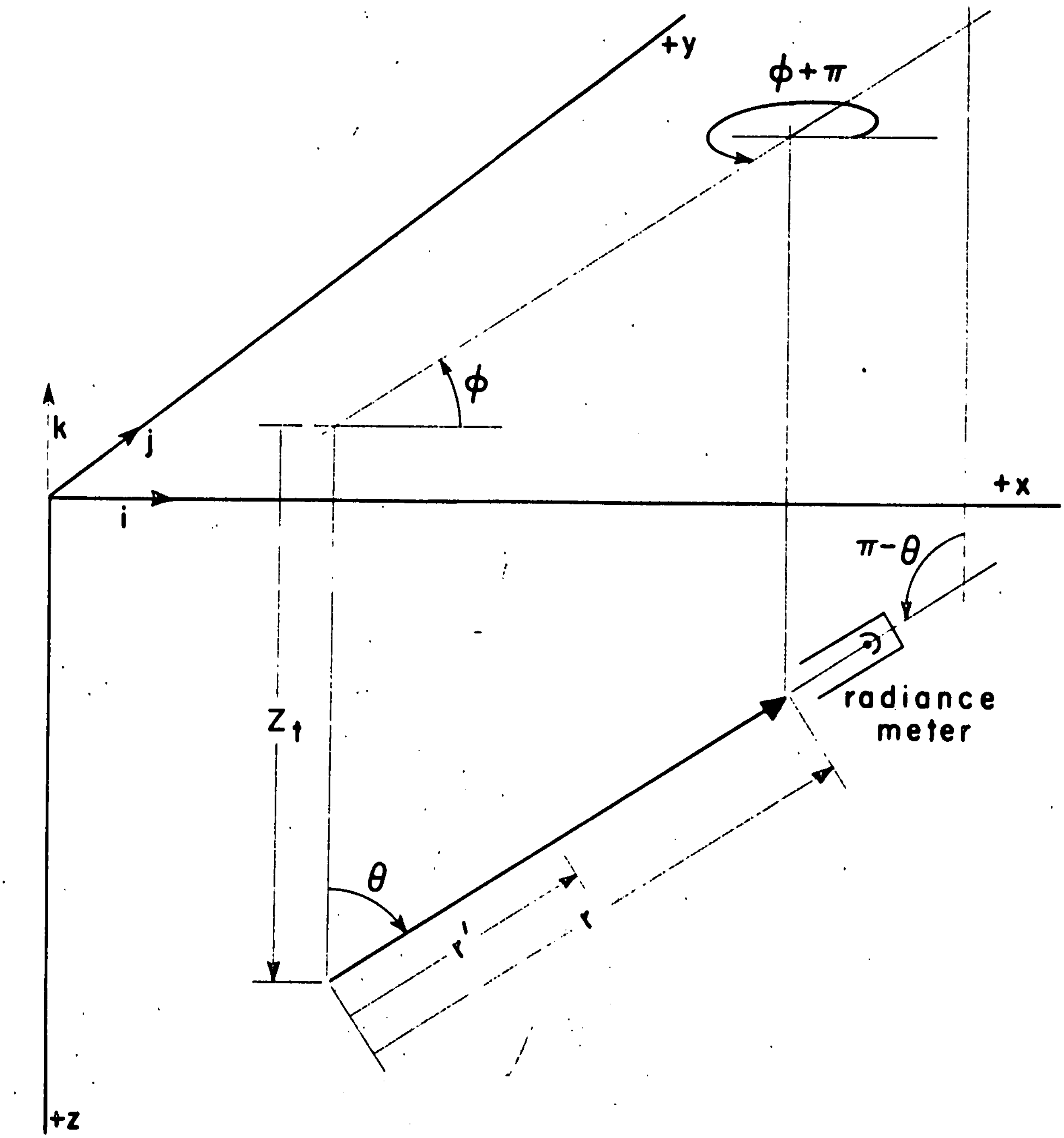

Figure 1. Rudolph W. Preisendorfer