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SOME MATHEMATICAL ASPECTS OF GEOPHYSICAL OPTICS CONCEPTS

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SOME MATHEMATICAL ASPECTS OF GEOPHYSICAL OPTICS CONCEPTS

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1. Optical Medium

An optical medium is a quintuple $(M, N, \alpha, \sigma, n)$. $M = X \times \Xi$. That is, M is a cartesian product of two sets: X, and Ξ. X is a subset of euclidean 3-space, where the latter is the collection of all triples $(x_1, x_2, x_3) = \underline{x}$, x_1, x_2, x_3 being real numbers. The triple $\underline{x}$ may be considered a vector, and may be assigned the length $(x_1^2 + x_2^2 + x_3^2)^{1/2}$. Ξ is the collection of all vectors which have unit length. X is the location space; members of X are location vectors. Ξ is the direction space, members of Ξ are direction vectors. We shall use $\underline{\xi}$ as the general symbol for an element of Ξ. A point in M is a pair $(\underline{x}, \underline{\xi})$. N is a function defined on M with values in the non negative subset of the real number system. N is the radiance function. The value of N at $(\underline{x}, \underline{\xi})$ is written as $N(\underline{x}, \underline{\xi})$ and is called the radiance at $(\underline{x}, \underline{\xi})$. The restriction of N to a point $\underline{x}$ in X is denoted by $N(\underline{x}, \ \cdot)$ and is called the radiance distribution at $\underline{x}$. (Technically, this restriction is called the $\underline{X}$-section of N determined by $\underline{x}$ in X.) α is a function

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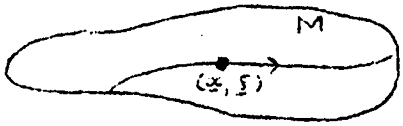
defined on X with values in the non negative subset of the real number system. α is the (volume) attenuation function. The value of α at $\underline{x}$ is written as $\alpha(\underline{x})$. σ is a function defined on $X \times \Xi \times \Xi$ with non negative real values. α is the (volume) scattering function. The value of σ at $(\underline{x}, \underline{\xi}', \underline{\xi})$ is written as $\sigma(\underline{x}, \underline{\xi}', \underline{\xi})$. n is defined on X with values in the real number interval $[1,\infty)$. n is the index of refraction function. The value of n at $\underline{x}$ is written $n(\underline{x})$, and is called the index of refraction of M at $\underline{x}$.

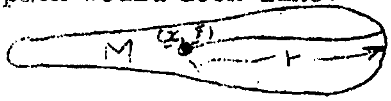
> (Scattering and absorption isotropy are implied by the notation for α. Isotropy (in the geometrical optical sense) is implied by the notation for n. Further assumptions about M such as unpolarized flux and time independence of N are implicit in the notation. Fixed wavelengths for N are implied.)

A ray through $(\underline{x}, \underline{\xi})$ in M is the (unique) set of points in M defined by the solution of the Euler equations of motion for photons in M with boundary condition given by $(\underline{x}, \underline{\xi})$. In other words, a ray is the natural trajectory taken by photons in M. A path is the subset $(\underline{x}, \underline{\xi}, r)$ of a ray comprised of all points at a distance less than or equal to r from $\underline{x}$, as measured along the ray and in the direction $\underline{\xi}$ from $\underline{x}$.

> (A more precise, but perhaps even less intelligible definition is possible. For the present, this definition will suffice. A diagram of a ray would look like:



> A diagram of a path would look like:



> A ray extends indefinitely on either "side" of $(\underline{x}, \underline{\xi})$. A path is a sensed curve through $(\underline{x}, \underline{\xi})$ of fixed length.)

If n is a constant function, rays in M are straight lines in M. In this case, the notation $(\underline{x}, \underline{\xi}, r)$ for paths in M is especially convenient: Any point in the path is uniquely determined by the form $\underline{x} + r' \underline{\xi}$, $0 \leq r' \leq r$. We call $\underline{x}$ the observation point, and $\underline{x}' = \underline{x} + r \underline{\xi}$ the target point; these definitions are to hold also in the general case, the target point being denoted by $\underline{x}'$. At times it will be convenient to replace $(\underline{x}, \underline{\xi}\ r)$ by the symbol p.

2. Principle of Relative Scattering Order (porso); Inherent Radiance.

The scattering order of a given sample of radiant flux may be arbitrarily fixed. The subsequent history of the sample as regards its scattering order is uniquely determined relative to the fixed order by means of the scattering properties of M as summarized by σ.

> (This will be illustrated later. It is assumed here what is meant by "scattering order," "radiant flux," and "given sample," though each may be defined without difficulty. The principle exploits the experimental indeterminateness of scattering order.)

Using the porso (usually implicitly) one derives the integral equation of transfer for N. This derivation may be made from first principles, or using the formal solution of the integrodifferential equation for N. In the customary representation of the equation, N is equated to two terms: N^o, and N^*. The former term is associated with the "o" group of photons comprising N. This group is presumed to have arrived at $(\underline{x}, \underline{\xi})$ without having suffered any scattering actions by M. The latter term is associated with the "*" group of photons comprising N. This group is presumed to have arrived at $(\underline{x}, \underline{\xi})$ only after having suffered at least one scattering action by M. The symbolic equation, $N = N^{0} + N^{*}$ represents the general decomposition of N into two components: the unscattered, and scattered component respectively. Specific examples may be exhibited which demonstrate that this decomposition is not unique. This non-unicity stems from the use of the porso. For once non-unicity is a blessing rather than an evil: it reflects (as will be demonstrated later) the freedom of choice of "inherent radiances," and thus the freedom of representation of "apparent radiances." From what has been developed so far, we may state the formal definition of inherent radiance: $N(\underline{x}, \underline{\xi})$ at $(\underline{x}, \underline{\xi})$ is an *inherent radiance* if and only if the porso has been invoked to assign 0 scattering order to $N(\underline{x}, \underline{\xi})$; this assignation is denoted by $N^{0}(\underline{x}, \underline{\xi})$.

(Webster's New Collegiate Dictionary defines "inherent" as, to be a fixed element or attribute" . . . "belonging by nature or settled habit." The choice of the term "inherent" in the light of the porso seems well-made.

We take the opportunity in this parenthetical pause to point up the existence of two classes of radiances: field radiances, and specific radiances. The radiance $N(\underline{x}, \underline{\xi})$ is a *specific radiance* (in the 1953 lecture notes, called surface radiance) if the photon flow is in the direction of $\underline{\xi}$. $N(\underline{x}, \underline{\xi})$ is a *field radiance* if the photon flow is in the direction $-\underline{\xi}$. Their numerical values at any point in M are of course equal; the use of one or the other is simply a matter of conceptual convenience and individual preference. The specific radiance is the geophysical counterpart to the astrophysical specific intensity, and in each branch appears to be preferred when a lagrangian analysis of the history of an ensemble of photons is made. Field radiance arises when a Gershun tube is pointed in the direction $\underline{\xi}$ and photons are funneled down the tube in the direction $-\underline{\xi}$ to register their presence over the photosensitive device at the base of the tube. The integro-differential equation, for example, is most easily *derived* using specific radiance. The *interpretation* of the *integral* equation of transfer more conveniently uses field radiance.)

3. Transmittance and Scattering Operators.

Let $p = (\underline{x}, \underline{\xi}, r)$ be a path in M. We may associate with p a real number $T_r(\underline{x}, \underline{\xi}) = \exp[-\int_p \alpha(\underline{x}'')\, dr']$. The integral is a line integral of α along p. T is the transmittance operator, and for any path, $T_r(\underline{x}, \underline{\xi})$ is the beam transmittance of p. We may use T_r for short if $\underline{x}$ and $\underline{\xi}$ are understood, i.e., if p is understood. If n is a constant function on M, the line integral may be written more explicitly: $T_r(\underline{x}, \underline{\xi}) = \exp[-\int_{\underline{x}}^{\underline{x} + r \underline{\xi}} \alpha(\underline{x} + r' \underline{\xi})\, dr']$. If, in addition to the assumption of the constancy of n, we assume α constant over p, then the form for the beam transmittance is even simpler: $T_r(\underline{x}, \underline{\xi}) = \exp[-\alpha r]$, where α denotes the constant value of the attenuation function over p. We observe that n need not be constant in order to deduce this simple form: the constancy of α on p suffices to deduce the simple exponential form of the beam transmittance of p.

Let $p = (\underline{x}, \underline{\xi}, r)$ be a path in M, let $(\underline{x}, \underline{\xi})$, and $(\underline{x}', \underline{\xi}')$ be observation and target points of p respectively. Using the porso, let $N^o(\underline{x}', \underline{\xi}')$ be the inherent radiance at the target point. Then $T_r(\underline{x}, \underline{\xi}) N^o(\underline{x}', \underline{\xi}')$ is the transmitted radiance of the target point (as seen) at the observation point of p. We may write the transmitted radiance as $N_r^o(\underline{x}, \underline{\xi})$, or N_r^o for short when p is understood. Thus, $N_r^o(\underline{x}, \underline{\xi}) = T_r(\underline{x}, \underline{\xi})\, N^o(\underline{x}', \underline{\xi}')$, or $N_r^o = T_r N^o$ for short.

If Ω is a subset of Ξ, we associate with the triple $(\underline{x}, \underline{\xi}, \Omega)$ the scattering operator $S_\Omega(\underline{x}, \underline{\xi})$, defined by, $S_\Omega(\underline{x}, \underline{\xi}) = \int [\cdot]\, \sigma(\underline{x}, \underline{\xi}', \underline{\xi})\, d\omega(\underline{\xi}')$. This operator has as domain the class of all radiance functions on M. In particular it operates on the radiance distribution $N(\underline{x}, \cdot)$ to yield a function $N_\Omega(\underline{x}, \cdot)$ on Ξ, which is the X-section determined by $\underline{x}$ of a function N_Ω defined on M, i.e., $S_\Omega(\underline{x}, \underline{\xi})\,[N(x, \cdot)] = \int_\Omega N(\underline{x}, \underline{\xi}')\sigma(\underline{x}, \underline{\xi}'; \underline{\xi})\, d_\omega(\underline{\xi}') = N_\Omega(\underline{x}, \underline{\xi})$. The triple $(\underline{x}, \underline{\xi}, \Xi)$ is of central importance and the most frequently used. In this case we will write $S_*(\underline{x}, \underline{\xi})\,[N(\underline{x}, \underline{\cdot})] = N_*(\underline{x}, \underline{\xi})$. Physically, N_* assigns a radiance per unit length to the ray through $(\underline{x}, \underline{\xi})$ at this point in M. N_* is the path function on M. If the radiance distribution is assigned a scattering order m at $\underline{x}$, the scattering operator $S_\Omega(\underline{x}, \underline{\xi})$ assigns a scattering order m + 1 to $N_\Omega(\underline{x}, \underline{\xi})$.

4. Path Radiance.

Let $p = (\underline{x}, \underline{\xi}, r)$ be a path in M. The radiance, $\int_p T_r{}'(\underline{x}, \underline{\xi})N_*(\underline{x}'', \underline{\xi}'')\, dr$ is the path radiance of p. If n is constant on M, we may write more explicitly, $\int_{\underline{x}}^{\underline{x}+r\underline{\xi}} T_{r'}(\underline{x}, \underline{\xi})N_*(\underline{x} + r'\underline{\xi}, \underline{\xi})\, dr'$. We designate the path radiance by the symbol $N_r^*(\underline{x}, \underline{\xi})$, or N_r^* for short if p is understood. If N along p is assigned the scattering order m, then N_r^* has the scattering order m + 1.

5. Apparent Radiance.

Suppose N is a radiance function on M and $p = (\underline{x}, \underline{\xi}, r)$ is a path in M. Then N assigns to $(\underline{x}, \underline{\xi})$ and $(\underline{x}', \underline{\xi}')$ (the observation and target points of p) the radiances $N(\underline{x}, \underline{\xi})$ and $N(\underline{x}', \underline{\xi}')$ respectively. Now we use the porso to assign 0 relative scattering order to $N(\underline{x}', \underline{\xi}')$, and write $N^0(\underline{X}', \underline{\xi}')$ Second, we use the transmittance operator to obtain the transmitted radiance $N_r^0(\underline{x}, \underline{\xi}) = T_r(\underline{x}, \underline{\xi})N^0(\underline{x}', \underline{\xi}')$. Third, the path radiance $N_r^*(\underline{x}, \underline{\xi})$ of p is determined. It then follows from the integral equation for N that $N(\underline{x}, \underline{\xi}) = N_r^0(\underline{x}, \underline{\xi}) + N_r^*(\underline{x}, \underline{\xi})$. If $p_1 = (\underline{x}, \underline{\xi}, r_1)$ is another path, repeating the above three steps would yield a different decomposition of the value $N(\underline{x}, \underline{\xi})$: $= N^{o}_{r_1}(\underline{x}, \underline{\xi}) + N^{*}_{r_1}(\underline{x}, \underline{\xi})$. This illustrates the earlier remark that the decomposition of values of N into the "o" and "*" groups is not unique. To indicate the particular decomposition under study, it is then recommended to append an "r" to N, and write $N_r(\underline{x}, \underline{\xi}) = N^{o}_{r}(\underline{x}, \underline{\xi}) + N^{*}_{r}(\underline{x}, \underline{\xi})$. Whenever N is decomposed in this way, we shall indicate it verbally by saying that $N_r(\underline{x}, \underline{\xi})$ is the <u>apparent radiance</u> of the target point (as seen) at the observation point of the path $(\underline{x}, \underline{\xi},$ Apparent radiance is always--by definition--associated with a given path in M. The symbol r in the abbreviated notation is the vestigal reminder of this important fact.

(At any point $(\underline{x}, \underline{\xi})$ in an optical medium M there is a path $(\underline{x}, \underline{\xi}, r)$ such that $\underline{x}'$ is on the boundary of M. Therefore, r may or may not be infinite. We may associate with $(\underline{x}', \underline{\xi}')$ an inherent radiance. It follows that the values of the radiance function in any medium may be interpreted as the apparent radiance of the boundary of the medium.)

6. N^{o}, N^{o}_{o}, and N_{o}.

For any path $(\underline{x}, \underline{\xi}, r)$ in M such that $r = 0$, we have $T_o(\underline{x}, \underline{\xi}) = 1$, and $N^{*}_{o}(\underline{x}, \underline{\xi}) = 0$. It follows from the expression for apparent radiance that $N^{o} = N^{o}_{o} = N_{o}$. In this case the symbols may be used interchangeably in numerical computations. For a path $(\underline{x}, \underline{\xi}, 0)$, however, their interpretations remain distinct: $\underset{N_o}{\text{(apparent radiance)}} = \underset{N^{o}_{o}}{\text{(transmitted radiance)}} = \underset{N^{o}}{\text{(inherent radiance)}}$.

It seems possible to employ the following rule governing the use of N_c and N^o: N^o shall be used as an inherent radiance being prepared for an intended transmission (i.e., an application of T_r); this leaves room for the resulting r subscript. N_c shall be used if no use of T_r is contemplated (e.g., in the definition and manipulations of inherent contrast requiring no applications of T_r).

[illegible] Contrast.

If $(\underline{x}, \underline{\xi})$ is a point in M, the background G of $\underline{\xi}$ is the set of all $\underline{\xi}'$ in Ξ such that $\underline{\xi}' \neq \underline{\xi}$. In general, the background of a set A of unit vectors is the set $G = \Xi - A$. If $(\underline{x}_1, \underline{\xi}_1)$ and $(\underline{x}_2, \underline{\xi}_2)$ are any two points of M, the real number $(N(\underline{x}_1, \underline{\xi}_1)/N(\underline{x}_2, \underline{\xi}_2))-1$ is the simple contrast of $N(\underline{x}_1, \underline{\xi}_1)$ with respect to $N(\underline{x}_2, \underline{\xi}_2)$. If $(\underline{x}, \underline{\xi})$ is a point of M, and $[\underline{\xi}_i]$ is a sequence of vectors in the background of $\underline{\xi}$ which converges to $\underline{\xi}$, then the limit, if it exists, of the sequence $[(N(\underline{x}, \underline{\xi})/N(\underline{x}, \underline{\xi}_i))-1]$ of simple contrasts is said to be the $[\underline{\xi}_i]$ - contrast of $N(\underline{x}, \underline{\xi})$ with respect to its background.

(It is not hard to concoct an example which yields contrast depending upon the choice of sequence used to approach the direction ξ. The complicated formulation of a (sequential) - contrast reflects the generally complicated problem of determining what one means by the contrast of a point or extended object in an actual scene when viewed against a non-uniform and non-continuous radiance background. The time is soon to come when we will reach the cross-road in the matter of the definition of contrast. As it appears now, at the cross-road at least one of two sacrifices must be made: (a) Sacrifice logical clarity; retain the intuitively simple but ambiguous notion of contrast. (b) Sacrifice the appealing simple and intuitive notion; retain logical clarity. Since the notion of contrast is designed primarily as the measure of the detectibility of the radiance under question, and since the detector (eye or machine) will do the task of dis- concerning, perhaps the notion of contrast can be tailored to the detector: one notion for the eye, another for the machine. The precision with which each notion can be defined will rest on the precision of knowledge of the properties of the detector. In this light, the notion of contrast for a machine can be put into logical and unambiguous form before that associated with the eye, presumably because the mechanics of the ordinary machine is more readily determinable than that for the eye.)

Suppose the radiance function has a simple discontinuity at $(\underline{x}, \underline{\xi})$ such that the background of $\underline{\xi}$ has uniform radiance, then the above definition yields a contrast independent of every sequence. This is the state of affairs usually assumed in order to obtain a meaningful definition of contrast, and in practice the assumption of uniform background, at least in the neighborhood of the target, can be made with some justification. If the radiance distribution $N(\underline{x}, .)$ is continuous, then every sequential-contrast is zero.

Let $\underline{x}$ be a point in X. An <u>extended target</u> with respect to $\underline{x}$ is the set of all target points of a given collection of paths through $\underline{x}$. Thus for a given point $\underline{x}$ in X, an extended target with respect to $\underline{x}$ defines a subset X' of X. Conversely, suppose $\underline{x}$ is a point of X and X' is a subset of X. If $\underline{x}'$ is any point of X', then the pair $(\underline{x}, \underline{x}')$ defines at least one path in M through both $\underline{x}$ and $\underline{x}'$. It follows that X' is an extended target with respect to $\underline{x}$. With each extended target X' with respect to $\underline{x}$, there is a collection Λ of unit vectors associated with X' (which point from $\underline{x}$ to points $\underline{x}'$ of X'). The <u>background</u> of X' is defined as the background G of Λ. For radiometric (and photometric) purposes Λ may without serious confusion be identified with X'. In other words, a target may be considered a collection of directions from a point $\underline{x}$ in X, and the background of the target is another collection of vectors in Ξ.

Let X' be an extended target with respect to $\underline{x}$ in X. Let Λ be the collection of direction vectors associated with X', and let G be the background of Λ. If $\underline{x}'$ is a point of X', and $(\underline{x}, \underline{\xi}, r)$ is the path between $\underline{x}$ and $\underline{x}'$, and $\underline{\xi}'$ is a point of G, then

$${}_{\underline{\xi}, \underline{\xi}'}C_r(\underline{x}) = \frac{N_r(x, \xi)}{N(\underline{x}, \underline{\xi}')} - 1$$

is the *apparent contrast* of X' (in the direction $\underline{\xi}$) with respect to its background G (in the direction $\underline{\xi}'$). This definition is made in terms of simple contrasts; in view of the remarks above (in the parentheses) it is unwise at present to attempt a comprehensive definition of "apparent contrast" in terms of sequences. The above definition is phrased so as to allow an apparent contrast to be assigned to such an intangible as a cloud (or even a transparent subset of X) and at the same time be applicable to more substantial targets as planes, ships, etc. Saying that $N_r(\underline{x}, \underline{\xi})$ is an apparent radiance commits us to write it in terms of a transmitted and a path radiance, yet in view of the conclusions of section 5, the target point of $(\underline{x}, \underline{\xi}, r)$ need not be associated with a material point in X. By the same analysis, the N in the denominator may be interpreted as an apparent radiance, and written explicitly as such if need be. This latter possibility will be exploited in the derivation of the expression for contrast transmittance.

A sequential definition of contrast for an extended target is possible if $N(\underline{x}, .)$ is constant over Λ and over some neighborhood of Λ in G. In this case we write:

$$AG^{Cr}(\underline{x}) = \frac{A^{N}r(\underline{x})}{G^{N}(\underline{x})} - 1,$$

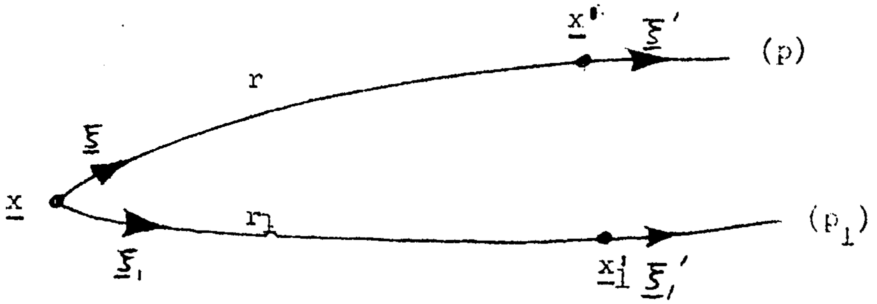
Strictly, the r should not be retained, but usually, with the initial assumption of uniform radiance over A and G is the added proviso that all points of X' are at the same distance r from $\underline{x}$, and with this in force, r may be meaningfully retained. The letter prescripts appear essential for an unambiguous notation which will serve to identify the object or region under consideration. This notation may be used even when X consists of a single vector $\underline{\xi}$. In accordance with the rules of use for N_o (section 6) we may write

$$AG^{Co}(\underline{x}) = \frac{A^{N}o(\underline{x})}{G^{N}(\underline{x})} - 1$$

for the inherent contrast of A with respect to G.

8. Contrast Transmittance.

In any optical medium of practical interest, n is assumed to have continuous derivatives of the first order for all its sections, and α, σ, and N are assumed integrable (in the sense of Lebesgue, if need be). It follows that paths $(\underline{x}, \underline{\xi}, r)$, associated transmittances T_r and path radiances N_r^{*} are all continuous in each of the variables $\underline{x}$, $\underline{\xi}$, and r. For more delicate arguments, N is required to have only discontinuities of the first kind: $\lim_{(\underline{x}, \underline{\xi}) \to (\underline{x}', \underline{\xi}')} N(\underline{x}, \underline{\xi})$ exists for every $(\underline{x}', \underline{\xi}')$ in H.



Let $(\underline{x}, \underline{\xi}, r) = p$, and $(\underline{x}, \underline{\xi}_1, r_1) = p_1$ be two paths in M (observe that they have the same observation point in X). Consider the simple contrast

$$_{\underline{\xi}, \underline{\xi}_1}C_r(\underline{x}) = \frac{N_r(\underline{x}, \underline{\xi})}{N(\underline{x}, \underline{\xi}_1)} - 1 .$$

Using p_1, the denominator may be written as an apparent radiance. Using the porso, $N(\underline{x}'_1, \underline{\xi}'_1)$ (the radiance at the target point of p_1) is assigned (relative) scattering order 0 and is designated by $N^0(\underline{x}'_1, \underline{\xi}'_1)$. Thus we have, $N_{r_1}(\underline{x}, \underline{\xi}_1) = T_{r_1}(\underline{x}, \underline{\xi}_1)N^{\bullet}(\underline{x}'_1, \underline{\xi}'_1) + N^*_{r_1}(\underline{x}, \underline{\xi}_1)$. We set $r = r_1$*, and examine the $\lim\limits_{(\underline{\xi}_1 \to \underline{\xi})} {}_{\underline{\xi}, \underline{\xi}_1}C_r(\underline{x})$. It is clear that the form for the simple contrast is: $_{\underline{\xi}, \underline{\xi}_1}C_r(\underline{x}) =$

$$\frac{[N^0(\underline{x}', \underline{\xi}') - N^0(\underline{x}'_1, \underline{\xi}'_1)]T_r(\underline{x}, \underline{\xi}) + [T_r(\underline{x}, \underline{\xi}) - T_r(\underline{x}, \underline{\xi}_1)]N^0(\underline{x}'_1, \underline{\xi}'_1) + [N^*_r(\underline{x}, \underline{\xi}) - N^*_r(\underline{x}, \underline{\xi}_1)]}{N_r(\underline{x}_1, \underline{\xi}_1)}$$

To emphasize the fact that $(\underline{x}', \underline{\xi}')$ is the target, we write $N^0(\underline{x}', \underline{\xi}')$ as $_AN^0(\underline{x}', \underline{\xi}')$, and since $\underline{\xi}_1$ is in the background of $\underline{\xi}$, we write $N^0(\underline{x}'_1, \underline{\xi}'_1)$ as $_GN^0(\underline{x}'_1, \underline{\xi}'_1)$.

Set, $_GN^0(\underline{x}', \underline{\xi}') = \lim\limits_{\underline{\xi}_1 \to \underline{\xi}} {}_GN^0(\underline{x}'_1, \underline{\xi}'_1)$, and

$$_GN_r(\underline{x}, \underline{\xi}) = \lim_{\underline{\xi}_1 \to \underline{\xi}} {}_GN^0(\underline{x}, \underline{\xi}_1).$$

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* Whenever possible. Otherwise the argument is trivially modified.

By continuity,

$$\lim_{\underline{\xi}_1 \to \underline{\xi}} [T_r(\underline{x}, \underline{\xi}) - T_r(\underline{x}, \underline{\xi}_1)] = 0, \text{ and}$$

$$\lim_{\underline{\xi}_1 \to \underline{\xi}} [N_r^*(\underline{x}, \underline{\xi}) - N_r^*(\underline{x}, \underline{\xi}_1)] = 0.$$

If we write,

$${}_{AG}C_r(\underline{x}) = \lim_{\underline{\xi}_1 \to \underline{\xi}} {}_{\underline{\xi}, \underline{\xi}_1}C_r(\underline{x}), \text{ then}$$

$${}_{AG}C_r(\underline{x}) = \frac{[\,{}_AN^o(\underline{x}', \underline{\xi}') - {}_GN^o(\underline{x}', \underline{\xi}')\,]}{{}_GN_r(\underline{x}, \underline{\xi})} T_r(\underline{x}, \underline{\xi}).$$

$$= \frac{{}_GN^o(\underline{x}', \underline{\xi}')}{{}_GN_r(\underline{x}, \underline{\xi})} \; \frac{[\,{}_AN^o(\underline{x}', \underline{\xi}') - {}_GN^o(\underline{x}', \underline{\xi}')\,]}{{}_GN^o(\underline{x}', \underline{\xi}')} \; T_r(\underline{x}, \underline{\xi}),$$

This is the general form for the *apparent contrast* (of the sequential typ *associated with the path p*. If $r = 0$, then $T_r = 1$, and ${}_GN^o(\underline{x}', \underline{\xi}')$ $= {}_GN_o(\underline{x}', \underline{\xi}')$ (section 6). Hence

$${}_{AG}C_o(\underline{x}') = \frac{{}_AN_o(\underline{x}', \underline{\xi}') - {}_GN_o(\underline{x}', \underline{\xi}')}{{}_GN_o(\underline{x}', \underline{\xi}')}$$

Thus we may write,

$${}_{AG}C_r(\underline{x}) = \left[\frac{{}_GN_o(\underline{x}', \underline{\xi}')}{{}_GN_r(\underline{x}, \underline{\xi})}\right] T_r(\underline{x}, \underline{\xi}) \; {}_{AG}C_o(\underline{x}') = T_r^* \; {}_{AG}C_o(\underline{x}')$$

As an interesting but idle notational curiosity, we exhibit in abbreviated form the following invariant (invariant along all rays in M):

$$ {}_{A}G^{C_r}\ {}_{A}{}^{N_o^o}\ G^{N_r} = {}_{A}G^{C_o}\ {}_{A}{}^{N_r^o}\ G^{N_o} \ . $$

The number in square brackets ((T_r^*) on the preceding page) is the <u>contrast transmittance</u> associated with the path p. Contrast transmittance, unlike beam transmittance, is dependent upon the "light field" along p. This fact is made more evident by writing the contrast transmittance of p in the following form:

$$ \text{contrast transmittance of } p = \frac{1}{\left[1 + \dfrac{N_r^*(\underline{x}, \underline{\xi})}{{}_{G}N_r^{o*}(\underline{x}, \underline{\xi})} \right]} \ , $$

where $p = (\underline{x}, \underline{\xi}, r)$, and N_r^* is the path radiance, and $G^{N_r^o}$ is the (limit of the) transmitted background radiance at the target point, as seen at the observation point of p.

Summary: Contrast is one of the more complicated concepts of Geophysical Optics both as an analytic and as an epistemic concept. It can be defined analytically with rigor, but perhaps at the expense of epistomological significance. At present we will be satisfied with the definition of <u>simple contrast</u>, and <u>sequential contrasts in light fields with simple discontinuities and uniform target and background radiances</u>. Definitions of contrast in these settings appear to simultaneously satisfy simple and meaningful interpretations both on an analytic and epistemic level. A derivation of the contrast transmittance associated with a path in M is made under the simple setting, with the proposed notation appearing in full dress.

9. Further Radiometric Notations.

Let $\underline{n}$ denote a fixed unit vector. Define $\Omega_+ = [\ \underline{\xi}: \underline{\xi}.\underline{n} \geq 0\ ]$, and $\Omega_- = [\ \underline{\xi}: \underline{\xi}.\underline{n} \leq 0\ ]$. Then for any pair $(\underline{x}, \underline{n})$ in M, we define $H(\underline{x}, \underline{n}) = \int_{\Omega_+} N(\underline{x}, \underline{\xi}')\underline{n} \cdot \underline{\xi}'\, d\omega(\underline{\xi}')$ as the <u>irradiance</u> at $(\underline{x}, \underline{n})$. The function H, like N, is defined on M. When $\underline{n}$ is fixed and known throughout a given discussion (as, e.g., the unit upward normal is a semi-infinite plane-parallel optical medium such as the customary model of a natural hydrosol) we may write H_+ for $H(\cdot,\underline{n})$, and H_- for $H(\cdot,-\underline{n})$. (In an expanded expository account care must be exercised in stipulating which radiance—specific or field—is being used in order to avoid confusion as to the directionality of flow of the radiant energy across the surface normal to $\underline{n}$.) In this case, H_+ and H_- are functions on X.

$\underline{H}(\underline{x}) = \int_{\Xi} N(\underline{x}, \underline{\xi}')\, \underline{\xi}'\, d\omega(\underline{\xi}')$ is the <u>vector irradiance</u> at $\underline{x}$. $\underline{H}$ is defined on X and has values in X (considered as a vector space), but to avoid confusion we may replace its range space by a copy of X with each triple assigned a dimension of irradiance.

$h(\underline{x}) = \int_{\Xi} N(\underline{x}, \underline{\xi}')\, d\omega(\underline{\xi}')$ is the <u>scalar irradiance</u> at $\underline{x}$. h is defined on X and has range in the real number system.

> (At present h serves both for scalar irradiance <u>and</u> scalar illuminance. It is suggested that e be used to denote scalar illuminance. This completes the analogy: H is to h, as E is to e. After some consideration, it is felt that no serious confusion can occur from the use of e as a scalar illuminance and in an exponential expression. From a typographical point of view, it is furthermore desirable to write exp[-kz] than e^{-kz}, and beam transmittance T_r already obviates to need for frequent use of exponentials.)

The remaining symbols W, U, J, P, and u need no further special attention in this discussion, other than the observation that W and u are functions of the type of H and h respectively, and that J can be dispensed with, it being at best a trouble-maker for beginners and experts alike. J cannot be defined in terms of N, and conversely. This fact in itself provides sufficient grounds for a mathematician or a mathematical physicist to exclude it from any discussion involving an optical medium (i.e., the quintuple (M, N, α, σ, n)). Anything that a J was supposed to do can be done better by an integral or a differential of N over a surface in X. (This can be convincingly illustrated in an expanded account.) Finally, we define $N_q = N_*/\alpha$, as the equilibrium radiance.

10. Reflectances, Transmittances.

$r(\theta)$, and $t(\theta)$ can be used for the values of Fresnel reflectances and transmittances, with n understood. r, and t can be used for general reflectances and transmittances; functional modifiers and subscripts may be added when needed. These lower case letters should be reserved for reflectances and transmittances associated with surfaces. In this way we are free to use T and R for transmittances and reflectances associated with regions (or subsets) of M with three-dimensional extent. It is felt that these symbols may be used in both radiometric and photometric contexts without any possibility of confusion.

11. Auxiliary Attenuating Functions.

(i) α, and σ taken as basic.

(ii) $s = \int_{\Xi} \sigma \, d\omega$, total scattering function.

(iii) $a = \alpha - s$, volume absorption function. Hence $\alpha = a + s$.

(iv) $f = \int_{\Omega_+} \sigma \, d\omega$, forward scattering function. (coll. flux)

[More precise def. in expanded accounts]

(v) $b = \int_{\Omega_-} \sigma \, d\omega$, backward scattering function. (coll. flux)

Hence $s = b + f$.

(vi) $D = h/H$, distribution factor.

(vii) k (lower case) irradiance attenuation function (arising from Schuster two-flux, fixed-radiance analysis).

(viii) $\tilde{\omega}_0 = s/\alpha$ scattering-attenuation ratio (albedo for single scattering).

In an expanded account, greater precision and clarity can be attained. For the present, we are interested only in the notational aspects as regards choice of symbols.

12. Ranges; Attenuating Lengths.

(i) If $p = (\underline{x}, \underline{\xi}, r)$ is a path in M, and α_0 is some fixed (reference) value of α, then $\bar{r}(\underline{x}, \underline{\xi}, r) = (1/\alpha_0)\int_p \alpha(\underline{x}'') \, dr'$ is the optical range of p. We may write $\bar{r}(\underline{x}, \underline{\xi}, r) = (-1/\alpha_0)\ln T_r(\underline{x}, \underline{\xi})$, or $\bar{r} = (-1/\alpha_0) \ln T_r$ for short when p is understood. If α is constant on p, then $\bar{r} = r$, if $\alpha_0 = \alpha(\underline{x})$.

(ii) Let $(\underline{x}, \underline{\xi})$ be a point in M. If there exists a path $(\underline{x}, \underline{\xi}, v)$ through $(\underline{x}, \underline{\xi})$ and in M, such that the contrast transmittance of $(\underline{x}, \underline{\xi}, v)$ is 0.02, then we say that the environmental range at $(\underline{x}, \underline{\xi})$ is v. If M is a natural aerosol (hydrosol) then v is the meteorological (respectively, hydrological) range at $(\underline{x}, \underline{\xi})$. If $N(\underline{x},.) = N(\underline{x}'',.)$ for all $\underline{x}''$ along the ray through $(\underline{x}, \underline{\xi})$, then it may be shown that $\bar{v}(\underline{x}, \underline{\xi}, v) = \ln 50/\alpha_0$. If in addition, α is constant on the ray, $v = \ln 50/\alpha_0$, if $\alpha_0 = \alpha(\underline{x})$: each of these cases hold providing the path $(\underline{x}, \underline{\xi}, v)$ is in M.

(iii) If w stands momentarily for any of the symbols ; α, s, a, k (and any other of the auxiliary attenuating functions) the quantity $1/w$ is a length, and will be designated by L_w, and is generally referred to as an attenuating length. In particular, $L\alpha$ is an attenuation length, L_s a scattering length, etc

13. Miscellaneous.

We have suggested notations for the most frequently occuring radiometric and attenuating functions, and in this way hope to stabilize them for future papers, reports, etc. which will require symbolic representations of these notions. Considerable study of the symbols has shown that virtually every case considered in radiative transfer can somehow be covered by the present adopted notation, with perhaps an errant subscript or other embellishment needed here and there in particularly complex situations, but these too, may be fixed after careful choice when they are encountered. Except for sea-state which is symbolized by S and is fairly well entrenched, we should be reasonably free to choose among x, y, and z for cartesian coordinates, and θ, and ϕ as polar coordinates (zenith and azimuth respectively).

14. Examples of Use of Notation

(i) The (integro-differential) Equation of Transfer.

$$\frac{dN(\underline{x}, \underline{\xi})}{dr} = -\alpha(\underline{x})N(\underline{x}, \underline{\xi}) + \int_{\Xi} N(\underline{x}, \underline{\xi}')\sigma(\underline{x}; \underline{\xi}', \underline{\xi})\, d\omega(\underline{\xi}')$$

$$= -\alpha(\underline{x})N(\underline{x}, \underline{\xi}) + N_*(\underline{x}, \underline{\xi}) .$$

This equation is always associated with a ray. d/dr is the Lagrangian derivative of N along the ray. Here the radiance function is conveniently thought of as specific radiance. The abbreviated form of the equation is:

$$\frac{dN}{dr} = -\alpha N + N_* .$$

(ii) The (integral) Equation of Transfer.

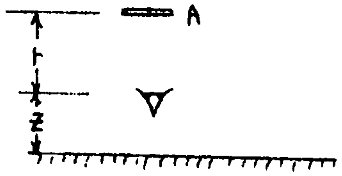
$$N_r(\underline{x}, \underline{\xi}) = N_r^o(\underline{x}, \underline{\xi}) + N_r^*(\underline{x}, \underline{\xi})$$

$$= T_r(\underline{x}, \underline{\xi})N^o(\underline{x}', \underline{\xi}') + \int_p T_{r'}(\underline{x}, \underline{\xi})N_*(\underline{x}'', \underline{\xi}'')\, dr'$$

This equation is always associated with a path $p = (\underline{x}, \underline{\xi}, r)$. Here the radiance is conveniently thought of as a field radiance. The abbreviated form of the equation is:

$$N_r = N_r^o + N_r^* .$$

(iii) Specific targets; Arguments understood.



Apparent radiance of target A:

$${}_A N_r(z) = {}_A N_r^o(z) + N_r^*(z)$$

Here the argument (i.e., the symbol in parentheses) indicates where the observer or instrument is. The subscript indicates symbolically the distance to the target A, and the prescript identifies the target. The need for the prescript is pointed up if, in addition to the target A, we wish to consider the radiance of its background G. We denote this by ${}_G N(z)$: we are at z, looking along a predetermined direction which is assumed common knowledge, hence it is not explicitly shown; we are looking at object G, which in this case is the background of A. If the need arises to consider ${}_G N(z)$ as an apparent radiance, then we append a subscript r, with the appropriate magnitude.

(iv) Schuster Analysis of radiant Flux.

One of the equations would look like:

$$\frac{dH_+^*}{dz} = -(a_+^* + b_+^*)H_+^*(z) + b_-^*H_-^*(z) + f_+^o H_+^o(z),$$

Here H_+^* and H_-^* represent down and upwelling irradiances comprised of radiant flux having suffered at least one scattering operation (hence the asterisk). H_+^o is the transmitted irradiance which is downwelling (cf., N_r^o). a_+^*, and b_+^* are absorption and backscattering coefficients for H_+^*; another set would be available for H_-^*. Without going into additional details, this example should demonstrate the appropriate use of the symbology, and its versatility through proper use of super-and subscripts tacked onto basic letters.