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SOME PRACTICAL CONSEQUENCES OF THE ASYMPTOTIC RADIANCE HYPOTHESIS*

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Some Practical Consequences of the Asymptotic Radiance Hypothesis*

by

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ABSTRACT

The asymptotic radiance hypothesis asserts that the angular distribution of radiance approaches a fixed form at great depths in natural waters. A simple proof of this hypothesis is given. The following consequences are deduced: the logarithmic derivatives (with respect to depth z) of radiance values $N(z, \theta, \phi)$ approach, with increasing z, a common fixed value k_∞ for all directions (θ, ϕ) ; further, the logarithmic derivatives of scalar irradiance $h(z)$, its up-and downwelling components $h(z,+)$ and $h(z,-)$, along with the derivatives of the up-and downwelling irradiances $H(z,+)$ and $H(z,-)$ all approach the common limit k_∞ as depth increases. Further consequences are that the classical Schuster two-flow equations for the light field in natural waters become exact with increasing depth. These and related results are illustrated by examples drawn from the special case of isotropic scattering. Finally, a formula is given which allows an estimate of the depth at and below which the actual radiance distributions differ from the asymptotic distribution by no more than a preassigned amount.

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INTRODUCTION

One of the simplest yet most important experimental facts one may cite about the light field in natural waters is its behavior at great depths: the amount (radiant density) of light decreases exponentially with increasing depth. This fact holds irregardless of the external lighting conditions and the optical state of the surface in any homogeneous or eventually homogeneous optically deep natural body of water. Furthermore the logarithmic rate of decrease appears to be governed only by the inherent optical properties of the water. Over the years the accumulated experimental and theoretical evidence has founded a firm basis for this fact.

The discussions in this note center on a related but vastly more striking experimental phenomenon. This phenomenon is associated with the form of the angular distributions of light at great depths in natural waters. Experimental evidence has pointed to the existence of a limiting, or asymptotic, radiance distribution which the radiance distributions appear to approach as depth is indefinitely increased. Furthermore, as in the case of radiant density, the eventual trend toward a regular behavior at great depths is apparently completely independent of the irregular optical conditions usually extant at and near the surface of the water, and dependent only on the inherent optical properties of the medium.

The first definitive recognition of this phenomenon appears to have been made by L. V. Whitney,[1, 2] who referred to the asymptotic radiance distribution as the "characteristic diffuse light." On the basis of his experimental evidence and that of others before him (sources cited in references 1 and 2) he formulated what we shall call the asymptotic radiance hypothesis. It may be stated as follows: The angular form of the radiance distribution $N(z,\theta,\phi)$ at a depth z in an optically deep plane-parallel medium approaches, with increasing z, a characteristic form, represented by a function $g(\theta,\phi)$, such that:

(i) $g(\theta,\phi_1) = g(\theta,\phi_2)$, i.e., g is independent of ϕ (or equivalently, the asymptotic radiance distribution is represented by a surface of revolution with vertical axis of symmetry);

(ii) $g(\theta,\phi)$ is independent of the external lighting conditions at the upper boundary of the medium;

(iii) $g(\theta,\phi)$ depends only on the inherent optical properties of the medium.

It has been proved[3] that the hypothesis holds in all plane-parallel media in which the phase function $\rho = 4\pi\sigma/\alpha$ (of radiative

[1] L. V. Whitney, "The angular distribution of characteristic diffuse light in natural waters," J. Marine Res. 4, 122-131 (1941).

[2] L. V. Whitney, "A general law of diminution of light intensity in natural waters and the percent of diffuse light at different depths," J. Opt. Soc. Am. 31, 714-722 (1941).

[3] R. W. Preisendorfer, "A proof of the asymptotic radiance hypothesis," SIO Reference 58-57, Scripps Institution of Oceanography, University of California, La Jolla, California, (1958).

transfer theory) approaches a limit with increasing depth. The proof has been designed to also cover closely related phenomena in the fields of neutron transport theory and astrophysical optics. The complementary relations between the present approach and the formal approaches used in neutron transport theory are developed in detail in reference 3. A simpler proof, which makes use of the experimentally documented eventual exponential behavior of radiant density and which is formulated explicitly for the hydrological optics context also has been devised.[4]

The practical consequences derivable from the hypothesis are of great importance to the experimental studies of the optical properties of natural waters. We particularly have in mind the consequences for the directly observable quantities in optically deep natural hydrosols, and for certain useful simple models used to describe the light fields in such media. The details of the necessary groundwork for the present discussion have been developed in some earlier notes.[5,6,7]

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[4] R. W. Preisendorfer, "On the existence of characteristic diffuse light in natural waters," SIO Reference 58-59, ibid. (1958).

[5] R. W. Preisendorfer, A model for radiance distributions in natural hydrosols," SIO Reference 58-42, ibid. (1957).

[6] R. W. Preisendorfer, "Unified irradiance equations," SIO Reference 58-43, ibid. (1957).

[7] R. W. Preisendorfer, "Directly observable quantities for light fields in natural hydrosols, SIO Reference 58-46, ibid. (1957).

BASIC FORMULAS

The Irradiance Quartet

The radiance function N is the basic radiometric quantity in terms of which all others can be defined. In particular the downwelling and upwelling irradiances $H(z,-)$ and $H(z,+)$ at depth z in a natural hydrosol are given by:

$$H(z,-) = -\int_{\Xi_-} N(z,\theta,\phi) \cos\theta \, d\Omega , \tag{1}$$

and

$$H(z,+) = \int_{\Xi_+} N(z,\theta,\phi) \cos\theta \, d\Omega , \tag{2}$$

where, for the present purposes, we define Ξ_- as the collection of all downward (or inward) directions (θ,ϕ) : $\pi/2 < \theta$, $0 \leq \phi < 2\pi$; and Ξ_+ is the collection of all upward (or outward) directions (θ,ϕ) : $\theta \leq \pi/2$, $0 \leq \phi < 2\pi$, where θ is measured as usual from the outward normal to the medium. We define $\Xi = \Xi_+ \cup \Xi_-$. For brevity we have written, $d\Omega = \sin\theta \, d\theta \, d\phi$.

In addition to $H(z,+)$ and $H(z,-)$, underwater optical experiments usually consider the following downwelling and upwelling scalar irradiances:

$$h(z,-) = \int_{\Xi_-} N(z,\theta,\phi)\, d\Omega \,, \tag{3}$$

and

$$h(z,+) = \int_{\Xi_+} N(z,\theta,\phi)\, d\Omega \,, \tag{4}$$

and their sum $h(z)$:

$$h(z) = h(z,-) + h(z,+) \,, \tag{5}$$

which is the scalar irradiance at depth z ($h(z)$ is equal to the product of the speed of light v and radiant density $u(z)$ at depth z).

The four quantities $H(z,\pm)$, $h(z,\pm)$ form the nucleus of a set of modern experimental quantities used to document the light field in natural waters. Of course, a complete documentation is obtained only through a systematic determination of the radiance values $N(z,\theta,\phi)$ at all depths z and over all directions (θ,ϕ) .

The D and R Functions

In the absence of detailed knowledge of $N(z,\theta,\phi)$ the basic quartet of irradiance functions defined above can be used to derive most of the information needed for the solution of underwater visibility problems, and image and flux transmission problems in general. In particular, an excellent index of the shape of the radiance distributions at depth z is given by the distribution functions $D(z,\pm)$ defined as:

$$D(z,\pm) = \frac{h(z,\pm)}{H(z,\pm)} . \tag{6}$$

Furthermore, information about the reflectance properties of the water at depth z is furnished by a study of the ratio:

$$R(z,-) = \frac{H(z,+)}{H(z,-)} , \tag{7}$$

which is the experimental counterpart to the classical R_∞ formula as given by the Schuster two-flow analysis of the light field. In fact, the D and R functions defined above, and the functions defined below are all either modern experimental counterparts or logical extensions of the tools provided by the classical two-flow theory of the light field in natural hydrosols. As noted above, the background of these quantities is considered in detail elsewhere,[7] so that the present discussion need not dwell further on their definitions and interrelations. We are concerned here only with the behavior of these quantities at great depths in media satisfying the requirement of the asymptotic radiance hypothesis.

The K- Functions

The essentially exponential behavior of the irradiance quantities defined above supplies the motivation for the following definitions:

$$K(z,\pm) = \frac{-1}{H(z,\pm)} \frac{dH(z,\pm)}{dz}, \qquad (8)$$

$$k(z,\pm) = \frac{-1}{h(z,\pm)} \frac{dh(z,\pm)}{dz}, \qquad (9)$$

$$k(z) = \frac{-1}{h(z)} \frac{dh(z)}{dz}. \qquad (10)$$

If the various irradiance quantities vary exactly in an exponential manner at all depths, then the corresponding K-functions would be constant functions each assuming a fixed value at all depths. In general, however, the depth-dependence of these quantities undergo irregular behavior before the exponential features eventually emerge. The preceding definitions are designed to characterize the depth-dependence of the irradiances under all conditions.

One of the main consequences derived from the asymptotic radiance hypothesis asserts that the five K-functions defined above all tend to a common limit with increasing depth. We prepare the groundwork leading to this conclusion by introducing a final K-function, namely that associated with the radiance function itself.

$$K(z,\theta,\phi) = \frac{-1}{N(z,\theta,\phi)} \frac{dN(z,\theta,\phi)}{dz} . \qquad (11)$$

Just as each of the various irradiance quantities may be expressed in terms of radiance, so can its corresponding K-function be expressed in terms of the K-function for radiance:

$$K(z,\pm) = \frac{\int_{\Xi_\pm} N(z,\theta,\phi)\, K(z,\theta,\phi) \cos\theta \; d\Omega}{\int_{\Xi_\pm} N(z,\theta,\phi) \cos\theta \; d\Omega} , \qquad (12)$$

$$k(z,\pm) = \frac{\int_{\Xi_\pm} N(z,\theta,\phi)\, K(z,\theta,\phi) \; d\Omega}{\int_{\Xi_\pm} N(z,\theta,\phi) \; d\Omega} , \qquad (13)$$

$$k(z) = \frac{\int_{\Xi} N(z,\theta,\phi)\, K(z,\theta,\phi)\, d\Omega}{\int_{\Xi} N(z,\theta,\phi)\, d\Omega} . \qquad (14)$$

A Reformulation of the Hypothesis

The K-function $K(z,\theta,\phi)$ for radiance is of fundamental importance in the present discussion of the asymptotic radiance hypothesis. In fact it is the function which gives rise to that form of the hypothesis which is most amenable to exact mathematical analysis. The desired reformulation reads as follows:[3] <u>for each $(\theta,\phi)\in\Xi$, the function $K(z,\theta,\phi)$ has a limit, as $z\to\infty$, and this limit is independent of (θ,ϕ).</u> In symbols:

$$k_\infty = \lim_{z\to\infty} K(z,\theta,\phi)$$

<u>exists for every $(\theta,\phi)\in\Xi$, and is independent of (θ,ϕ).</u>

The preceding formulation is made plausible by the following observations: For every depth z, $N(z,\theta,\phi)$ may be represented exactly by

$$N(z,\theta,\phi) = N(0,\theta,\phi)\, exp\left\{-\int_0^z K(z',\theta,\phi)\, dz'\right\}.$$

Suppose there is some depth z_0 below which we have $K(z,\theta,\phi) = k_\infty$, a fixed number for all (θ,ϕ) . Then

$$\begin{aligned} N(z,\theta,\phi) &= N(0,\theta,\phi)\, exp\left\{-\int_0^{z_0} K(z',\theta,\phi)\, dz' - \int_{z_0}^{z} K(z',\theta,\phi)\, dz'\right\} \\ &= N(z_0,\theta,\phi)\, exp\left\{-k_\infty(z-z_0)\right\}. \end{aligned}$$

Let

$$g(z_0, \theta, \phi) = N(z_0, \theta, \phi) \exp\{ k_\infty z_0 \} ,$$

then for all $z \geq z_0$,

$$N(z, \theta, \phi) = g(z_0, \theta, \phi)\, e^{-k_\infty z} . \tag{15}$$

It follows that below z_0, $N(z, \theta, \phi)$ has a fixed angular structure given by $g(z_0, \theta, \phi)$.

The Basic Transfer Equations

One final relation needed below is the reformulation of the equation of transfer in terms of the K-function for radiance. This is easily obtained from the standard form of the transfer equation for stratified source-free plano-parallel media:

$$- \cos\theta \, \frac{dN(z, \theta, \phi)}{dz} = - \alpha(z)\, N(z, \theta, \phi) + N_*(z, \theta, \phi) ,$$

where

$$N_*(z, \theta, \phi) = \int_{\Xi} \sigma(z; \theta, \phi ; \theta', \phi')\, N(z, \theta', \phi')\, d\Omega .$$

By means of the definition of $K(z, \theta, \phi)$, the above equation may be rewritten as:

$$N(z,\theta,\phi) = \frac{N_*(z,\theta,\phi)}{\alpha(z) + K(z,\theta,\phi)\cos\theta}, \tag{16}$$

which is the canonical form of the equation of transfer.

The equation of transfer governing $K(z,\theta,\phi)$ is also easily found. From (16), the definition of $K(z,\theta,\phi)$, and the following definition of an analogous K-function:

$$K_q(z,\theta,\phi) = \frac{-1}{N_q(z,\theta,\phi)} \frac{dN_q(z,\theta,\phi)}{dz}, \tag{17}$$

where

$$N_q(z,\theta,\phi) = N_*(z,\theta,\phi)/\alpha(z), \tag{18}$$

we have:

$$\frac{dK(z,\theta,\phi)}{dz} = \left[K(z,\theta,\phi) - K_q(z,\theta,\phi)\right]\left[K(z,\theta,\phi) + \alpha(z)\sec\theta\right]. \tag{19}$$

This formulation is analogous to the following formulation of the equation of transfer for $N(z,\theta,\phi)$ in which $N_q(z,\theta,\phi)$ is used:

$$\frac{dN(z,\theta,\phi)}{dz} = \left[N(z,\theta,\phi) - N_q(z,\theta,\phi)\right]\left[+\alpha(z)\sec(\theta)\right]$$

These formulations point up the following physical significance of the equilibrium radiance N_q and its K-function K_q : for $\theta > \frac{\pi}{2}$ we observe that if $N(z,\theta,\phi) \lesseqgtr N_q(z,\theta,\phi)$ then $dN(z,\theta,\phi)/dz \gtreqless 0$. This follows immediately from the preceding equation. Thus N, $\theta > \pi/2$, always tends toward the equilibrium radiance N_q. Now a similar phenomenon exists between K and K_q. To see this, we observe that the second factor on the right in (19) has the property that

$$K(z,\theta,\phi) + \alpha(z)\sec\theta < 0$$

for all z and all downward directions (θ,ϕ). Therefore if $K(z,\theta,\phi) \lesseqgtr K_q(z,\theta,\phi)$ then $dK(z,\theta,\phi)/dz \gtreqless 0$, showing that K always tends toward K_q for these directions. This property of the function $K(z,\theta,\phi)$ provides the key to a rigorous proof of the existence of an asymptotic radiance distribution. An example of such a use of (19) is given in the final sections below.

CONSEQUENCES FOR DIRECTLY OBSERVABLE QUANTITIES

The Equation for the Asymptotic Radiance Distribution

An application of the asymptotic radiance hypothesis to (16) yields the formula for the asymptotic radiance distribution g. In view of the heuristic discussion leading to (15) and the statement of the hypothesis in terms of $K(z,\theta,\phi)$, we see that

$$g(\theta,\phi) = \lim_{z\to\infty} g(z,\theta,\phi) = \lim_{z\to\infty} N(z,\theta,\phi)\exp\{k_\infty z\}$$

exists for all (θ,ϕ). Hence multiplying each side of (16) by $\exp\{k_\infty z\}$ and passing to the limit as $z\to\infty$, we have

$$g(\theta,\phi) = \frac{\frac{1}{4\pi}\int_{\Xi} p(\theta,\phi;\theta',\phi')\, g(\theta',\phi')\, d\Omega}{1+\left(\frac{k_\infty}{\alpha}\right)\cos\theta} \tag{20}$$

where

$$p(\theta,\phi;\theta',\phi') = \lim_{z\to\infty} 4\pi\sigma(z;\theta,\phi;\theta',\phi')/\alpha(z)$$

and

$$k_\infty = \lim_{z\to\infty} K(z,\theta,\phi)$$

and*

$$\alpha = \lim_{z \to \infty} \alpha(z) .$$

The integral equation (20) has the property that the values of its solution g are independent of ϕ . Thus we may set

$$g(\theta,\phi) = \frac{1}{2\pi}\, g_0(\theta),$$

and (20) may be simplified to read:

$$g_0(\theta) = \frac{\frac{1}{2}\int_{\theta'=0}^{\pi} p^{(0)}(\theta;\theta')\, g_0(\theta') \sin\theta' d\theta'}{1 + \left(\frac{k_\infty}{\alpha}\right)\cos\theta} \quad (21)$$

where we have set

$$p^{(0)}(\theta;\theta') = \frac{1}{2\pi}\int_{\phi'=0}^{2\pi} p(\theta,\phi;\theta',\phi')\, d\phi'.$$

The function g_0 defines the asymptotic radiance distribution. Conditions (i), (ii), and (iii) of the asymptotic radiance hypothesis are satisfied by the function g_0 as determined by the equation (21). A graph of g_0 is clearly a surface of revolution with vertical axis (in the coordinate system of the plane-parallel medium). Furthermore, the structure of g_0 and the value of k_∞ are completely determined by the phase function p (k_∞/α plays

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* For most practical situations, the medium is homogeneous or eventually homogeneous, so that this limit exists. Actually, as noted above, the asymptotic radiance distribution exists whenever $\lim_{z\to\infty} \sigma/\alpha$ exists, without necessarily requiring that the individual limits $\lim_{z\to\infty} \sigma$ and $\lim_{z\to\infty} \alpha$ exist. This more general situation is discussed in reference 3.

the role of an eigenvalue of the equation (21)). Thus g_0 is determined completely by the inherent optical properties of the medium by means of equation (21) and therefore is independent of the external lighting conditions.

The Limits of the K-Functions

From the relations (12) - (14), and the statement of the hypothesis, we conclude that

$$\lim_{z \to \infty} K(z,\pm) = k_\infty , \tag{22}$$

$$\lim_{z \to \infty} k(z,\pm) = k_\infty , \tag{23}$$

$$\lim_{z \to \infty} k(z) = k_\infty . \tag{24}$$

The limit (24) is interpreted as follows: the logarithmic derivative (with respect to z) of scalar irradiance $h(z)$ eventually approaches the common limit k_∞ of the logarithmic derivatives of radiance distribution $N(z,\theta,\phi)$. The limit (23) shows that the logarithmic derivatives of the up-and down-welling irradiances (which are measurably distinct at all small depths z) approach a common value, namely k_∞ . A similar interpretation holds for the K -functions of the up-and down-welling scalar irradiances.

The Limits of the D and R Functions

From (6) and the hypothesis, we have immediately:

$$D(\pm) = \lim_{z\to\infty} D(z,\pm) = \frac{\int_\pm g_o(\theta)\sin\theta\, d\theta}{\int_\pm g_o(\theta)[\pm\cos\theta]\sin\theta\, d\theta}, \quad (25)$$

where

$$\int_+ = \int_{\theta=0}^{\pi/2} \;, \qquad \int_- = \int_{\theta=\pi/2}^{\pi} \;.$$

In an earlier note[7] it was shown that $R(z,-)$ can be represented quite generally in terms of the K-functions and the distribution functions as follows:

$$R(z,-) = \frac{K(z,-) - a(z,-)}{K(z,+) + a(z,+)},$$

where

$$a(z,\pm) = D(z,\pm)\, a(z),$$

and where $a(z)$ is the value of the volume absorption function of the medium at depth z. It follows that

$$R_\infty = \lim_{z \to \infty} R(z,-)$$

exists and is given by

$$R_\infty = \frac{k_\infty - a(-)}{k_\infty + a(+)}, \qquad (26)$$

where

$$a(\pm) = \lim_{z \to \infty} a(z,\pm) = \lim_{z \to \infty} D(z,\pm)\, a(z) = D(\pm)\, a.$$

Further limit relations may be determined by systematically going through the set of directly observable quantities discussed in reference 7. The preceding results will serve to illustrate the general procedure of obtaining the desired limit expressions.

We observe that (26) is similar to the classical expression for R_∞ as given by the Schuster two-flow equations. This similarity is not coincidental; it is, rather, a consequence of the fact that under the asymptotic radiance hypothesis, the general two-flow equations become exact with increasing depth. We now consider this fact in more detail.

CONSEQUENCES FOR SOME SIMPLE THEORETICAL MODELS

The Two-Flow Analysis of the Light Field

In an earlier investigation[6] a study of the classical equations for $H(z,+)$ and $H(z,-)$ showed that these equations were exact if and only if the distribution functions $D(z,+)$ and $D(z,-)$ were independent of depth. Under the asymptotic radiance hypothesis it was seen that the distribution functions become independent of depth at great depths (cf (25)). It follows that the equations for $H(z,+)$ and $H(z,-)$ become exact at great depths whenever the hypothesis holds.

In the same investigation a formulation of the equations for $H^*(z,+)$ and $H^*(z,-)$ (the irradiances associated with diffuse light) was made in which each stream of flux was assigned a fixed distribution factor $D^*(+)$, $D^*(-)$ (the two-D theory). This formulation was justified on the basis of experimental evidence which showed that $D(z,+)$ and $D(z,-)$ were essentially fixed (generally distinct) constants. In the light of the present analysis, the use of the two-D theory is thus given further justification on theoretical grounds whenever the asymptotic radiance hypothesis holds.

The two-D model gives explicit formulas for $H^*(z,+)$ and $H^*(z,-)$. In view of the preceding observations, these expressions become exact with increasing depth z. Using the equations developed in reference 6 one may show that for every depth z,

$$H^*(z,-) = N^\circ C(\mu_0,-)\left[e^{-k_\infty z} - e^{-\alpha z/\mu_0}\right], \qquad (27)$$

$$H^*(z,+) = N^\circ \left[C(\mu_0,-)\frac{g_-(+)}{g_-(-)}e^{-k_\infty z} - C(\mu_0,+)\, e^{-\alpha z/\mu_0}\right]. \qquad (28)$$

Observe that we have set $k_\infty = -k_-$, where k_- is given in reference 6. The physical setting associated with (27) and (28) is an infinitely deep plane-parallel slab irradiated by collimated flux incident at the upper boundary at an angle $\theta_0 = \arccos \mu_0$ from the outward normal. The response to an arbitrary incident distribution is obtained by integrating (27) and (28) over Ξ_-. $C(\mu_0,\pm)$ are constants determined by the optical parameters and boundary conditions; and

$$g_-(\pm) = 1 \mp \frac{a(\mp)}{k_\infty},$$

where $a(\pm)$ are as defined in (26). The observable irradiances $H(z,\pm)$ are, by definition,

$$H(z,\pm) = H^\circ(z,\pm) + H^*(z,\pm),$$

where

$$H^{\circ}(z,+) = 0$$
$$H^{\circ}(z,-) = N^{\circ}\mu_0 e^{-\alpha z/\mu_0}.$$

The preceding model yields the following prediction of the limit of $R(z,-)$:

$$R_\infty = \lim_{z\to\infty} R(z,-) = \lim_{z\to\infty} \frac{H(z,+)}{H(z,-)} = \frac{g_-(+)}{g_-(-)} = \frac{k_\infty - a(-)}{k_\infty + a(+)},$$

which is equal to (2.6), the exact limit given by general radiative transfer theory. These observations show that in any medium in which the asymptotic radiance hypothesis holds, if we restrict ourselves to the class of all possible two-flow models of the light field, the model which attains maximal accuracy is that given by the two-D theory.

Critique of Whitney's "General Law"

After conjecturing that the radiance distributions assume a fixed shape at great depths, L. V. Whitney made use of the conjecture to deduce a so-called "general law of the diminuation of light intensity in natural waters."[2] An examination of the differential equations formulating this law reveals that they are incomplete:

they fail to account for the contribution to the downwelling irradiance by the back-scattered fraction of the upwelling irradiance. As a result, the solutions of the differential equations are generally inadequate to cope with the contribution from one half of the light field, namely the component associated with the upwelling flux. Furthermore, some (convenient, but incorrect) assumptions were made about the depth rate of change of the mean free path for unscattered light at various depths. On this basis the equations were integrated, holding the mean free path for directly transmitted light fixed. Both of these inadequacies of an otherwise satisfactory theory have been remedied in the two-D theory of the light field. The equations (27) and (28) represent the concomitant effects of both up-and down welling streams. Furthermore, the awkwardness of stemming from the change with depth of the mean free path of directly transmitted light has been avoided by considering only collimated incident flux of radiance N^{o} at the upper boundary.

The Simple Model for Radiance Distributions

In an earlier note[5] a simple model for radiance distributions was derived from the classical two-flow analysis of the light field. In view of the preceding observations, it is concluded that the proposed simple model of reference 5 becomes exact with increasing depth in all media in which the asymptotic radiance hypothesis holds.

EXAMPLES

We conclude with some examples drawn from the case of a plane-parallel medium which exhibits isotropic scattering and in which the asymptotic radiance hypothesis holds. In this way we obtain some general ideas about the shape of g_o , and the order of magnitudes of the quantities $D(\pm)$, R_∞ , and k_∞, one may expect in real media. Finally, it is possible to give, in the present context, a simple heuristic proof of the hypothesis, and at the same time derive a formula which will provide a means of determining the depth in a medium below which asymptoticity has essentially been attained.

The Standard Ellipsoid

When scattering is isotropic, the phase function takes on the form:

$$p(\theta,\phi;\theta',\phi') = \tilde{\omega}_0 = s/\alpha$$

where s is the total scattering coefficient. Using this phase function in (21), we see that $g_o(\theta)$ takes on a particularly simple form,

$$g_o(\theta) = \frac{\tilde{\omega}_0}{2} \frac{g_o}{1+\epsilon\cos\theta}\,, \tag{29}$$

where

$$\epsilon = k_\infty / \alpha \ ,$$

and

$$g_0 = \int_{\theta=0}^{\pi} g_0(\theta) \sin\theta \, d\theta .$$

Physical significance can be attached to g_0 by returning to the definition of $g(\theta,\phi)$ and integrating over Ξ . The result is

$$g_0 = \lim_{z\to\infty} h(z) e^{k_\infty z} .$$

Hence if there is some depth z_0 below which one may consider that for practical purposes asymptoticity has been attained, then the preceding relation can be written:

$$g_0 = h(z_0) e^{k_\infty z_0} .$$

Expression (29) defines a prolate spheroid of revolution whose axis of symmetry is vertical. The eccentricity of the ellipsoid is $\epsilon = k_\infty/\alpha$. This ellipsoid may serve as a convenient reference against which distributions from real media may be compared. To effect a comparison one must know the $\tilde{\omega}_0$ and ϵ of the medium. Since ϵ and $\tilde{\omega}_0$ are generally related, it suffices in principle to know only $\tilde{\omega}_0$ and the phase function. This is illustrated below after a necessary preliminary discussion of $D(\pm)$ and R_∞ .

Expressions for $D(\pm)$ and R_∞

By means of (25) and (29) we find that

$$D(\pm) = \frac{\epsilon \ln(1 \pm \epsilon)}{\epsilon \mp \ln(1 \pm \epsilon)} . \qquad (30)$$

Furthermore, from (7) and (29) (i.e., (29) replaces $N(z,\theta,\phi)$ in (7)), we have:

$$R_\infty = \frac{\ln(1+\epsilon) - \epsilon}{\ln(1-\epsilon) + \epsilon} . \qquad (31)$$

The same result could be obtained by using (26) and the preceding forms for $D(\pm)$.

Values of $D(\pm)$ and R_∞ as functions of ϵ, $0 < \epsilon < 1$ are given in Table I. It is easy to verify that for the extreme values 0 and 1 of ϵ the corresponding values of $D(\pm)$ and R_∞ are:

$$\lim_{\epsilon \to 0} D(\pm) = 2 \qquad \lim_{\epsilon \to 0} R_\infty = 1$$

$$\lim_{\epsilon \to 1} D(+) = \frac{\ln 2}{(1 - \ln 2)} = 2.259 \qquad \lim_{\epsilon \to 1} R_\infty = 0$$

$$\lim_{\epsilon \to 1} D(-) = 1$$

TABLE I

DISTRIBUTION AND REFLECTANCE FACTORS
FOR STANDARD ELLIPSOID

ϵ	$D(-)$	$D(+)$	R_∞
0.100	1.9664	2.0319	0.8750
0.200	1.9286	2.0622	0.7640
0.300	1.8881	2.0911	0.6642
0.400	1.8438	2.1185	0.5733
0.500	1.7943	2.1143	0.4895
0.600	1.7381	2.1692	0.4110
0.700	1.6722	2.1928	0.3361
0.800	1.5906	2.2157	0.2622
0.900	1.4775	2.2377	0.1841
0.950	1.3911	2.2483	0.1379

Table II of reference 6 gives values of $D(z,\pm)$ for a real medium under varying external conditions. A comparison of these real values with those summarized in Table I above reveals the following information: the $D(z,-)$ values are significantly less than the standard $D(-)$ values; the $D(z,+)$ values are significantly greater than the standard $D(+)$ values. Since all natural waters exhibit anisotropic scattering we can infer the following features of the structure of asymptotic radiance distributions in all natural waters: when compared with the standard ellipsoid, the plots of $g_o(\theta)$ for real media must necessarily be narrower in the angular range $\theta > \pi/2$ (downwelling light) and must necessarily be broader in the angular range $\theta \leq \pi/2$ (upwelling light).

The amount of departure of the $q_0(\theta)$ for a real medium from the standard ellipsoid may be taken as a measure of the anisotropy of scattering in the medium.

The Determination of ϵ

The quantity $\epsilon = k_\infty/\alpha$ is functionally related to $\tilde{\omega}_0$. In the case of isotropic scattering the relation is well known and of a particularly simple structure.[8] In general, ϵ is determined by viewing it as an eigenvalue of the integral equation (20). There is an alternate way, however, to characterize ϵ which, while not the most analytically direct way, is perhaps of greatest value in generating an insight into the physical significance of ϵ and also of supplying a link between ϵ and the directly observable quantities of the light in real media. This alternate characterization of ϵ stems from the following functional relation which holds between $K(z,\pm)$ and the various scattering and absorption functions of an arbitrary medium[7]:

$$1 = \frac{b(z,-)}{K(z,-) - a(z,-)} - \frac{b(z,+)}{K(z,+) + a(z,+)} .$$

[8] S. Chandrasekhar, Radiative Transfer (Clarendon Press, Oxford, 1950), p.19.

As depth is increased each term tends toward a well defined limit, so that as $z \to \infty$, the above relation tends to

$$1 = \frac{b(-)}{k_\infty - D(-)a} - \frac{b(+)}{k_\infty + D(+)a} .$$

This may be rewritten as

$$1 = \frac{\beta(-)}{\epsilon - (1-\tilde{\omega}_0) D(-)} - \frac{\beta(+)}{\epsilon + (1-\tilde{\omega}_0) D(+)} , \qquad (32)$$

which is the general characteristic equation for ϵ. Here

$$\beta(\pm) = \frac{\frac{1}{4\pi} \int_{\Xi_\mp} \left[\int_{\Xi_\pm} p(\theta,\phi;\theta',\phi')\, g_0(\theta')\, d\Omega' \right] d\Omega}{\pm \int_{\Xi_\pm} g_0(\theta') \cos\theta' \, d\Omega'} .$$

In the case of isotropic scattering,

$$\beta(\pm) = \frac{\tilde{\omega}_0}{2} D(\pm) ,$$

and (32) reduces to the following simple form after the explicit expressions for $D(\pm)$, as given by (30), are substituted in it:

$$\tilde{\omega}_0 = \frac{2\epsilon}{\ln\left[\frac{1+\epsilon}{1-\epsilon}\right]} . \tag{33}$$

This is the well known characteristic equation for ϵ in the isotropic case. As $\tilde{\omega}_0$ varies from 0 to 1, ϵ varies from 1 to 0. Hence, for all $\tilde{\omega}_0$, $0 \leq \tilde{\omega}_0 \leq 1$; $0 \leq \epsilon \leq 1$. Whenever scattering is present, i.e., whenever $\tilde{\omega}_0 > 0$, then the useful inquality $k_\infty < \alpha$ holds. Actually, the inequalities $0 \leq k_\infty/\alpha \leq 1$ hold in general.[3] This fact is made plausible by an inspection of (21) keeping in mind that the function g_0 is bounded in all physically meaningful situations, so that the denominator cannot vanish.

An Heuristic Proof of the Hypothesis

We now present a brief argument which makes plausible the assertion of the hypothesis, namely that $K(z,\theta,\phi) \longrightarrow k_\infty$ for all (θ,ϕ). For simplicity we will assume that the space is homogeneous and that scattering is isotropic. The resulting line of argument, while restricted to this special setting, can be made completely rigorous.

Under the present assumptions, we see that (18) may be written

$$N_q(z,\theta,\phi) = \frac{1}{4\pi}\,\tilde{\omega}_o\, h(z),$$

so that

$$K_q(z,\theta,\phi) = k(z).$$

Thus (19) reduces to

$$\frac{dK(z,\theta,\phi)}{dz} = \left[K(z,\theta,\phi) - k(z)\right]\left[K(z,\theta,\phi) + \alpha \sec\theta\right].$$

The preceding discussion of this equation showed that $K(z,\theta,\phi)$ always tends toward $k(z)$ for downward directions. Hence if $k(z)$ approaches a limit, $K(z,\theta,\phi)$ also tends toward this limit. More explicitly, suppose there is some depth z_o below which $k(z)$ is essentially constant and equal to k_∞. Then the above equation is a simple Riccati equation for $K(z,\theta,\phi)$ whose general solution is:

$$K(z,\theta,\phi) = \frac{k_\infty + \alpha\sec\theta\; C\exp\{(k_\infty + \alpha\sec\theta)z\}}{1 - C\exp\{(k_\infty + \alpha\sec\theta)z\}} \quad (34)$$

where

$$C = \frac{K(0,\theta,\phi) - k_\infty}{K(0,\theta,\phi) + \alpha\sec\theta}.$$

Since

$$k_\infty + \alpha\sec\theta < 0$$

for all $\theta > \pi/2$, it follows immediately from (34) that

$$\lim_{z \to \infty} K(z,\theta,\phi) = k_\infty$$

for all $\theta > \pi/2$. This means that the shape of the downwelling radiance distribution becomes fixed at great depths. It follows that the reflected upwelling radiance distribution also becomes fixed, so that the shape of the entire radiance distribution becomes fixed at great depths.

A Criterion for Asymptoticity

According to (34), $K(z,\theta,\phi)$ approaches k_∞ with least speed when $\theta = \pi$ (i.e., for the directly downward direction). Hence when $K(z,\pi,\phi)$ has come within a given distance of k_∞, we can conclude that the other values $K(z,\theta,\phi)$, $\pi/2 \leq \theta < \pi$ are within the same neighborhood of k_∞. From (34) it follows that

$$K(z,\pi,\phi) - k_\infty = \frac{(k_\infty - \alpha)\, C \exp\{(k_\infty - \alpha) z\}}{1 - C \exp\{(k_\infty - \alpha) z\}} . \tag{35}$$

Thus a preassigned value of the difference on the left side determines an associated value of z. Although (35) is exact only at great depths, and applies only in the present (isotropic) context, it nevertheless supplies a useful approximate method for estimating the depths at which $K(z,\pi,\phi) - k_\infty$ has attained a given small value.