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PHOTIC FIELD THEORY FOR NATURAL HYDROSOLS

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Rudolph W. Preisendorfer[2]

INTRODUCTION

The purpose of this note is to present an example of the application of the vector theory of the photic field ("light field") to an important class of scattering-absorbing optical media, namely the class of natural hydrosols consisting, e.g., of oceans, harbors, and lakes. The application is at the same time of practical value in that it yields explicit expressions for the depth-dependence of the light vector in terms of its components at the surface and certain of the optical properties of these media. Furthermore, the discussion presents particularly simple interpretations of the quasipotential and related functions. These interpretations emerge naturally from the geometry and physics of the present application. In this way we add to the evidence that the formalism of the photic field as developed by Moon, Spencer, and others (**1**), (**2**), [3] is of more than academic

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[3] The boldface numbers in parentheses refer to references appended to this paper.

interest, and in fact provides an elegant tool for the study of the light vector in the practical settings encountered in the study of hydrological optics.

While the practical context of the present discussion is limited specifically to that of natural hydrosols, the mathematical arguments apply equally well to any arbitrary plane-parallel scattering-absorbing medium in which the light vector possesses a quasipotential.

THE QUASI-IRROTATIONAL LIGHT FIELD IN NATURAL WATERS

The present discussion makes use of the concept of a quasi-irrotational light field, i.e., a light field in which at each depth z of a natural hydrosol, the light vector satisfies the integrability condition:

$$\underset{\sim}{H}(z) \cdot \operatorname{curl} \underset{\sim}{H}(z) = 0 . \tag{1}$$

In general, $\underset{\sim}{H}$, the light field (or vector irradiance function) is defined at each point $\underset{\sim}{x} = (x, y, z)$ of an optical medium by the relation:

$$\underset{\sim}{H}(\underset{\sim}{x}) = \int_{\Xi} \underset{\sim}{\xi}\, N(\underset{\sim}{x}, \underset{\sim}{\xi})\, d\Omega(\underset{\sim}{\xi}) , \tag{2}$$

where Ξ is the unit sphere (the collection of all unit vectors) in euclidean three-space E_3, and $N(\xi, \cdot)$ is the radiance (or helios) distribution at point $\underline{x}$.

As shown in (12) below, the justification of the use of (1) rests on the following two vectorial statements of well-known experimental facts about the spatial and directional distribution of light in natural hydrosols:

(I) For every given $z \geq 0$, $\underline{H}(x, y, z)$ is independent of x and y.

(II) For every given pair (x, y), $\underline{H}(x, y, z)$ is parallel to a fixed vertical plane for all $z \geq 0$.

Of course, some variations of $\underline{H}$ on horizontal planes, and some oscillations of the vertical plane containing $\underline{H}$ do occur in all natural hydrosols. However, properties (I) and (II) summarize the two most readily apparent and permanent gross features of the light field in natural waters.

INTERPRETATIONS OF THE INTEGRATING FACTOR

The general theory of vector fields asserts that to each quasi-irrotational light field one may associate two real-valued functions Φ and ζ, defined on the appropriate subset of E_3 representing the optical medium. These functions have the property that:

$$\underset{\sim}{H}(\underset{\sim}{x}) = \frac{1}{f(\underset{\sim}{x})} \nabla \Phi(\underset{\sim}{x}) . \tag{3}$$

Φ is the quasipotential function, and f is the integrating factor. Equation (3) is the necessary and sufficient condition that

$$\underset{\sim}{H}(\underset{\sim}{x}) \cdot \text{curl}\, \underset{\sim}{H}(\underset{\sim}{x}) = 0$$

at each $\underset{\sim}{x}$ of the medium (3).

In the present context the function f has particularly simple and interesting geometrical and physical interpretations.

The Geometrical Interpretation

Figure 1 defines the rectangular coordinate system usually adopted for the discussion of the light fields in natural hydrosols. The fixed plane of property (II) is the xz-plane. The unit vectors $\underset{\sim}{i}$ and $\underset{\sim}{k}$ are as shown. The unit vector $\underset{\sim}{j}$ along the positive y-axis is normal to the plane of the figure and directed away from the reader.

Consider an arbitrary rectangular path ABCD in the plane such that its sides are parallel to the coordinate axes. According to (3) and properties (I) and (II) of the light field in natural hydrosols, it follows that

$$\int_{ABCD} \zeta(x)\, \underline{H}(z)\, d\underline{s} = 0$$

and that

$$\int_{AB} \zeta(x)\, \underline{H}(z_1)\, d\underline{s} = \int_{DC} \zeta(x)\, \underline{H}(z_2)\, d\underline{s} .$$

Since the integrating factor is unique to within a multiplicative constant, the preceding expression requires that ζ be independent of x and y. Letting

$$\bar{H}(z, \underline{s}) = \underline{H}(z) \cdot \underline{s} ,$$

we have

$$\zeta(z_1)\, \bar{H}(z_1, \underline{s}) = \zeta(z_2)\, \bar{H}(z_2, \underline{s}) \qquad (4)$$

From this and the preceding observation on the uniqueness of ζ, we may write:

$$\zeta(z)\, \bar{H}(z, \underline{s}) = \bar{H}(0, \underline{s}) \qquad (5)$$

for all $z \geq 0$. Thus ζ is a dimensionless quantity which normalizes the horizontal component $\bar{H}(z, \underline{s})$ to $\bar{H}(0, \underline{s})$ at every depth $z \geq 0$.

The Physical Interpretation

The invariance with depth of the product

$$h(z)\,\bar{H}(z,-)$$

shows that the depth dependence of $h(z)$ is such that its logarithmic derivative is equal, to within an algebraic sign, to the logarithmic derivative of the net horizontal irradiance (pharosage) $\bar{H}(z,-)$. That is

$$\frac{1}{h(z)}\,\frac{dh(z)}{dz} = \frac{-1}{\bar{H}(z,-)}\,\frac{d\bar{H}(z,-)}{dz} \equiv \bar{K}(z,-) \qquad (6)$$

Now the logarithmic derivative $\bar{K}(z,-)$ of $\bar{H}(z,-)$ is but one member of an important family of apparent optical properties used in modern hydrological optics (4). This family of optical properties includes such well-known quantities as

$$k(z) = \frac{-1}{h(z)}\,\frac{dh(z)}{dz} , \qquad (7)$$

where

$$h(z) = \int_{\Xi} N(z,\xi)\, d\Omega(\xi) \qquad (8)$$

is the scalar irradiance (space pharosage) at depth z. Thus the logarithmic derivative of $\mathfrak{J}$ is none other than the K -function for the net horizontal irradiance in the $\underset{\sim}{i}$ -direction. According to (5) we may represent $\mathfrak{J}(z)$ as

$$\mathfrak{J}(z) = e^{C(z)} \tag{9}$$

where we have set

$$C(z) = \int_0^z \overline{K}(z', \underset{\sim}{i})\, dz'. \tag{10}$$

THE CURL AND DIVERGENCE OF THE SUBMARINE LIGHT FIELD

The Curl

Under the present assumptions (I) and (II) about the light field, and with the adopted coordinate system, the curl of $\underset{\sim}{H}$ takes the form:

$$\operatorname{curl} \underset{\sim}{H}(z) = -\underset{\sim}{j}\, \frac{d\bar{H}(z, \underset{\sim}{i})}{dz},$$

so that

$$\operatorname{curl} \underset{\sim}{H}(z) = \underset{\sim}{j}\, \bar{K}(z,\underset{\sim}{L})\, \bar{H}(z,\underset{\sim}{L}). \tag{11}$$

It follows that:

$$\underset{\sim}{H}(z) \cdot \operatorname{curl} \underset{\sim}{H}(z) = 0 \tag{12}$$

The Divergence

The derivation of the divergence relation for the light field in (source-free) scattering-absorbing media will require the use of the equation of transfer for radiance:

$$\underset{\sim}{\xi} \cdot \nabla N(\underset{\sim}{x},\underset{\sim}{\xi}) = -\alpha(\underset{\sim}{x}) N(\underset{\sim}{x},\underset{\sim}{\xi}) + N_*(\underset{\sim}{x},\underset{\sim}{\xi}), \tag{13}$$

where the path function (heliosent) N_* is defined as:

$$N_*(\underset{\sim}{x},\underset{\sim}{\xi}) = \int_{\Xi} \sigma(\underset{\sim}{x};\underset{\sim}{\xi};\underset{\sim}{\xi}') N(\underset{\sim}{x},\underset{\sim}{\xi}')\, d\Omega(\underset{\sim}{\xi}'). \tag{14}$$

Here σ and α are, respectively, the volume scattering function and the volume attenuation function.

If we set

$$s(\underline{x}) = \int_{\Xi} \sigma(\underline{x};\underline{\xi}';\underline{\xi})\, d\Omega(\underline{\xi}'), \tag{15}$$

which is the (volume) total scattering function, then from general radiative transfer theory,

$$\alpha(\underline{x}) = a(\underline{x}) + s(\underline{x}), \tag{16}$$

where a is the volume absorption function.

Returning to (13) and integrating over Ξ, we have:

$$\operatorname{div} \underline{H}(\underline{x}) = -\alpha(\underline{x})\, h(\underline{x}) + s(\underline{x})\, h(\underline{x}),$$

which reduces to the required divergence relation:

$$\operatorname{div} \underline{H}(\underline{x}) = -a(\underline{x})\, h(\underline{x}). \tag{17}$$

For the present geometry, (17) reduces to

$$\frac{d\bar{H}(z,\underset{\sim}{k})}{dz} = a(z)h(z), \tag{18}$$

where

$$\bar{H}(z,\underset{\sim}{k}) = \underset{\sim}{H}(z)\cdot\underset{\sim}{k} \tag{19}$$

is the vertical component (the net upward irradiance) of the light vector at depth z .

GENERAL REPRESENTATION OF THE SUBMARINE LIGHT FIELD

The starting point of the present derivation is taken as theorem 9 of reference (1) which asserts (in the present notation) that if

$$\operatorname{div}\underset{\sim}{H}(z) = -a(z)h(z)$$

and

$$\underset{\sim}{H}(z)\cdot\operatorname{curl}\underset{\sim}{H}(z) = 0,$$

then a quasipotential Φ and integrating factor ζ exist such that

$$\nabla^2 \Phi(x,y,z) = \frac{1}{\zeta(z)}\left[-a(z)h(z) + \nabla\left(\frac{1}{\zeta(z)}\right)\cdot\nabla\Phi(x,y,z)\right]. \tag{20}$$

It follows from properties (i) and (II) and the preceding equation that Φ can be at most linear in x and y, so that from (3):

$$H(z) = \frac{1}{\zeta(z)}\left[\underset{\sim}{i}\frac{\partial\Phi}{\partial x} + \underset{\sim}{j}\frac{\partial\Phi}{\partial y} + \underset{\sim}{k}\frac{\partial\Phi}{\partial z}\right], \tag{21}$$

we have

$$\frac{\partial\Phi}{\partial x} = A, \tag{22}$$

a constant. Furthermore, by virtue of the present coordinate system,

$$\frac{\partial\Phi}{\partial y} = 0. \tag{23}$$

Hence Φ may be represented in the present context by

$$\Phi(x,y,z) = A x + f(z).$$

According to (20) it remains to determine A and $f'(z) = df(z)/dz$. Equation (20) may then be written:

$$f''(z) - \bar{K}(z,\underset{\sim}{i}) - \zeta'(z)\,a(z)\,h(z) = 0.$$

The integrating factor for this differential equation is clearly

$$\frac{1}{\zeta(z)} = e^{-c(z)},$$

so that

$$\left(f(z)/\zeta(z)\right)' = a(z)\,h(z),$$

and

$$f'(z)/\zeta(z) = f'(0) + \int_0^z a(z')\,h(z')\,dz'.$$

Finally, from (21) it follows that

$$\underset{\sim}{H}(z) = \underset{\sim}{i}\,\bar{H}(0,\underset{\sim}{i})\,e^{-c(z)} + \underset{\sim}{k}\left[\bar{H}(0,\underset{\sim}{k}) + \int_0^z a(z')h(z')\,dz'\right], \tag{24}$$

which is the desired general representation of $\underset{\sim}{H}$.

EXAMPLES

The Case of Isotropic Scattering

For the remaining portion of this note we will assume that the medium is homogeneous, i.e., that α (and hence σ , s , and a) is independent of depth z . The present paragraph will be concerned with an illustration of the particular form that $\underset{\sim}{H}(z)$ takes in a medium that scatters isotropically and which is irradiated at the upper boundary by collimated flux incident at an angle $\theta_0 = \arccos \mu_0, \phi_0 = 0$, where ϕ_0 is measured in a horizontal plane from the positive x -axis.

Now it is easy to verify that the diffuse component of the light field (i.e., that part consisting of all radiant flux scattered one or more times) is symmetrical about the z -axis. Hence the net horizontal irradiance receives no contribution from the diffuse light field. Therefore

$$\bar{H}(z, \underset{\sim}{i}) = \bar{H}(0, \underset{\sim}{i}) e^{-\alpha z/\mu_0} , \qquad (25)$$

so that $\bar{K}(z, \underset{\sim}{L})$ in this case is represented by

$$\bar{K}(z, \underset{\sim}{L}) = \alpha/\mu_0 \, , \tag{26}$$

and

$$c(z) = \alpha z/\mu_0 \, . \tag{27}$$

Asymptotic Form of the Light Field

In optically infinitely deep media the values $k(z)$ of the function k defined in (7) rapidly approach, with increasing z, a fixed magnitude k_∞ which is independent of the external lighting conditions and which depends only on the inherent optical properties of the medium (5). In view of this it is permissible, for most engineering calculations, to assume that there is a depth $z_0 \geq 0$ below which $k(z) = k_\infty$. From the divergence relation (18) we see that in general

$$\bar{H}(z_2, \underset{\sim}{k}) - \bar{H}(z_1, \underset{\sim}{k}) = \int_{z_1}^{z_2} a(z) h(z) \, d(z) \, ; \tag{28}$$

so that in particular,

$$\bar{H}(z,\underset{\sim}{k}) - \bar{H}(0,\underset{\sim}{k}) = \int_0^z a(z')\,h(z')\,dz'.$$

Furthermore, in the present case,

$$\bar{H}(z,\underset{\sim}{k}) - \bar{H}(z_0,\underset{\sim}{k}) = a\int_{z_0}^{z} h(z')\,dz' = a\,h(z_0)\int_{z}^{z} e^{-k_{00}(z'-z_0)}\,dz'$$

$$= \frac{-a\,h(z_0)\,e^{k_{00}z_0}}{k_{00}}\left[e^{-k_{00}z} - e^{-k_{00}z_0}\right]$$

$$= \frac{a\,h(z_0)}{k_{00}}\left[1 - e^{-k_{00}(z-z_0)}\right].$$

It follows that the $\underset{\sim}{k}$ -component of $\underset{\sim}{H}(z)$ in (24) may be written

$$\bar{H}(z,\underset{\sim}{k}) = \bar{H}(0,\underset{\sim}{k}) + \int_0^z a(z')\,h(z')\,dz'$$

$$= \bar{H}(z_0,\underset{\sim}{k}) + \int_{z_0}^{z} a(z')\,h(z')\,dz'$$

$$= \bar{H}(z_0,\underset{\sim}{k}) + \frac{a\,h(z_0)}{k_{00}}\left[1 - e^{-k_{00}(z-z_0)}\right].$$

Now since $\bar{H}(z,\underset{\sim}{k}) \to 0$ as $z \longrightarrow \infty$, it follows that we may set:

$$\bar{H}(z_0,\underset{\sim}{k}) = -\left(\frac{a}{k_{00}}\right) h(z_0),$$

so that (24) reduces to

$$H(z) = \underset{\sim}{\iota}\, \bar{H}(z_0, \underset{\sim}{\iota})\, e^{-c(z-z_0)} + \underset{\sim}{k}\, \bar{H}(z_0, \underset{\sim}{k})\, e^{-k_{oo}(z-z_0)} \quad , \tag{29}$$

for $z \geqq z_0$.

A further simplification is effected if we observe that for $z \gg z_0$, the diffuse component of the light field is essentially summetrical about the z-axis (5), so that, as in the isotropic case,

$$\bar{K}(z, \underset{\sim}{\iota}) = D^0 \alpha$$

where we take

$$D^0 = \frac{1}{\mu_0}$$

for clear sunny skies with sun at $\theta_0 = \arccos \mu_0$ from the zenith, or

$$D^0 = 1$$

for overcast days. Thus, with these assumptions, (24) takes the particularly simple approximate form:

$$H(z) = \underset{\sim}{\iota}\, \bar{H}(z_0, \underset{\sim}{\iota})\, e^{-D^0 \alpha (z-z_0)} + \underset{\sim}{k}\, \bar{H}(z_0, \underset{\sim}{k})\, e^{-k_{oo}(z-z_0)} \quad , \tag{30}$$

where D^o may take either of the above special values. For most engineering applications it is permissible to take $z_o = 0$ in (29) or (30).

We conclude by making a few observations on the limiting directions of $\underset{\sim}{H}$ as $z \to \infty$. First, if $s \neq 0$, then $k_\infty < \alpha, (\underset{\sim}{\xi})$. If, in addition, we also have $a \neq 0$ then from (30) and the fact that $\overline{H}(z, \underset{\sim}{k}) < 0$ it is clear that

$$\lim_{z \to \infty} \frac{\underset{\sim}{H}(z)}{|\underset{\sim}{H}(z)|} = -\underset{\sim}{k}.$$

If, on the other hand we have $a = 0$, then $\overline{H}(z, \underset{\sim}{k}) = 0$ and

$$\frac{\underset{\sim}{H}(z)}{|\underset{\sim}{H}(z)|} = \pm \underset{\sim}{i}'$$

for all z; the sign being that of $\overline{H}(0, \underset{\sim}{i}')$. Finally, if $s = 0$, then the problem of the explicit determination of $\underset{\sim}{H}(z)$ for all z reduces to a relatively trivial (although sometimes tedious) calculation. In this case the limiting direction of $\underset{\sim}{H}$ depends in a simple way only on the external lighting conditions. If $N^o(0, \underset{\sim}{\xi})$ represents the incident radiance distribution at $z = 0$, suppose $|\underset{\sim}{\xi}_o \cdot \underset{\sim}{k}|$ is the largest value for which $N^o(0, \underset{\sim}{\xi}_o) \neq 0$. Then the limiting direction as $z \to \infty$, of $\underset{\sim}{H}(z)$ is along the line defined by $\underset{\sim}{\xi}_o$.

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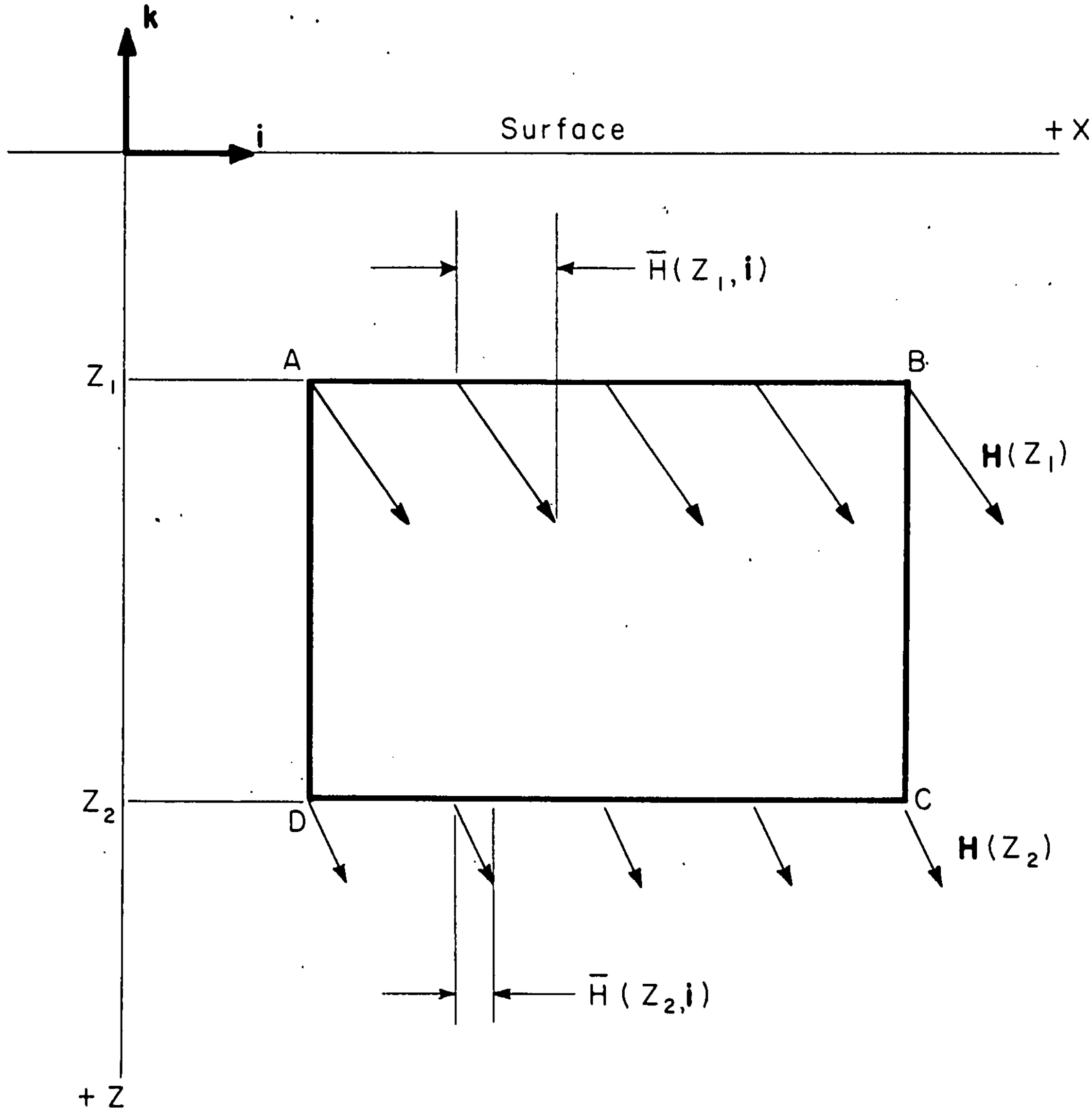

Figure 1

Rudolph W. Preisendorfer