Visibility Laboratory
University of California
Scripps Institution of Oceanography
San Diego 52, California

A STUDY OF LIGHT-STORAGE PHENOMENA IN SCATTERING MEDIA

Rudolph W. Preisendorfer

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Approved:

Seibert Q. Duntley
Seibert Q. Duntley, Director
Visibility Laboratory

Approved for Distribution:

Roger Revelle
Roger Revelle, Director
Scripps Institution of Oceanography

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Rudolph W. Preisendorfer

Scripps Institution of Oceanography, University of California

La Jolla, California

INTRODUCTION

Everyday Examples of Light Storage

Those who have looked out of a window of an airplane as it descended into a sun-bathed cloud layer may recall the sudden transition to a brilliant ambient field of light, and how the sensation of brightness in every direction increased to dazzling proportions as the airplane descended further into the upper regions of the cloud. This phenomenon is but one of many common examples of the storage of light by the mechanism of scattering. One can also see evidences of light storage on overcast nights on the outskirts of large cities: the cloud layer hovering low over the city is deeply and extensively illuminated from the street and building lights below. Flashes of lightning in storm clouds can light up an extensive cloud layer from horizon to horizon even though the actual volume taken up by the network of electrical discharges is a minute fraction of the illuminated volume. Lighthouses on densely fogged nights pour a well-defined beam of light into a surrounding fog with the result that the beam and lighthouse are imbedded in a field of scattered light which, under suitable conditions, may be observed by mariners far sooner than the light of the revolving beam.

These examples illustrate the phenomenon of the storage of light in scattering media. The sense of the word "storage" as used in its every-day sense: the accumulation or building up of radiant energy in the scattering material that surrounds the source of the energy. If one were to quickly extinguish the light source, the stored light would not immediately disappear with the extinction of the source; rather, the scattered light energy would disappear at a much slower rate. If the sun were to suddenly go out, the visible scattered light stored in the earth's atmosphere would take on the order of a score of microseconds to be lost into space, converted into longer wavelengths of radiation and other forms of energy.

The decaying atmospheric light field is like the diminishing reverberation of organ notes in a spacious auditorium in which the acoustical energy is momentarily entrapped and redirected by the walls of the auditorium. In the case of light, the walls of the auditorium are replaced by multitudes of tiny scattering centers comprising clouds, fogs, or even the entire atmosphere, and the hydrosphere of the earth: the light impinges on the scattering centers and is redirected again and again by scattering.

Thus, the energy of a pencil of photons, which ordinarily traverses a given volume of empty space in one microsecond, could, in principle, be cycled and recycled within the confines of the volume for a period of several dozens of microseconds before it escapes or is transformed.

Therefore, if a continuous steady beam of light is poured into such a volume, the steady state density of scattered light stored within the volume could be dozens of times greater than the average density of the light ordinarily within the beam.

Do all these phenomena have a common simple description? Is there a small set of properties of the medium and of the source that, when isolated, can serve as the salient parameters in an analytical description of the stored light field?

The Purpose and Scope of the Present Study

In this paper we embark on a preliminary attempt to describe the phenomenon of light storage in precisely defined terms. Once we have decided on an exact radiometrical definition of "stored light energy," we go on to formulate a simple mathematical model of the light field in a scattering-absorbing medium which can describe how the stored light energy depends on the inherent optical properties of the medium, the geometry of the medium, and the properties of the light source.

It turns out that there are several ways in which we may formulate the description of "stored light energy". The form of the description depends on one's choice of the radiometric quantity used in the description. For example, we find that there is a description associated with the radiometric concept of radiance, another description with irradiance, another with radiant density, and still another with radiant energy.

In this paper we will limit our attention exclusively to the description of stored light energy by means of the concept of radiant energy. The resulting description is by far the most natural of all the various possibilities; it is, by a happy coincidence, also the most simple to deal with, and the easiest to draw examples from.

In the event that more detailed descriptions of storage phenomena than those developed in the present study are ever required, we have formulated the requisite transport equations of those radiometric quantities which will most likely form the basis for such descriptions, namely the time-dependent transport equations for radiant density and radiance. A preliminary investigation of the time-dependent radiant flux problem made some time ago[1] supplements the results of the present study by providing detailed numerical and graphical illustrations of the solution of the n-ary radiant energy equations.

Summary of Results

The paper begins with a phenomenological definition of multiply scattered light, or more precisely, n-ary radiance. The definition employs the basic concepts of the volume scattering function and radiance. A time-dependent transport equation for n-ary radiance is then an immediate consequence of the definition, and the transport equations for n-ary radiant density and n-ary radiant energy follow immediately from that of n-ary radiance.

Detailed time-dependent solutions of the n-ary radiant energy equations are obtained for the case of a steady source turned on and turned off in an infinite homogeneous medium. The solutions yield a multitude of properties of the time-dependent n-ary radiant energy, and in particular a set of useful facts about the steady state magnitudes of primary scattered radiant energy, secondary scattered radiant energy, etc., in the medium. These relations give immediate insight into many natural light storage phenomena.

For example, the steady state solutions allow an estimate of the fraction of the total radiant energy within an extensive scattering medium consisting of scattered radiant energy. Specifically, let U represent the total steady state radiant energy; let U^o represent the amount of U consisting of <u>reduced radiant energy</u> from the source (associated with photons which have not yet been scattered or absorbed); and finally, let U^* represent the amount of U consisting of <u>diffuse radiant energy</u> within the medium (associated with photons which have undergone at least one scattering operation). The ratio U^*/U is then a measure of the relative amount of scattered radiant energy in the medium. It is a number which lies between zero and one and will be referred to as the <u>storage capacity</u> of the medium.

In the case of an infinite homogeneous medium the storage capacity has a particularly simple representation in terms of the total volume scattering coefficient s, and the volume attenuation coefficient α of the medium:

$$\text{storage capacity} = \frac{U^*}{U} = \frac{s}{\alpha} \equiv \omega_0 \,, \tag{1}$$

where ω_0 is the albedo for single scattering. In the case of nonhomogeneous or finite media, the storage capacity is a more complicated function of ω_0 and the geometry of the medium. (Examples of more general storage capacity formulas are given in Equations (43) and (44).) But even in this simple context, we gain important insight into storage phenomena in general: the storage capacity depends basically on the <u>relative</u> magnitudes of s and α. Thus if we consider two media, one in which $s = 0.01$/meter, $\alpha = 0.02$ /meter, and another in which $s = 0.10$/meter, $\alpha = 0.20$ /meter, we see that the former medium has an attenuation length of $1/\alpha = 50$ meters while the latter medium is an order of magnitude more optically dense with an attenuation length of $1/\alpha = 5$ meters. However, the albedo for single scattering for each medium is $\omega_0 = 0.5$. Thus, despite the great disparity in optical density of these media, their storage capacities have a common value, namely $U^*/U = 0.5$, indicating that in the steady state in each medium, the stored radiant energy (in scattered form) is 50% of the total energy within each medium.

Further results about the storage capacity of infinite homogeneous media are obtained by considering the scattering-order decomposition of U^*. Since U^* is, by definition, the radiant energy in the medium which has been scattered at least once, we may write

$$U^* = U^1 + U^2 + \cdots \,, \tag{2}$$

where U^n, $n = 1, 2, \ldots$, is that part of U^* which has been scattered precisely n times. U^n is called the n-ary radiant energy. The total energy in the medium may then be represented as the sum of the infinite series:

$$U = U^0 + U^1 + U^2 + \cdots \quad . \tag{3}$$

One of the results of the present study is the fact that

$$\frac{U^n}{U^0} = \omega_0^n \, , \quad n = 0, 1, 2, \ldots , \tag{4}$$

i.e., the ratio of steady state n-ary radiant energy U^n to reduced radiant energy U^n is simply the nth power of ω_0 (or in this special context, the nth power of the storage capacity of the medium, by virtue of Equation (1)).

A few simple corollaries of (4) are:

$$\frac{U^{n+1}}{U^n} = \omega_0 \, , \qquad \frac{U^{n+k}}{U^n} = \omega_0^k \, , \tag{5}$$

$$\frac{U}{U^0} = 1 + \frac{U^1}{U^0} + \frac{U^2}{U^0} + \cdots = 1 + \omega_0 + \omega_0^2 + \cdots = \frac{1}{1-\omega_0} , \tag{6}$$

$$\frac{U^*}{U^0} = \frac{U^1}{U^0} + \frac{U^2}{U^0} + \cdots = \omega_0 + \omega_0^2 + \cdots = \frac{\omega_0}{1-\omega_0} \, . \tag{7}$$

The last two results check with Equation (1) when one recalls that $U = U^0 + U^*$. The first part of Equation (5) states that the ratio of the amount of (n+1)-ary radiant energy to n-ary radiant energy is independent of n and equal to ω_0; the second part of (5) is an immediate consequence of the first part of (5).

Equations (4) - (7) constitute the fine structure properties of the radiation field in that they give detailed information about the relative magnitudes of radiant energy comprised of various n-ary components. It should be pointed out that the quantities U^n, $n = 0, 1, 2, \ldots$, are non-observable quantities: no present-day instrument can isolate and measure the individual component U^n of U. Of all the quantities discussed above, only U is directly measurable.

Some Practical Consequences of the Results

Although the components U^n of U are not directly measurable, knowledge of the relative magnitudes of U^n and U^{n+1}, as summarized in (5), is immediately applicable to many practical problems in which observables are of primary interest. One such problem is the computation of the steady state radiance distributions in the medium by means of the n-ary approach which decomposes radiance N into its n-ary components:

$$N = N^0 + N^* \tag{8}$$

where

$$N^* = N^1 + N^2 + \cdots \quad . \tag{9}$$

In this approach to the multiple scattering problem N^0 is known once the α and σ of the medium are given along with the boundary radiance distributions. From N^0, N^1 is obtained by a well-defined integral operation $\mathcal{Q}$, and once N^1 is known, N^2, etc., is determinable by the same operation:

$$
\begin{aligned}
N^{n+1} &= \mathcal{Q}(N^n) \\
&= \int_{X \times \Xi} N^n R \, dv, \quad n = 0, 1, 2, \ldots,
\end{aligned}
\tag{10}
$$

where R is a known function of α and σ, and $X \times \Xi$ is the phase space associated with the medium. Now, by definition,

$$
U^n = \int_{X \times \Xi} N^n \, dv, \tag{11}
$$

so that, from (10) and (11),

$$
U^{n+1} = \bar{R} \, U^n \tag{12}
$$

where $\bar{R}$ depends in general on the geometry of the medium, α, and σ. In the case of an infinite homogeneous medium (12) reduces to the left-hand side of (5). The point to observe here is that by virtue of (10) and (11) the <u>average</u> values $\bar{N}^{n+1}$ and $\bar{N}^n$ over $X \times \Xi$ are then related the same way as the values U^{n+1} and U^n. This fact leads to a method of approximeting (9) by a finite series:

$$
N^* \cong N^1 + N^2 + \cdots + N^n, \tag{13}
$$

with a known truncation error. For example, if $\omega_0 = 0.5$, then from the preceding observation,

$$\bar{N}^3 = 0.5\,\bar{N}^2 = 0.025\,\bar{N}^1 \cong 0.0125\,\bar{N}^0 .$$

Hence the average contribution to (9) by each radiance term higher than scattering order 3 is on the order of less than one percent of the average magnitude $\bar{N}^0$ of N^0. From this follows the truncation error estimate:

$$\bar{N}^4 + \bar{N}^5 + \cdots \cong \frac{\omega_0^4\,\bar{N}^0}{1-\omega_0} = 0.125\,\bar{N}^0 .$$

Thus terminating (9) at N^3 will, on the average, induce a truncation error of about 12% in the estimate of N for this particular medium. If a greater accuracy in the estimate of N is needed, (9) may be truncated at N^4 with a resulting truncation error of about 6%, and so on. The general expression for the truncation error resulting from terminating (8) at N^n is:

$$N^{n+1} + N^{n+2} + \cdots \cong \frac{\omega_0^{n+1}\,\bar{N}^0}{1-\omega_0} . \tag{14}$$

In this way, knowledge of the storage capacity of an extensive, practically homogeneous medium can be used to verify and further important transfer calculations associated with the medium.

We shall now go on to the detailed development of those transport equations and solutions which form a basis for the above results.

DEFINITIONS

n-ary Radiance and Related Concepts

Suppose a source of radiant flux irradiates a region of space filled with scattering-absorbing material. For concreteness, we may choose any one of the examples of sources and media cited in the Introduction. Suppose we choose the lighthouse example. Let its beam have an <u>inherent radiance</u> N^o just outside the tower window. As the beam just enters the fog it is partially scattered in all directions. The magnitude $N_*^1(\theta)$ of the primary scattered radiance in the beam generated per unit length in a direction making an angle θ with the beam is given by:

$$N_*^1(\theta) = N^o \sigma(\theta) \Omega \ , \tag{1}$$

where $\sigma(\theta)$ is the value of the volume scattering function for the angle of scattering θ, and Ω is the solid angle subtense of the lighthouse reflector (which is presumed to have a uniform inherent radiance N^o over its extent). Equation (1) defines the <u>primary path function</u>, and it is associated with the radiance of primary scattered light generated per unit length in the θ-direction. If we move out along the beam and compute $N_*^1(\theta)$ at some distance r from the lighthouse, we must replace N^o in (1) by $N^o T_r$, the <u>reduced radiance</u>, where T_r is the <u>beam transmittance</u> for the path of length r (in homogeneous spaces $T_r = e^{-\alpha r}$).

If we had a more extensive source of inherent radiance, such as the brightly lit city under the cloud, or an extensive network of lightning flashes within the cloud, then N^o may arrive at a given point from virtually all directions. To cover this possibility we may generalize (1) to the following form:

$$N_*^1(\underset{\sim}{x},\underset{\sim}{\xi}) = \int_{\Xi} \sigma(\underset{\sim}{x};\underset{\sim}{\xi};\underset{\sim}{\xi}')\, N^0(\underset{\sim}{x},\underset{\sim}{\xi}')\, d\Omega(\underset{\sim}{\xi}'). \quad (2)$$

Here $N^0(\underset{\sim}{x},\underset{\sim}{\xi}')$ is the reduced radiance of the source at point $\underset{\sim}{x}$ in the direction $\underset{\sim}{\xi}'$ (which refers to the direction of motion of the unscattered photons arriving at $\underset{\sim}{x}$ from the source). Ξ is the collection of all directions, and $\sigma(\underset{\sim}{x};\underset{\sim}{\xi};\underset{\sim}{\xi}')$ is the value of the volume scattering function at $\underset{\sim}{x}$ for the incident $\underset{\sim}{\xi}'$ and scattered $\underset{\sim}{\xi}$ directions. Equation (2) defines the <u>primary path function</u> $N_*^1(\underset{\sim}{x},\underset{\sim}{\xi})$ at $\underset{\sim}{x}$ for the direction $\underset{\sim}{\xi}$, and is interpreted as the radiance of single scattered light generated at $\underset{\sim}{x}$ per unit length in the direction $\underset{\sim}{\xi}$.

Now primary scattered light is generally produced at all points of the medium, (for example, at each point of the cloud above the city). To find the total amount of primary radiance at $\underset{\sim}{x}$ in the direction $\underset{\sim}{\xi}$, we must integrate over the entire path in the medium defined by this direction. Specifically, if $\underset{\sim}{x}'$ is a general point along this path, then $N_*^1(\underset{\sim}{x}',\underset{\sim}{\xi})$ is the primary radiance generated at $\underset{\sim}{x}'$ per unit length in the direction $\underset{\sim}{\xi}$. If dr' denotes an increment of length along the path, $N_*^1(\underset{\sim}{x}',\underset{\sim}{\xi})\,dr'$ then represents the amount of primary radiance generated over the length dr' at $\underset{\sim}{x}$ in the direction $\underset{\sim}{\xi}$. This primary radiance is then transmitted a distance $r - r'$ (the distance between $\underset{\sim}{x}$ and $\underset{\sim}{x}'$ along the path) to arrive at $\underset{\sim}{x}$ with magnitude

$$N_*^1(\underset{\sim}{x}',\underset{\sim}{\xi})\, T_{r-r'}(\underset{\sim}{x}',\underset{\sim}{\xi})\, dr',$$

where

$$T_{r-r'}(\underset{\sim}{x}', \underset{\sim}{\xi}) = \exp\left\{-\int_0^{r-r'} \alpha(r'')\, dr''\right\}$$

is the beam transmittance of the path with initial point $\underset{\sim}{x}'$, direction $\underset{\sim}{\xi}$, and length $r-r'$. Hence the primary radiance $N_r^1(\underset{\sim}{x}, \underset{\sim}{\xi})$ at $\underset{\sim}{x}$ in the general direction $\underset{\sim}{\xi}$ is clearly

$$N_r^1(\underset{\sim}{x}, \underset{\sim}{\xi}) = \int_0^r N_*^1(\underset{\sim}{x}', \underset{\sim}{\xi})\, T_{r-r'}(\underset{\sim}{x}', \underset{\sim}{\xi})\, dr', \tag{3}$$

where r is the length of the path from x to the boundary of the medium in the direction $-\underset{\sim}{\xi}$. In the interests of brevity we will omit explicit reference to r in the subsequent notation; it will be understood that $N^1(\underset{\sim}{x}, \underset{\sim}{\xi})$ is obtained by integrating over a path which extends from $\underset{\sim}{x}$ to the boundary of the medium.

We now have gone through a complete cycle of definitions, starting from N^0, the inherent radiance, going to N_*^1, the primary radiance path function, and ending with N^1, the primary radiance function. This cycle may be repeated ad infinitum: starting with N^1, we define N_*^2 by increasing by unity the superscript integers occurring in (2) and then increasing the superscripts in (3) by unity. The interpretations of the resulting N_*^2 and N^2 are thus exactly analogous to N_*^1 and N^1. N_*^2 is the <u>secondary path function</u>, and N^2 is the <u>secondary radiance function</u>. Suppose we have defined N^{n-1}, $n \geq 1$, then the n-ary path function N_*^n is defined as:

$$N_*^n(\underset{\sim}{x}, \underset{\sim}{\xi}) = \int_{\Xi} \sigma(\underset{\sim}{x}; \underset{\sim}{\xi}; \underset{\sim}{\xi}'')\, N^{n}(\underset{\sim}{x}, \underset{\sim}{\xi}')\, d\Omega(\underset{\sim}{\xi}'), \tag{4}$$

and the n-ary radiance is defined as:

$$N_r^n(\underset{\sim}{x}, \underset{\sim}{\xi}) = \int_0^r N_*^n(\underset{\sim}{x}', \underset{\sim}{\xi})\, T_{r-r'}(\underset{\sim}{x}', \underset{\sim}{\xi})\, dr' \,. \tag{5}$$

As agreed above, we omit explicit reference to r, and write $N_r^n(\underset{\sim}{x}, \underset{\sim}{\xi})$ as $N^n(\underset{\sim}{x}, \underset{\sim}{\xi})$, or N^n for short.

The observable radiance $N(\underset{\sim}{x}, \underset{\sim}{\xi})$ at $\underset{\sim}{x}$ in the direction $\underset{\sim}{\xi}$ is evidently representable as the sum of the following infinite series:

$$\begin{aligned} N(\underset{\sim}{x}, \underset{\sim}{\xi}) &= \sum_{n=0}^{\infty} N^n(\underset{\sim}{x}, \underset{\sim}{\xi}) \\ &= N^0(\underset{\sim}{x}, \underset{\sim}{\xi}) + N^*(\underset{\sim}{x}, \underset{\sim}{\xi}) \,, \end{aligned} \tag{6}$$

where

$$N^*(\underset{\sim}{x}, \underset{\sim}{\xi}) = \sum_{n=1}^{\infty} N^n(\underset{\sim}{x}, \underset{\sim}{\xi}) \,, \tag{7}$$

is the <u>path radiance</u>.

The <u>storage capacity for radiance</u> of the medium at $\underset{\sim}{x}$ in the direction $\underset{\sim}{\xi}$ is defined as

$$\frac{N^*(\underset{\sim}{x}, \underset{\sim}{\xi})}{N(\underset{\sim}{x}, \underset{\sim}{\xi})} \,. \tag{8}$$

n-ary Radiant Density

We define n-ary radiant density $u^n(\underset{\sim}{x})$ at $\underset{\sim}{x}$ as

$$u^n(\underset{\sim}{x}) = \frac{1}{v(\underset{\sim}{x})} \int_{\Xi} N^n(\underset{\sim}{x}, \underset{\sim}{\xi})\, d\Omega(\underset{\sim}{\xi}), \tag{9}$$

where $v(\underset{\sim}{x})$ is the speed of light at $\underset{\sim}{x}$ in X.

The observable radiant density $u(\underset{\sim}{x})$ is

$$\begin{aligned} u(\underset{\sim}{x}) &= \sum_{n=0}^{\infty} u^n(\underset{\sim}{x}) \\ &= u^0(\underset{\sim}{x}) + u^*(\underset{\sim}{x}), \end{aligned} \tag{10}$$

where

$$u^*(\underset{\sim}{x}) = \sum_{n=1}^{\infty} u^n(\underset{\sim}{x}), \tag{11}$$

is the <u>diffuse radiant density</u> at $\underset{\sim}{x}$ and $u^0(\underset{\sim}{x})$ is the <u>reduced radiant density</u> at $\underset{\sim}{x}$.

The <u>storage capacity</u> for radiant density of the medium at $\underset{\sim}{x}$ is defined as:

$$\frac{u^*(\underset{\sim}{x})}{u(\underset{\sim}{x})}. \tag{12}$$

n-ary Radiant Energy

We define the n-ary radiant energy content $U^n(X)$ of the medium X as:

$$U^n(X) = \int_X u^n(\underset{\sim}{x})\, dV(\underset{\sim}{x}), \tag{13}$$

and U^n for short if X is understood.

The observable radiant energy $U(X)$ is

$$U(X) = \sum_{n=0}^{\infty} U^n(X) \tag{14}$$
$$= U^0(X) + U^*(X),$$

where

$$U^*(X) = \sum_{n=1}^{\infty} U^n(X), \tag{15}$$

is the <u>stored radiant energy</u> and $U^0(X)$ is the <u>reduced radiant energy</u> in X.

The <u>storage capacity</u> for radiant energy of X is

$$\frac{U^*(X)}{U(X)} . \tag{16}$$

The storage capacities (8), (12) and (16) all refer to a steady state condition of the light field throughout X.

EQUATION OF TRANSFER FOR n-ARY RADIANCE

Suppose a beam of n-ary scattered photons ($n = 1, 2, \ldots$) is at $\underset{\sim}{x}$, moving in the direction $\underset{\sim}{\xi}$ through a region of constant index of refraction. Some of the n-ary photons are lost by scattering and absorption processes. The amount of $N^n(\underset{\sim}{x}, \underset{\sim}{\xi})$ lost by attenuation in an increment of path length dr is:

$$\alpha(\underset{\sim}{x}) N^n(\underset{\sim}{x}, \underset{\sim}{\xi})\, dr .$$

The population of n-ary photons is increased by the scattering in of (n-1)-ary photons in accordance with (4). The amount of this increase of $N^n(\underset{\sim}{x}, \underset{\sim}{\xi})$ over each increment of path length dr is:

$$N_*^n(\underset{\sim}{x}, \underset{\sim}{\xi})\, dr .$$

Therefore, the <u>net</u> space rate of change of $N^n(\underset{\sim}{x}, \underset{\sim}{\xi})$ is governed by

$$\frac{dN^n(\underset{\sim}{x}, \underset{\sim}{\xi})}{dr} = -\alpha(\underset{\sim}{x}) N^n(\underset{\sim}{x}, \underset{\sim}{\xi}) + N_*^n(\underset{\sim}{x}, \underset{\sim}{\xi}), \qquad (17)$$

which is the desired steady state, source-free equation of transfer for N^n.

Equation (17) may easily be put into time-dependent form. We write the total space derivative occurring in (17) as:

$$\frac{dN^n}{dr} = \frac{1}{v} \frac{dN^n}{dt}$$

where v is the speed of light at x and t is time measured from some epoch.

Then, in cartesian coordinates,

$$\frac{dN^n}{dt} = \frac{\partial N^n}{\partial t} + \frac{\partial N^n}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial N^n}{\partial x_2}\frac{dx_2}{dt} + \frac{\partial N^n}{\partial x_3}\frac{dx_3}{dt}.$$

Now the vector

$$\frac{1}{v}\left(\frac{dx_1}{dt}, \frac{dx_2}{dt}, \frac{dx_3}{dt}\right)$$

is simply the unit vector $\underset{\sim}{\xi}$, so that the time-dependent form of (17) is

$$\frac{1}{v}\frac{\partial N^n(\underset{\sim}{x},\underset{\sim}{\xi},t)}{\partial t} + \underset{\sim}{\xi}\cdot\nabla N^n(\underset{\sim}{x},\underset{\sim}{\xi},t) = -\alpha(\underset{\sim}{x},t)N^n(x,\xi,t) + N_*^n(\underset{\sim}{x},\underset{\sim}{\xi},t), \quad n \geq 1, \tag{18}$$

where ∇ is the operator:

$$\underset{\sim}{i}_1\frac{\partial}{\partial x_1} + \underset{\sim}{i}_2\frac{\partial}{\partial x_2} + \underset{\sim}{i}_3\frac{\partial}{\partial x_3}.$$

The set of equations (18) is readily solved in principle--no tricks or esoteric techniques are needed to crack it--just a clear head, a will of iron, and some computation aid such as a large automatic computor. For, the set (18) is simply a denumerably infinite collection of simultaneous first order linear differential equations which is solved by successively evaluating the N^n's, starting with $n=1$. Our present study does not require the explicit solution of the set (18), but rather the solution of a related set derived from (18), which describes the time-dependent n-ary radiant energies $U^n(X, t)$. The preliminary step in this solution is the derivation of the transport equations for n-ary radiant density, to which we now turn.

TRANSPORT EQUATION FOR n-ARY RADIANT DENSITY

From the definition of n-ary radiant density $u^n(\underset{\sim}{x})$ (Equation (9)), it is clear that the transport equation for $u^n(\underset{\sim}{x})$ is obtained from (18) by an integration of (18) over Ξ :

$$\int_{\Xi} \frac{1}{v} \frac{\partial N^n(\underset{\sim}{x},\underset{\sim}{\xi},t)}{\partial t}\, d\Omega(\underset{\sim}{\xi}) + \int_{\Xi} \underset{\sim}{\xi}\cdot\nabla N^n(\underset{\sim}{x},\underset{\sim}{\xi},t)\, d\Omega(\underset{\sim}{\xi})$$

$$= -\alpha(\underset{\sim}{x},t)\int_{\Xi} N^n(\underset{\sim}{x},\underset{\sim}{\xi},t)\, d\Omega(\underset{\sim}{\xi}) + \int_{\Xi} N_*^n(\underset{\sim}{x},\underset{\sim}{\xi},t)\, d\Omega(\underset{\sim}{\xi}) .$$

The first term on the left of this equation may be written, by virtue of (9), as

$$\frac{\partial}{\partial t}\int_{\Xi} \frac{N^n(\underset{\sim}{x},\underset{\sim}{\xi},t)}{v}\, d\Omega(\underset{\sim}{\xi}) = \frac{\partial u^n(\underset{\sim}{x},t)}{\partial t} .$$

The second term on the left of the equation may be written

$$\nabla\cdot\int_{\Xi} \underset{\sim}{\xi} N^n(\underset{\sim}{x},\underset{\sim}{\xi},t)\, d\Omega(\underset{\sim}{\xi}) = \nabla\cdot \underset{\sim}{H}^n(\underset{\sim}{x},t)$$

where $\underset{\sim}{H}^n$ is the n-ary <u>irradiance vector</u> function.

The first term on the right of the above equation is clearly

$$-\alpha(\underset{\sim}{x},t)\, v\, u^{n}(\underset{\sim}{x},t)\,,$$

and the final term may be written as:

$$\int_{\Xi}\left[\int_{\Xi}\sigma(\underset{\sim}{x};t;\underset{\sim}{\xi};\underset{\sim}{\xi}')\,N^{n-1}(\underset{\sim}{x},\underset{\sim}{\xi},t)\,d\Omega(\underset{\sim}{\xi}')\right]d\Omega(\underset{\sim}{\xi}) = s(\underset{\sim}{x},t)\, v\, u^{n-1}(\underset{\sim}{x},t)$$

where

$$s(\underset{\sim}{x}\ t) = \int_{\Xi}\sigma(\underset{\sim}{x};t,\underset{\sim}{\xi};\underset{\sim}{\xi}')\,d\Omega(\underset{\sim}{\xi})\,,$$

is the <u>total volume scattering function</u>.

Hence the required transport equation for $u^{n}(\underset{\sim}{x},t)$ is:

$$\frac{1}{v}\ \frac{\partial u^{n}(\underset{\sim}{x},t)}{\partial t} + \frac{1}{v}\nabla\cdot\underset{\sim}{H}^{n}(\underset{\sim}{x},t) = -\alpha(\underset{\sim}{x},t)\,u^{n}(\underset{\sim}{x},t) + s(\underset{\sim}{x},t)\,u^{n-1}(\underset{\sim}{x},t)\,. \tag{19}$$

The radiant density function $u^{n}(\underset{\sim}{x},t)$ contains less information than the radiance function $N^{n}(\underset{\sim}{x},t)$, but yet its transport equation (17) is intrinsically more difficult to solve, chiefly because of the presence of the divergence term on the left side. To overcome the obstacle presented by this term the solver must resort to various tricks which range in sophistication from simply setting $\nabla\cdot\underset{\sim}{H}^{n}(\underset{\sim}{x},t)=0$, up to assuming a Fick's law behavior for $\underset{\sim}{H}^{n}$. Fortunately for our present purposes, this obstacle vanishes (literally) when we consider n-ary radiant energy.

TRANSPORT EQUATION FOR n-ARY RADIANT ENERGY

By definition (13), we must integrate each side of (19) over X in order to obtain the transport equation for $U^n(X,t)$. More generally, the transport equation for $U^n(X',t)$ for the n-ary radiant energy content of any subset X' of X is obtained by integrating (19) over X':

$$\frac{1}{v}\frac{\partial}{\partial t}\int_{X'} u^n(\underset{\sim}{x},t)\,dV(\underset{\sim}{x}) + \frac{1}{v}\int_{X'} \nabla\cdot \underset{\sim}{H}^n(\underset{\sim}{x},t)\,dV(\underset{\sim}{x})$$

$$= -\int_{X'} \alpha(\underset{\sim}{x},t)\,u^n(\underset{\sim}{x},t)\,dV(\underset{\sim}{x}) + \int_{X'} s(\underset{\sim}{x},t)\,u^{n-1}(\underset{\sim}{x},t)\,dV(\underset{\sim}{x}),$$

The first term on the left-hand side gives

$$\frac{1}{v}\;\frac{\partial U^n(X',t)}{\partial t}\;;$$

the second term, by the divergence theorem, gives

$$\frac{1}{v}\int_{X'} \nabla\cdot \underset{\sim}{H}^n(\underset{\sim}{x},t)\,dV(\underset{\sim}{x}) = \frac{1}{v}\,\overline{P}^n(X',t)$$

where $\overline{P}^n(X',t)$ is the <u>net outward</u> n-ary flux across the boundary of X' at time t.

The remaining two terms on the right do not generally simplify any further. However, there is an important special case in which these terms do simplify, namely when the medium is homogeneous (α and s independent of $\underset{\sim}{x}$). We will now adopt the assumption of temporal and spatial homogeneity of X, and that X is infinite in extent in all directions. This results in a relatively simple set of equations for $U^n(X',t)$:

$$\frac{1}{v}\frac{\partial U^n(X',t)}{\partial t} + \frac{1}{v}\bar{P}^n(X',t) = -\alpha U^n(X',t) + s U^{n-1}(X',t). \tag{20}$$
$$(n \geq 1)$$

We now can eliminate the term involving $\bar{P}^n(X',t)$ by adopting the following simple device: suppose the source in X is at the origin and is turned on at $t = 0$. The radiant energy in X from the source (both its diffuse and reduced components) is restricted at every instant $t > 0$ to a spheroid $\mathcal{S}(0,t)$ of radius vt, (the characteristic spheroid--see reference 1). Hence for all times t, $\mathcal{S}(0,t)$ is properly contained in X. Thus, if we integrate over any subset X' of X which properly contains $\mathcal{S}(0,t)$ we necessarily have

$$U^n(X',t) = U^n(X,t),$$

and

$$\bar{P}^n(X',t) = 0, \quad n \geq 1.$$

The resulting form of (20) is:

$$\frac{dU^n(t)}{dt} = -\frac{U^n(t)}{T_\alpha} + \frac{U^{n-1}(t)}{T_s}, \quad n \geq 1, \tag{21}$$

where we have set $T_\alpha = 1/v\alpha$, $T_s = 1/vs$, $U^n(x,t) \equiv U^n(t)$. Each equation of (21) is readily solved in principle by specifying the time dependent radiant flux output $P^0(t)$ of the source. $P^0(t)$ may vary in any manner over a period of time of arbitrary length. The remainder of this paper will be devoted to the study of the general solutions of the equations in (21). To make certain that the reader understands the conditions under which (21) holds, we repeat them below:

Equation (21):

(i) is valid on a space X with temporally and spatially homogeneous attenuation function α volume scattering function σ and index of refraction n.

(ii) describes the evolution of U^n in any subregion X' of X which at time $t>0$ properly contains the characteristic spheroid $\mathcal{S}(0,t)$ of the source of radiant energy.

Before we go on to solve the set (21) we observe that it does not explicitly include the case $n = 0$. We obtain the equation for reduced radiant energy $U^0(t)$ by noting that the "source term" $U^{n-1}(t)/T_s$ for $U^n(t)$ in (21) must be replaced by $P^0(t)$:

$$\frac{dU^0(t)}{dt} = -\frac{U^0(t)}{T_\alpha} + P^0(t) . \tag{22}$$

This equation may also be obtained by a formal application of (20) to the case $n=0$ in which, clearly, $U^{-1}(X',t) \equiv 0$, and the term $\bar{P}^{\circ}(X',t)$ (for $X' \supset \mathcal{S}(0,t)$) is now simply the negative of the radiant power supplied to the medium by the source in X' .

SOLUTIONS OF n-ARY RADIANT ENERGY EQUATIONS

General Formula for U^0 and Discussion of T_α

The general solution of (22) is:

$$U^\circ(t) = U^\circ(0)e^{-t/T_\alpha} + \int_0^t e^{-(t-t')/T_\alpha} P^\circ(t')\,dt' , \qquad (23)$$

where $T_\alpha = 1/v\alpha$ is the time constant for reduced radiant energy U° in X . Hence, given $P^\circ(t)$ on an interval $(0,T)$, and the boundary condition $U^\circ(0)$, $U^\circ(t)$ is determinable for all times $t > 0$.

The physical significance of the time constant T_α is readily discerned if we imagine X to have an amount $U^\circ(0)$ of reduced radiant energy at time $t=0$, and if we imagine $P^\circ(t) = 0$ for all $t > 0$. Then from (23)

$$U^\circ(t) = U^\circ(0)\,e^{-t/T_\alpha} . \qquad (24)$$

Hence in the absence of any source of radiant power the reduced radiant energy content of X decays exponentially with time t , a situation which is analogous to a discharging condenser in a simple R-C DC circuit, or a radioactively decaying collection of isotope atoms, etc.

The magnitude of T_α is very small for natural aerosols and hydrosols. A representative value of $T_\alpha = n/c\alpha$ for such media is obtainable from their index of refraction n and typical α-values. For example, in oceans and lakes $n = 4/3$, and α is almost always somewhere between 10^{-2}/meter to 1.0 /meter. Thus T_α is almost always somewhere between 5×10^{-7} and 5×10^{-9} seconds. For the atmosphere near the earth's surface, we may take $n = 1$, and α may range from about 10^{-2}/meter to about 10^{-5}/meter, so that the corresponding range of T_α is from 3×10^{-7} seconds to 3×10^{-4} seconds. In interstellar space, T_α can be on the order of $10^5 - 10^6$ years, but as low as 10 - 100 minutes in dense interstellar clouds. These are all rough estimates, and should be understood as such. They merely serve to give some feeling for the typical magnitudes of in various media.

Further insight into the role of T_α in the time-dependent radiant energy problem can be obtained by returning to Equation (23) and now setting $U^o(0) = 0$ and $P^o(t) = P^o > 0$, a fixed rate for all $t > 0$. The solution of $U^o(t)$ is then:

$$U^o(t) = P^o T_\alpha \left(1 - e^{-t/T_\alpha}\right) . \tag{25}$$

In this case the reduced radiant energy content $U^o(t)$ of X starts with zero magnitude at time $t = 0$ and grows to a finite limit:

$$U^o(\infty) = P^o T_\alpha \, . \tag{26}$$

Thus for a given infinite optical medium X and a steady finite input rate P^o of radiant energy for this medium, there is an upper limit on the steady state amount of reduced radiant energy in X. This upper limit, given by (26), may be interpreted as follows: imagine an alternate infinite space X_o in which $\alpha = 0$ and in which there is a steady source of radiant flux of output P^o. If this source emits for a period of time equal to T_α and then is shut off, the reduced radiant energy content of X_o is equal to the steady state content of X defined above.

General Formula for U^n

We now solve the general differential equation (21) for $U^n(t), n \geq 1$. We begin with the case $n = 1$, and then use the known function $U^1(t)$ to solve (21) for the case $n = 2$. Once these solutions have been obtained we will be able to discern the general analytical pattern of $U^n(t)$ and then proceed to the formula for $U^n(t)$ by an inductive guess, which is then easily checked.

By setting $n = 1$ in (21) we have:

$$\frac{dU^1(t)}{dt} = -\frac{U^1(t)}{T_\alpha} + \frac{1}{T_\alpha}\left[U^0(0)e^{-t/T_\alpha} + \int_0^t e^{-(t-t')/T_\alpha} P^o(t')\,dt'\right],$$

where the expression in the square brackets is the general formula for $U^0(t)$ as given by (23). The integrating factor for this equation is e^{t/T_α}. Hence

$$\frac{d\left[e^{t/T_\alpha} U^1(t)\right]}{dt} = \frac{1}{T_\alpha}\left[U^0(0) + \int_0^t e^{t'/T_\alpha} P^o(t')\,dt'\right],$$

Integrating each side between the time limits 0 and $t > 0$,

$$e^{t'/T_\alpha} U^1(t')\Big|_0^t = U^0(0)\left(\frac{t}{T_s}\right) + \frac{1}{T_s}\int_0^t \left[\int_0^{t'} e^{t''/T_\alpha} p^0(t'')\,dt''\right] dt$$

so that

$$U^1(t) = \left[U^1(0) + \left(\frac{t}{T_s}\right)U^0(0)\right]e^{-t/T_\alpha} + \frac{1}{T_s}\int_0^t (t-t')\,e^{-(t-t')/T_\alpha} p^0(t')\,dt \qquad (27)$$

We now use this expression for $U^1(t)$ in:

$$\frac{dU^2(t)}{dt} = -\frac{U^2(t)}{T_\alpha} + \frac{U^1(t)}{T_s}$$

which is (21) with $n = 2$. Again, the integrating factor is e^{t/T_α}, so that

$$\frac{d\left[e^{t/T_\alpha} U^2(t)\right]}{dt} = \frac{1}{T_s}\left[U^1(0) + \left(\frac{t}{T_s}\right)U^0(0)\right]$$

$$+ \frac{1}{T_s^2}\int_0^t \left[\int_0^{t'} (t'-t'')\,e^{t''/T_\alpha} p^0(t'')\,dt''\right] dt'.$$

Integrating this between the time limits 0 and $t > 0$,

$$e^{t'/T_\alpha} U^2(t')\Big|_0^t = \left(\frac{t}{T_s}\right)U^1(0) + \frac{1}{2}\left(\frac{t}{T_s}\right)^2 U^0(0) +$$

$$+ \frac{1}{2}\frac{1}{T_s^2}\int_0^t (t-t')^2\, e^{-(t-t')/T_\alpha} p^0(t')\,dt'.$$

Hence

$$U^2(t) = \left[U^2(0) + \left(\frac{t}{T_s}\right)U^1(0) + \frac{1}{2}\left(\frac{t}{T_s}\right)^2 U^0(0)\right] e^{-t/T_\alpha} + \tag{28}$$

$$+ \frac{1}{2T_s^2}\int_0^t (t-t')^2 e^{-(t-t')/T_\alpha} p^0(t')\,dt'.$$

The general analytic form of $U^n(t)$ is now evident. On the basis of (27) and (28) we write:

$$U^n(t) = \left[U^n(0) + \left(\frac{t}{T_s}\right)U^{n-1}(0) + \dots + \frac{1}{n!}\left(\frac{t}{T_s}\right)^n U^0(0)\right] e^{-t/T_\alpha} +$$

$$+ \frac{1}{n!\,T_s^n}\int_0^t (t-t')^n e^{-(t-t')/T_\alpha} p^0(t')\,dt'. \tag{29}$$

Our guess can be proved by means of mathematical induction: write the differential equation of $U^{n+1}(t)$ with the above hypothesized form of $U^n(t)$ substituted in place of $U^n(t)$. Formally solving the equation will result in the form (29) for the value n replaced throughout by $n+1$. However, we will automatically obtain an alternate check of a different kind on the validity of (29) when we derive the formula for $U^*(t)$. Therefore, the details of the inductive proof just sketched will be left to the reader.

The Standard Growth and Decay Formulas for U^n

Our preceding study of the properties of $U^o(t)$ was facilitated by considering the two special cases: (a) $U^o(0) = 0$, $P^o(t) = P^o$ for all $t > 0$; (b) $P^o(t) = 0$ for all $t > 0$, $U^o(0) > 0$. In case (a), we studied the pure growth of $U^o(t)$ in a medium with a steady source turned on at $t = 0$. Case (b) represents the pure decay of $U^o(t)$ from some finite initial value in a space with no sources. It turns out that these two cases applied to $U^n(t)$ also uncover the salient properties of the higher-order scattered radiant energy components.

We therefore consider the following two special cases of the general formula (29):

(a) <u>The Standard Growth Formula for $U^n(t)$</u>

Initial Condition: $U^n(0) = 0 \;, \; n \geq 0$

Source Condition: $P^o(t) = P^o$ for all $t \geq 0$

$$U^n(t) = \frac{P^o}{n!\,T_s^n} \int_0^t (t-t')^n e^{-(t-t')/T_\alpha} \, dt'$$

$$= U^n(\infty) \left\{ 1 - \left[\sum_{j=0}^{n} \frac{(t/T_\alpha)^j}{j!} \right] e^{-t/T_\alpha} \right\}, \quad (30)$$

where $U^n(\infty) = \omega_0^n \, U^o(\infty) = \omega_0^n \, P^o T_\alpha$ and $\omega_0 = s/\alpha = T_\alpha / T_s$.

(b) <u>The Standard Decay Formula for $U^n(t)$</u>

Initial Condition: $U^n(0) = \omega_0^n \, U^0(0)$, $n \geq 0$

Source Condition: $P^0(t) = 0$ for all $t \geq 0$

$$U^n(t) = U^n(0) \, e^{-t/T_\alpha} \sum_{j=0}^{n} \frac{(t/T_\alpha)^j}{j!} \, , \qquad (31)$$

The choice of the initial condition for case (b) must be explained. An examination of the standard growth formula (or the <u>general growth formula</u> obtained from (29) by setting $U^n(0) = 0$, $n \geq 0$, but leaving $P^0(t)$ arbitrary), will show that any two steady state values $U^n(\omega)$ and $U^m(\omega)$ are <u>not</u> independent. Their interrelations depend on the function $P^0(t)$ (and the geometry of the medium in general). In the present simple context (infinite, homogeneous space), their interrelations are given by (30) for $t = \omega$. The standard decay formula starts with the light field in steady state, which has been attained under standard growth conditions. Therefore, the initial conditions of (b) necessarily are governed by the steady state conditions of (a).

We can combine the two cases (a) and (b) into one standard formula as follows:

(c) <u>The Standard Formula for $U^n(t)$</u>

Initial Condition: $U^n(0)$, $n \geq 0$ given as the steady state value attained under a previous standard growth condition.

Source Condition: $P^0(t) = P^0$ for all $t \geq 0$, where P^0 is in general different than the source condition giving rise to the $U^n(0)$ of the initial condition.

Then:

$$U^n(t) = U^n(\infty) + \left[U^n(0) - U^n(\infty)\right] F_n(t/T_\alpha) \tag{32}$$

where $U^n(\infty)$ is determined by (30) (by setting $t = \infty$) for the present source condition, and where

$$F_n(t/T_\alpha) = e^{-t/T_\alpha} \sum_{j=0}^{n} \frac{(t/T_\alpha)^j}{j!} \ .$$

The standard formula (32) contains both (30) and (31) as special cases. For, under conditions (a) above, $U^n(0) = 0$ and we obtain (30); under conditions (b) above, $U^n(\infty) = 0$, and (32) reduces to (31). Equation (32) shows in simplest terms the essential structure of the standard growth and decay characteristics of $U^n(t)$ under the standard conditions. In particular, since $F_n(t/T_\alpha)$ is a monotonic decreasing function of t , we conclude that $U^n(t)$ decreases monotonically for all $n \geq 0$, under standard decay conditions; and that $U^n(t)$ increases monotically for all $n \geq 0$ under standard growth conditions.

EQUATIONS FOR STORED RADIANT ENERGY AND RELATED CONCEPTS

Transport Equation for Stored Radiant Energy

By definition (15) the stored radiant energy in X at time t is

$$U^*(t) = \sum_{n=1}^{\infty} U^n(t) ,$$

where, as usual, specific reference to X in the notation has been dropped. We may obtain the required time-dependent transport equation for $U^*(t)$ by formally summing over the denumerably infinite set of transport equations for $U^n(t)$, $n \geq 1$. Thus, we may start with (21):

$$\sum_{n=1}^{\infty} \frac{dU^n(t)}{dt} = -\frac{\sum_{n=1}^{\infty} U^n(t)}{T_\alpha} + \frac{\sum_{n=1}^{\infty} U^{n-1}(t)}{T_s} ,$$

which reduces to:

$$\frac{dU^*(t)}{dt} = -\frac{U^*(t)}{T_\alpha} + \frac{U(t)}{T_s}$$

$$= -\frac{U^*(t)}{T_\alpha} + \frac{[U^0(t) + U^*(t)]}{T_s} .$$

Hence

$$\frac{dU^*(t)}{dt} = -\frac{U^*(t)}{T_a} + \frac{U^0(t)}{T_s} , \tag{33}$$

which is the required differential equation for U^*. $U^\circ(t)$ is the reduced radiant energy at time t, and is given by (23) in terms of the fundamental source output $P^\circ(t)$. Observe that the decay of U^* is governed by the volume absorption coefficient a, and not the volume attenuation coefficient α.

General Formula for Stored Radiant Energy

The general solution of (33) under the present conditions of infinite homogeneous X with an arbitrary source is as follows. The integrating factor of (33) is e^{t/T_a}, where $T_a = 1/va$, so that

$$\frac{d\left[e^{t/T_a} U^*(t)\right]}{dt} = \frac{1}{T_s} e^{t/T_a} U^\circ(t).$$

Observe that

$$\frac{1}{T_\alpha} = \frac{1}{T_a} + \frac{1}{T_s},$$

which follows from the definitions of the various time constants and the fact that $\alpha = a + s$. Hence the preceding equation may be written:

$$\frac{d\left[e^{t/T_a} U^*(t)\right]}{dt} = \frac{1}{T_s}\left[e^{-t/T_s} + e^{-t/T_s}\int_0^t e^{t'/T_\alpha} P^\circ(t')\,dt'\right].$$

Integrating each side between time limits 0 and $t > 0$, and simplifying, we have

$$U^*(t) = U^*(0)e^{-t/T_a} + \int_0^t \left[e^{-(t-t')/T_a} - e^{-(t-t')/T_\alpha}\right] P^\circ(t')\,dt', \tag{34}$$

which is the desired general formula for $U^*(t)$. This may also be obtained by summing (29) over all $n \geq 1$, which gives the alternate check on (29).

Standard Growth and Decay Formulas for Stored Energy

Under the standard growth conditions: $U^*(0) = 0$, $P^\circ(t) = P^\circ$ for all $t > 0$, (34) becomes

$$U^*(t) = P^\circ T_a \left(1 - e^{-t/T_a}\right) - U^\circ(t). \tag{35}$$

Under the standard decay conditions, $U^*(0)$ is understood to be the steady state value of a previous standard growth condition, and $P^\circ(t) = 0$, $t \geq 0$. Hence (34) becomes

$$U^*(t) = U^*(0)\, e^{-t/T_a} . \tag{36}$$

Formulas for Observable Radiant Energy

The observable radiant energy $U(t)$ is defined in (14). Its differential equation may then be obtained by adding the differential equations for $U^*(t)$ and $U^\circ(t)$. The result is:

$$\frac{dU(t)}{dt} = -\frac{U(t)}{T_a} + P_o(t) \tag{37}$$

This is a particularly interesting equation in that it contains time-dependent form of the integrated divergence relation for the irradiance vector.[2] This formula has great potentialities in experimental studies in which the volume absorption function is sought. It is also of use in determining energy contents of volumes of known optical properties. We will show later how a general steady state version of (37) can be used to arrive at practical estimates of storage capacity of finite homogeneous media.

The general solution of (37) is straightforward:

$$U(t) = U(0)\, e^{-t/T_a} + \int_0^t e^{-(t-t')/T_a}\, P^0(t')\, dt' \qquad (38)$$

The standard growth formula is obtainable directly from (35):

$$U(t) = P^0 T_a \left(1 - e^{-t/T_a}\right), \qquad (39)$$

The standard decay formula is clearly

$$U(t) = U(0)\, e^{-t/T_a}, \qquad (40)$$

which is identical in form with (36), so that, for example, the time-dependent storage capacity of χ is constant under standard decay conditions.

Steady State Storage Capacities

We have now uncovered all the basic information required in the derivation of the storage capacity expressions given in the Introduction. Specifically, from the standard growth formulas for $U^*(t)$ and $U(t)$ we have

$$\frac{U^*(t)}{U(t)} = \frac{p^o T_a(1-e^{-t/T_a}) - U^o(t)}{p^o T_a(1-e^{-t/T_a})}$$

$$= 1 - \frac{T_\alpha(1-e^{-t/T_\alpha})}{T_a(1-e^{-t/T_a})} .$$

The limit of this, as $t \to \infty$, is the storage capacity of X, i.e.,

$$\lim_{t\to\infty} \frac{U^*(t)}{U(t)} = \frac{U^*}{U} = 1 - \frac{T_\alpha}{T_a} = 1 - \frac{a}{\alpha} = \frac{\delta}{\alpha} \equiv \omega_o . \quad (41)$$

The additional relations given in the Introduction follow immediately from the standard growth formulas for $U^n(t)$, $n \geq 0$, and they need not be rederived here.

METHODS OF MEASURING STORAGE CAPACITY

Is there some way in which the storage capacity of a given real medium can be determined without having to measure the various radiant densities at each point in the medium? If the medium were optically infinite (or very extensive optically) the preceding theory provides a good estimate of the storage capacity by means of (41), which requires the determination of ω_0 at only one point. But what about media of finite extent such as clouds, local fog banks, finite subregions of the atmosphere or hydrosphere? The formulas developed above (such as (19) and (20) are too complicated to solve analytically for each case encountered--granted that they can be solved at all. The presence of the divergence terms and net flux terms are the epicenters of these analytical complexities. But these terms, which prevent neat simple analytical estimates of U^*/U, are the very key to simple measuring techniques which lead to estimates of U^*/U without the necessity of extensive point by point radiometric probing within the medium of interest.

General Measurement Formula for Storage Capacity

Consider the steady state form of (20):

$$\frac{1}{v} \bar{P}^n(x') = -\alpha U^n(x') + s\, U^{n-1}(x') ,$$
$$(n \geq 1)$$

We sum this set of equations over all $n \geq 1$:

$$\frac{1}{v} \sum_{n=1}^{\infty} \bar{P}^n(x') = -\alpha \sum_{n=1}^{\infty} U^n(x') + s \sum_{n=1}^{\infty} U^{n-1}(x') ,$$

which reduces to

$$\frac{1}{v}\ \overline{P}^{*}(X') = -\alpha\, U^{*}(X') + s\, U(X'), \qquad (42)$$

where,

$$\overline{P}^{*}(X') = \sum_{n=1}^{\infty} \overline{P}^{n}(X'),$$

is the net outward flux of the stored energy across the boundary of X'.

In accordance with our preceding remarks, we are interested in estimating the quantity $U^{*}(X')$ with the ultimate goal in mind of estimating the ratio $U^{*}(X')/U(X')$. But any such estimation must be couched in terms of <u>observable</u> or <u>simply calculable</u> quantities. $U^{*}(X')$ is not observable, and $U(X')$, while observable, is not simply calculable. (It requires a determination of observable radiant density $u(\underset{\sim}{x}')$ at each point $\underset{\sim}{x}'$ of X'.) In casting about for easily observable and simply calculable quantities, the observable net flux $\overline{P}(X')$, the reduced net flux $\overline{P}^{o}(X')$ and the reduced energy $U^{o}(X')$ immediately come to mind. If we can obtain an expression for $U^{*}(X')/U(X')$ in terms of $\overline{P}(X')$, $\overline{P}^{o}(X')$ and $U^{o}(X')$ we will have obtained the best solution possible to the problem of empirically determining the storage capacity of a <u>finite</u> homogeneous medium.

It turns out that the characterization of $U^{*}(X')/U(X')$ in terms of $\overline{P}(X')$, $\overline{P}^{o}(X')$ and $U^{o}(X')$ is relatively easy to achieve.

Starting with (42), and noting that

$$\bar{P}(X') = \bar{P}^{o}(X') + \bar{P}^{*}(X'),$$

we can recast (42) into the form:

$$\frac{1}{v}\bar{P}(X') - \frac{1}{v}\bar{P}^{o}(X') = -aU^{*}(X') + sU^{o}(X').$$

We can then represent the nonobservable $U^{*}(X')$ in terms of observable and calculable quantities:

$$U^{*}(X') = \frac{s}{a}U^{o}(X') + \frac{1}{av}\left[\bar{P}^{o}(X') - \bar{P}(X')\right],$$

Hence

$$\boxed{\frac{U^{*}(X')}{U(X')} = 1 - \frac{a}{\alpha - \left[\dfrac{\bar{P}(X') - \bar{P}^{o}(X')}{vU^{o}(X')}\right]}} \qquad (43)$$

Equation (43) gives the desired general formulation of the storage capacity of a finite homogeneous medium in terms of the directly observable net outward flux $\bar{P}(X')$ over the boundary of X', the calculable net outward reduced flux $\bar{P}^{o}(X')$ over the boundary of X', and the calculable reduced energy content $U^{o}(X')$ of X'. The volume absorption coefficient a and the volume attenuation coefficient α are the inherent optical properties of X' which enter into the calculation and which are assumed known.

It should be remarked that Equation (43) is an <u>exact</u> and determinate formula for $U^*(X')/U(X')$ whenever X' is any finite homogeneous medium with $a > 0$, irradiated by sources in an arbitrary manner and in which the resultant light field is in steady state. If X' is infinite such that its boundaries are at infinity, then $\bar{P}(X') = \bar{P}^o(X')$ and (43) reduces to (41).

Example

To illustrate how (43) is used in particular contexts, consider, for example, a horizontally extensive cloud stratum which is of finite geometric depth under a clear sunlit sky (or clear moonlit sky). We agree that the principal source of flux is to be the sun (or moon) with negligible auxiliary sources associated with the sky and ground. Suppose the sun cannot be seen through the cloud layer looking up from below. It may be checked that the difference $\bar{P}(X') - \bar{P}^o(X')$ in (43) then reduces to $P^*(X',+)$, the total outward rate of flow of stored energy across the two boundaries of X'. Suppose the outward rate of flow from X' over its lower boundary is small compared to that out of its upper boundary (which is compatible with the assumptions above). Then

$$U^o(X') = \frac{N^o \Omega A}{v \alpha \sec\theta} = \frac{P^o(X',-)}{v\alpha},$$

where N^{o} is the radiance of the sun (moon) at the upper boundary of X', θ its angle from the zenith, Ω is its solid angle subtense, and A is the area of the upper boundary of the cloud. The second equality follows from the definition of inward reduced flux over the upper boundary of X'. Hence (43) becomes

$$\frac{U^{*}(X')}{U(X')} = \frac{\omega_0 - R(X')}{1 - R(X')} , \tag{44}$$

where $R(X') = P^{*}(X',+)/P^{o}(X',-)$ is the reflectance of X' at its upper boundary, a directly measurable quantity.

Equation (44) illustrates but one of the many practical formulas which may be deduced--under various hypotheses--from the general formula (43). The preceding derivation will suffice to indicate the general outline of such procedures, and we leave the exploration of other possibilities to the interested reader.

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