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TEMPORAL METRIC SPACES IN RADIATIVE TRANSFER THEORY

IV. Temporal Diameters

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IV. Temporal Diameters

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INTRODUCTION

When a steady point source of radiant energy is turned on in optical medium such as a cloud, a lake, or some given region of an ocean, there is a small yet finite period of time in which the light field produced by the source is in the process of building up to a steady state. During this period the radiant energy from the source is transmitted at the speed of light to the farthest reaches of the medium while simultaneously being diffused and rediffused within the characteristic spheroid (the interior of the region defined by the wavefront and the boundaries of the medium). Even though this transmission, diffusion, and final attainment of the steady state configuration of the light field take place in a relatively small interval of time, an enormous collection of complex radiometric events takes place.

In this note we systematically isolate some of these radiometric events and show that they may be arrayed in orderly and informative sequence in time and space. The results of such a painfully detailed analysis of apparently imperceptible transient phenomena are immediately applicable to all radiative transfer processes, including those in the so-called steady state, since these are viewable as superpositions of instantaneous transient states. Furthermore, the approach to the time-dependent problem presented here requires no explicit model of the light field such as that represented by the equation of transfer. Specifically, the approach of the present paper is based on the idea of the n-ary temporal diameter of an optical medium.

The n-ary temporal diameter T_n, $n = 1, 2, 3, \ldots,$ of an optical medium is defined as the epoch time required for the n-ary scattered radiance from some fundamental source to reach its steady state in the medium. Since the observable radiance field may be represented as the sum of an infinite series of n-ary radiances, knowledge of the various times of steady state attainment of its components can lead to practical methods of determining the time at which the entire field will be in steady state.

In this study the main emphasis will be on the precise definitions of the n-ary temporal diameters, and to a brief systematic development of their salient properties. The background for the concepts and terminology used here will be found in the references.

TEMPORAL DIAMETER

The setting for the entire paper is as follows: a classical carrier space $(\Phi, \underset{\sim}{\Phi}, \nu)$ with a temporal metric function t on $\Phi \times \Phi$. Φ is closed and bounded with respect to t, and optically convex. The term, optically convex, means that if p_1, p_2 are in Φ, then the track of the extremal with these endpoints lies in Φ. According to Theorem 3, paper III, we then have $t = T$ on $\Phi \times \Phi$, where T is the least local epoch time metric. Therefore, we may work solely with the metric t which is derivable from the classical transfer process $\mathcal{T}_\psi$ on $(\Phi, \underset{\sim}{\Phi}, \nu)$ in general, or which may be obtained--if specific information is given on the index of refraction function n on Φ --from the Euler equations for extremals. The results of the present study may be elevated to more general settings, but the essential parts of the assertions of the theorems remain unchanged. Therefore, the results are presented in the simplest possible context in order that the basic ideas emerge clearly and with some intuitive content.

We begin with the definition of the temporal diameter $T(\Phi)$ of Φ.

Definition 1. The temporal diameter $T(\Phi)$ of Φ is defined as:

$$T(\Phi) = \sup\{t(p';p) : (p';p) \in \Phi \times \Phi\}.$$

This is exactly the way the diameter of a set is defined in general metric space theory. $T(\Phi)$ is then the time associated with the points of greatest temporal separation in Φ. Since Φ is closed and bounded with respect to t, $T(\Phi)$ is bounded and we may actually find two points p_1 and p_2 on the termini of a natural path $P(p_1,p_2)$ in Φ such that

$$T(\Phi) = t(p_1,p_2)\,. \tag{1}$$

Relation (1) would not generally be possible if any one of the boundedness, closure, or convexity properties were dropped. Actually, the requirement that Φ be closed is no restriction at all from the physical point of view. The difference between an open set $R \subset \Phi$ and its closure $\bar{R}$ is such that $v(\bar{R}-R)=0$ i.e., a set of optical measure zero. Hence the added convenience of having Φ closed so that (1) holds is bought at virtually no cost to the generality of Φ. The requirement that Φ be bounded is slightly more restrictive, however. For while we do not actually have truly unbounded carrier spaces in nature, many of the mathematician's models of carrier spaces are infinite subsets of E_3 (slabs, half-spaces, etc.) which do have infinite extent in one or more directions about each point. In the interests of completeness, the definition of temporal diameters for unbounded carrier spaces will be briefly covered in paper V of this series, wherein it will be shown that unbounded carrier spaces under an appropriate point of view may have associated with them well defined <u>finite</u> temporal diameters.

Therefore, results concerning bounded spaces may also be applicable to unbounded spaces.

Definition 2. Let p be a point of Φ, and let p_1 and p_2 be the endpoints of the temporal diameter of Φ. Then define $t_m(p)$ as:

$$t_m(p) = \max\{t(p,p_1),\ t(p,p_2)\}.$$

Theorem 1. $2t_m(p) \geq T(\Phi)$ for all $p \in \Phi$.

Proof: By the triangle inequality, $T(\Phi) \leq t(p_1,p) + t(p,p_2)$. Further, by definition of $t_m(p)$, $t(p_1,p) + t(p,p_2) \leq 2t_m(p)$. Hence $T(\Phi) \leq 2t_m(p)$.

PRIMARY TEMPORAL DIAMETER

Definition 3. Let p^* be a local source in Φ. Then the primary temporal diameter $T_1(p^*)$ of Φ with respect to the local source p^* is defined as

$$T_1(p^*) = \sup\{t(p^*, p') + t(p', p) : (p', p) \in \Phi \times \Phi\}$$

Since the function F defined by

$$F(p^*, p', p) = t(p^*, p') + t(p', p)$$

for each p^* is a continuous function on $\Phi \times \Phi$, a closed bounded set with the product topology induced by t, there exist (not necessarily unique) points $p_1, p_2 \in \Phi$ such that

$$T_1(p^*) = t(p^*, p_1) + t(p_1, p_2). \tag{2}$$

For such a triple of points as that in (2), p_2 is among the last points of Φ to receive primary scattered flux from the source p^*. $T_1(p^*)$ may be interpreted as the local epoch time, measured at p^*, at which, for all points p_1 in Φ, primary scattered flux from p^* has reached a steady state. $T_1(p^*)$ is clearly dependent on the geometric structure of Φ, and of course on n, the index of refraction function on Φ. $T_1(p^*)$ is bounded on Φ. The proof of this fact will be deferred until we have defined the n-ary temporal diameter $T_n(\rho^*)$. At that time we will prove the boundedness of $T_n(\rho^*)$ once and for all $n \geq 1$, and exhibit explicit bounds.

Examples of Primary Temporal Diameters

Example 1. Let Φ be a rectangular parallelepiped in E_3 with constant n and non zero volume scattering function σ on Φ (see Figure 1). In this case, $T_1(\rho^*) = t(\rho^*, V_1) + t(V_1, V_2)$, where V_1 and V_2 are diagonally opposite vertices of Φ such that $t_m(\rho^*)$ is $t(\rho^*, V_1)$. Here we may write $t_m(\rho^*)\, v = r_m(\rho^*)$, where v is the speed of light in Φ . If D is the diameter of Φ (in the usual metric), then, $T(\Phi) = D/v$, and

$$T_1(\rho^*) = [r_m(\rho^*) + D]/v.$$

We observe that if $\rho^* = V_2$, then $T_1(V_2) = 2T(\Phi)$.

Example 2. Let Φ be a spheroid in E_3 with diameter D (Figure 2) and with an $r_m(\rho)$ as defined in Example 1. Then

$$T_1(\rho^*) = [r_m(\rho^*) + D]/v.$$

Example 3: Let Φ be a general bounded closed convex body, as in Figure 3. In analogy with the results of Examples 1 and 2, we may tentatively write:

$$T_1(\rho^*) = [r_m(\rho^*) + D]/v,$$

where D is the diameter (in the usual metric) of Φ. We must observe, however, that while this expression for $T_1(p^*)$ is true for a multitude of convex bodies (e.g., those in Examples 1, 2), it is in general only an approximate formula. (Can the reader find counter examples to the above expression for $T_1(p^*)$?) Later (Theorems 4, 5, 6) attempts are made to see just how good such an approximate rule is in the case of the general n-ary temporal diameter, $T_n(p^*)$. The basic reason that the preceding formula for $T_1(p^*)$ is only of limited validity may be found in the following theorem, when one recalls the definition of $t_m(p)$.

Theorem 2. If (p_1, p_2) is a point pair in an arbitrary Φ such that $T_1(p^*) = t(p^*, p_1) + t(p_1, p_2)$ then $t_m(p^*) \leq t(p^*, p_1)$.

Proof: We observe that $t_m(p^*) + T(\Phi)$ is a sum of the kind in Definition 3, therefore,

$$t(p^*, p_1) + t(p_1, p_2) \geqq t_m(p^*) + T(\Phi)$$

This inequality may be strengthened by writing:

$$t(p^*, p_1) + T(\Phi) \geqq t_m(p^*) + T(\Phi)$$

whence the statement of the theorem follows.

n-ARY TEMPORAL DIAMETER

We now consider the general definition of the n-ary temporal diameter and deduce some of its important properties. We will say that two points (ρ,ρ'), (ρ'',ρ''') of $\Phi\times\Phi$ are <u>adjacent</u> in $\Phi\times\Phi$ if $\rho'=\rho''$. Let $S_n=(\Phi\times\Phi)^n$, i.e., S_n is the n-fold cartesian product of $\Phi\times\Phi$. Let S_n' be the set of all points of S_n whose ith and (i+1)th coordinates are adjacent in $\Phi\times\Phi$, $i=1,\ldots,n-1$.

<u>Definition 4</u>. Let ρ^* be a local source in Φ. Then the <u>n-ary temporal diameter</u> $T_n(\rho^*)$ of Φ with respect to the source ρ^* is defined as:

$$T_n(\rho^*) = \sup_{S_n'}\left\{t(\rho^*,\rho_1)+\sum_{j=1}^{n} t(\rho_j,\rho_{j+1})\right\}.$$

For the same reasons as those in the case of $n=1$, it is possible to find a point in S_n', $n>1$ say $\{(\rho_1,\rho_2),\ldots,(\rho_n,\rho_{n+1})\}$, such that

$$T_n(\rho^*) = t(\rho^*,\rho_1)+\sum_{j=1}^{n} t(\rho_j,\rho_{j+1}). \qquad (3)$$

This point in S_n' is not necessarily unique, but the time $T_n(\rho^*)$ associated with all such points is, of course, unique.

Example of n-ary Temporal Diameter

Example 4. Figure 2 shows a spheroidal Φ with diameter D (in the usual metric). For any ρ^* in Φ, it is clear that

$$T_n(\rho^*) = [r_m(\rho^*) + nD]/v,$$

where $r_m(\rho^*)$ is as shown in the figure.

Theorems on n-ary Temporal Diameters

We now prove that $T_n(\rho^*)$ is bounded on Φ.

Theorem 3. If $T_n(\rho^*)$ is the n-ary temporal diameter of Φ with respect to the local source ρ^*, then

$$T_n(\rho^*) \leq t_m(\rho^*) + (n+\tfrac{1}{2})T(\Phi).$$

Proof: Suppose that the conclusion were false, i.e., that $T_n(\rho^*) > t_m(\rho^*) + (n+\frac{1}{2})T(\Phi)$. Now there exist n adjacent point pairs such that $T_n(\rho^*) = t(\rho^*, \rho_1) + \dots + t(\rho_n, \rho_{n+1})$, and for each such pair, $t(\rho_i, \rho_{i+1}) \leq T(\Phi)$. It follows that

$$t(\rho^*, \rho_1) + n\,T(\Phi) \geq T_n(\rho^*) > t_m(\rho^*) + (n+\tfrac{1}{2})T(\Phi).$$

Hence

$$t(\rho^*, \rho_1) > t_m(\rho^*) + T(\Phi)/2,$$

so that, by Theorem 1,

$$t(\rho^*, \rho_1) > T(\Phi)$$

which is impossible, by definition of $T(\Phi)$. Hence the stated bound is true, and the theorem is proved.

<u>Corollary</u>: The first summand $t(\rho^*, \rho_1)$ in the representation of $T_n(\rho^*)$ satisfies the inequality $t_n(\rho^*) \leq t(\rho^*, \rho_1)$.

The bound given in Theorem 2 is adequate for our present purposes, but it is not the sharpest, as may be seen by an examination of either Examples 1, 2 or 4 above. Consider Example 4. In particular, let $\rho^* = 0$, the center of the diameter of Φ . Then $t_n(0) = T(\Phi)/2$, and $T_n(0) = T(\Phi)/2 + nT(\Phi) \doteq (n + \frac{1}{2})T(\Phi)$, whereas the bound formula states that $T_n(0) \leq (n+1)T(\Phi)$ in this case.

A convenient rule of thumb for $T_n(\rho^*)$ is obtained by replacing $t_m(\rho^*)$ by $T(\Phi)/2$, so that $T_n(\rho^*) \cong (n+1)T(\Phi)$. However, for large n, this rule of thumb may be sharpened, and the basis for the rule lies in the following theorem which shows that $T_n(\rho^*) \rightarrow nT(\Phi)$ for every $\rho^* \varepsilon \Phi$.

<u>Theorem 4. If $T_n(\rho^*)$ is the n-ary temporal diameter of Φ with respect to the local source ρ^*, and $T(\Phi)$ is the temporal diameter of Φ , then</u>

$$\lim_{n \rightarrow \infty} \frac{T_n(\rho^*)}{n} = T(\Phi),$$

for all $\rho^* \varepsilon \Phi$.

Proof: We observe first that $t_m(\rho^*) + n\,T(\Phi)$ is a sum of the kind occurring in Definition 4. Therefore, $T_n(\rho^*) \geq t_m(\rho^*) + n\,T(\Phi)$. Further, by Theorem 3, $T_n(\rho^*)$ is bounded by $t_m(\rho^*) + (n+\frac{1}{2})T(\Phi)$. Hence

$$t_m(\rho^*) + n\,T(\Phi) \leq T_n(\underline{\Phi}) \leq t_m(\rho^*) + (n+\tfrac{1}{2})\,T(\Phi)$$

Dividing this set of inequalities by n and taking the limit, as $n \to \infty$, we have the desired result.

Corollary: An estimate of the difference $[T_n(\rho^*)/n] - T(\Phi)$ for each n and ρ^* is obtained by the following inequalities:

$$\frac{t_m(\rho^*)}{n} \leq \frac{T_n(\rho^*)}{n} - T(\Phi) \leq \frac{t_m(\rho^*)}{n} + \frac{T(\Phi)}{2n} .$$

The preceding theorem appears to imply that the individual summands of $T_n(\rho^*)$ approach $T(\Phi)$ as n increases. The precise form of this fact is proved in:

Theorem 5. Let $\{T_n(\rho^*), n = 1, 2, \ldots\}$ be a sequence of n-ary temporal diameters of Φ with respect to a fixed point ρ^*. Then the sequence of terminal summands $\{t(\rho_n, \rho_{n+1})\}$ of $T_n(\rho^*)$ is convergent with limit $T(\Phi)$.

<u>Proof</u>: From the definition of $T_n(\rho^*)$, we have, for all $n \geq 1$,

$$T_n(\rho^*) \geq t_m(\rho^*) + nT(\Phi).$$

Using the summation representation of $T_n(\Phi)$, this inequality may be written as:

$$t(\rho^*, \rho_1) - t_m(\rho^*) + [t(\rho_1, \rho_2) - T(\Phi)] + \cdots + [t(\rho_n, \rho_{n+1}) - T(\Phi)] \geq 0$$

By definition of $T(\Phi)$, we have $t(\rho_i, \rho_{i+1}) - T(\Phi) \leq 0$ for $i = 1, \ldots, n$. Furthermore, $0 \leq t(\rho^*, \rho_1) - t_m(\rho^*) \leq T(\Phi)$. Hence the preceding inequality may be strengthened to read

$$T(\Phi) + \sum_{i=1}^{n} [t(\rho_i, \rho_{i+1}) - T(\Phi)] \geq 0 \tag{4}$$

We now hypothesize that

$$T(\Phi) - t(\rho_i, \rho_{i+1}) \geq \epsilon > 0 \tag{*}$$

for more than a finite number of integers $i \geq 1$, where ϵ is an arbitrary positive number. From the hypothesis, it follows that for this ϵ there is an integer N_ϵ such that

$$T(\Phi) + \sum_{i=1}^{N_\epsilon} [t(\rho_i, \rho_{i+1}) - T(\Phi)] < 0$$

which contradicts (4) for each $T_n(\rho^*)$ with $n \geq N_\epsilon$. Hence (*) is untenable and the alternate possibility implies

$$\lim_{n\to\infty} t(\rho_n, \rho_{n+1}) = T(\Phi)$$

which proves the theorem.

We conclude the present set of theorems with the following, which is helpful in the construction of n-ary temporal diameters.

Theorem 6. If ρ and ρ' are the endpoints of a temporal diameter of Φ and $t(\rho^*, \rho_1) + \cdots + t(\rho_n, \rho_{n+1})$ is a representation of $T_n(\rho^*)$ such that either point ρ_j or ρ_{j+1} of the summand $t(\rho_j, \rho_{j+1})$ has an endpoint in common with ρ or ρ', then all summands subsequent to $t(\rho_j, \rho_{j+1})$ have magnitude $T(\Phi)$.

Proof: For if any subsequent term is not of magnitude $T(\Phi)$, a new finite sum representation $T_n'(\rho^*)$ of $T_n(\rho^*)$ may be constructed starting with $t(\rho_{j+1}, \rho_{j+2})$ such that the points of the subsequent components of $T_n'(\rho^*)$ are coincident with the endpoints ρ, ρ' of $T(\Phi)$. It follows that $T_n'(\rho^*) > T_n(\rho^*)$, contradicting the definition of $T_n(\rho^*)$, which proves the theorem.

FURTHER PROBLEMS

We will indicate in this concluding section some further directions of research into the properties of $T_n(\rho^*)$ which will result in deeper and more detailed knowledge of n-ary temporal diameters. Some questions of immediate interest are:

1. Under what conditions are the summands of $T_n(\rho^*)$ non-decreasing? It is conjectured that this property holds for all convex Φ.

2. When may $T_{n+1}(\rho^*)$ be obtained from $T_n(\rho^*)$ by simply adding the summand $t(\rho_{n+1}, \rho_{n+2})$ to $T_n(\rho^*)$, leaving all other summands invariant?

3. In the representation of $T(\Phi)$, if we set $d_i = T(\Phi) - t(\rho_i, \rho_{i+1})$ what are the full properties of convergence of the sequence $\{d_i\}$ (i.e., rapidity of convergence, truncation error estimates, etc.)?

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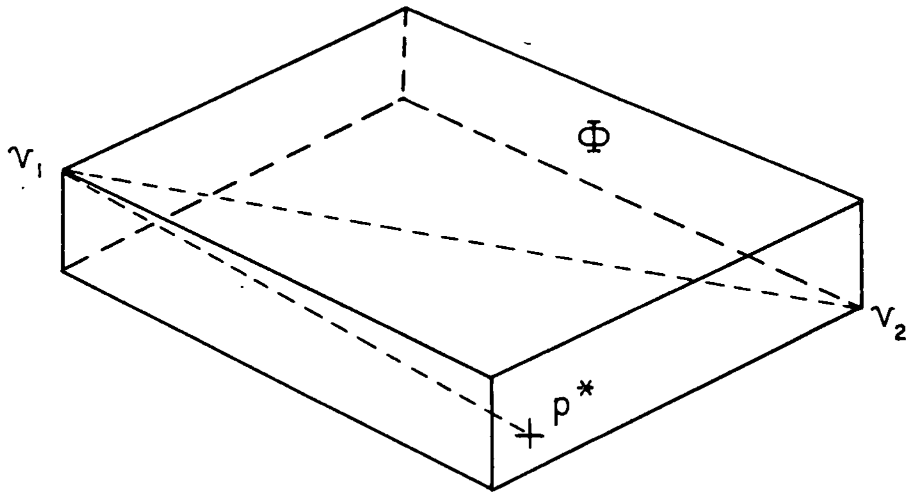

Figure 1

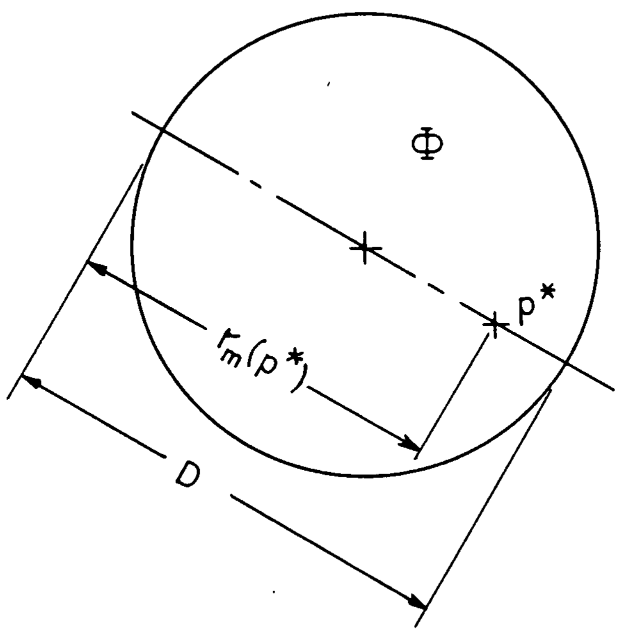

Figure 2

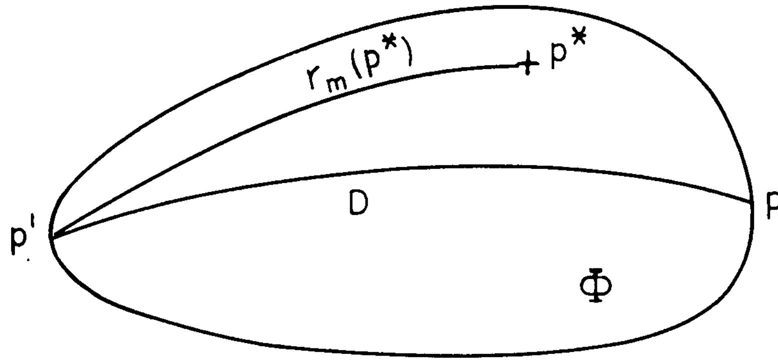

Figure 3

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