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THE UNIVERSAL RADIATIVE TRANSPORT EQUATION

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The Universal Radiative Transport Equation

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ABSTRACT

The various radiative transport equations used in general radiative transfer theory (for astrophysical and geophysical applications) over the past fifty-four years can be unified by a single transport equation—the universal transport equation. The 34 types of transport equations discussed in this work range from the transfer equation for radiance to the time-dependent transport equation for n-ary scattered radiant energy. Besides serving as a mathematical focal point of unification of the various transport equations, the universal transport equation supplies several new practical results which are beyond the capabilities of the individual standard transport equations for the radiometric concepts; for example, the proofs of the asymptotic theorems about the light field in deep optical media, and the basis for the equilibrium principle (enunciated below).

INTRODUCTION

The universal radiative transport equation is an equation which, by suitable choice of its parameters, yields in turn such equations as the general equation of transfer for radiance, the general Schuster two-flow transport equations for irradiance, the transport equation for scalar irradiance and the transport equations governing the apparent optical properties of an optical medium.

The primary purpose of the universal radiative transport equation is to formulate in a single mathematical package all the important transport equations which have been developed during the past fifty-four years in the theoretical studies of the steady state transfer of radiant energy through scattering-absorbing media of the stratified plane-parallel type. In this way a unification of all these important transport equations is achieved.

A second purpose of the universal transport equation is to provide a new useful tool in the study of radiative transfer theory. For, example, certain special forms of the universal transport equation have already been successfully used to obtain a solution to the long-standing practical problem of the existence of the asymptotic light field in deep stratified hydrosols, a feat which appears to be impossible without the introduction of the general type of functions associated with the universal transport equation. Further evidence of the usefulness of the universal transport equation as a tool which leads to new practical results will be illustrated below.

Before we go into the details of how the universal transport equation can achieve a semblance of unity in the classification of modern radiative transport equations, and of how it leads in some cases to new results which are beyond the capabilities of the classical transport equations, it may be of help to the reader to indicate the steps in the development of modern radiative transfer theory which have led to the idea of the universal transport equation. With such information in mind the reader can then easily follow the steps of the synthesis.

There are four well defined steps in the development of modern radiative transfer theory which form the immediate background to the formulation of the universal transport equation. These are, in chronological order: the adoption of the general equation of transfer for radiance and the development of the notion of equilibrium radiance;[1] the development of the unified irradiance equations and the notion of equilibrium irradiance;[2] the development of the canonical equation of transfer and the notion of the radiance K-function;[3] the development of the theory of the asymptotic light field and the transport equation for the radiance K-function.[4]

In the following two sections we will illustrate these steps in detail and add still further illustrations which have been uncovered subsequent to the time of the fourth step. In this way we will systematically build up evidence for the existence of a universal transport equation and for the equilibrium principle with which it is closely associated. After these concrete examples of the various transport equations have been assembled, the general form of the universal transport equation is presented.

This is followed by an illustration of the use of the universal transport equation in a discussion of the reflectance function $R(z,-)$. The paper closes with a brief survey of less common but equally important examples of transport equations which are also subsumed by the universal transport equation.

TRANSPORT EQUATIONS FOR RADIOMETRIC CONCEPTS

In this section we will present the transport equations for the following six radiometric quantities used in the study of plane-parallel media: radiance function $N(z,\theta,\phi)$, up and downwelling irradiance functions $H(z,\pm)$, up and downwelling scalar irradiance functions $h(z,\pm)$, and the scalar irradiance function $h(z)$.

Each of these transport equations is cast into a form which explicitly exhibits a certain attenuation function and equilibrium function associated with the radiometric concept it governs. Thus, for example, the customary form of the equation of transfer for radiance is recast so that it explicitly exhibits the special attenuation function $-\alpha(z)/\cos\theta$ and the equilibrium function $N_q(z,\theta,\phi) = N_*(z,\theta,\phi)/\alpha(z)$. Similarly, the unified irradiance equations governing $H(z,\pm)$ are recast into forms which explicitly exhibit the corresponding attenuation functions $\mp[a(z,\pm)+b(z,\pm)]$ and equilibrium functions $H_q(z,\pm)$. These two reformulations for the transport equations of $N(z,\theta,\phi)$ and $H(z,\pm)$ are already known;[1,2] however, the reformulations are now viewed with the purpose of seeing what mathematical and physical characteristics are held in common by these transport equations. It turns out that the common characteristics are the <u>attenuation</u> and <u>equilibrium</u> functions associated with each of the radiometric concepts governed by these equations and that each of these transport equations is but a special case of a more general equation.

The discussion of the present section continues with the derivation of the exact transport equations for $h(z,\pm)$ and $h(z)$. It is shown that each of these functions also may have associated with it an <u>attenuation</u> function and an <u>equilibrium</u> function. In this way we show that the six radiometric quantities used in the study of plane-parallel media have an important set of properties common to all: the notion of an associated attenuation function and an associated equilibrium function, and finally, that the transport equation for each of these six radiometric concepts is subsumed under one general equation.

We now proceed to substantiate the preceding assertions by considering in turn each of the six radiometric concepts and its associated transport equation.

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Equation (3) is the desired reformulation of (1).* For our present purposes we draw special attention to the two functions:

$$\text{(i)} \quad -\frac{\alpha(z)}{\cos\theta} \qquad\qquad \text{(ii)} \quad N_q(z,\theta,\phi) \tag{4}$$

Function (i) is <u>attenuation function</u> for $N(z,\theta,\phi)$ for a fixed direction (θ,ϕ).

Function (ii) is <u>equilibrium function</u> for $N(z,\theta,\phi)$ for a fixed direction (θ,ϕ).

Transport Equations for $H(z,\pm)$

The transport equations for $H(z,\pm)$ (or more accurately, the general Schuster equations for the two-flow analysis of the light field) are of the form:[2]

$$\mp\frac{dH(z,\pm)}{dz} = -\left[a(z,\pm)+b(z,\pm)\right]H(z,\pm)+b(z,\mp)H(z,\mp) \tag{5}$$

Associated with $H(z,-)$ and $H(z,+)$ is an equilibrium function $H_q(z,-)$ and $H_q(z,+)$, respectively. These equilibrium functions are

* An alternate formulation of (3) is possible by adopting the optical depth parameter $\tau = \int_0^z \alpha(z')\,dz'$. Such a formulation using τ has been found of especial use in an earlier study.[4] However, for our present purposes, equation (3) is more appropriate.

defined as:[2]

$$H_q(z,\pm) = \frac{b(z,\mp)\,H(z,\mp)}{a(z,\pm) + b(z,\pm)} . \tag{6}$$

By means of these functions the equations in (5) may be written:

$$\mp \frac{dH(z,\pm)}{dz} = -\left[a(z,\pm) + b(z,\pm)\right]\left[H(z,\pm) - H_q(z,\pm)\right]. \tag{7}$$

The equations in (7) are the desired reformulations of (5). For our present purposes we draw special attention to the two sets of functions:

(i) $\mp\left[a(z,\pm) + b(z,\pm)\right]$

(ii) $H_q(z,\pm)$ (8)

Set (i) gives the attenuation function for the upwelling $(+)$ and downwelling $(-)$ irradiances $H(z,\pm)$.

Set (ii) gives the equilibrium function for the upwelling $(+)$ and downwelling $(-)$ irradiances $H(z,\pm)$.

Transport Equations for $h(z,\pm)$

The exact transport equations for $h(z,+)$ and $h(z,-)$ apparently have never been discussed in the literature. The reason for this gap in the family transport equations for the common radiometric concepts is two fold. First and perhaps most important, there has never been an explicit need for the transport equations for $h(z,\pm)$; the ordinary irradiances

$H(z,\pm)$ were considered adequate in the early studies of the light field in stratified media. However, with the advent of more precise and detailed studies of the light field, the functions $h(z,\pm)$ have finally assumed a legitimate and useful role in modern radiative transfer theory.[7] Second, there is no simple way of obtaining the exact transport equations for $h(z,\pm)$ from first principles (de novo derivations starting only with the definition of $h(z,\pm)$ and the basic volume absorption and volume scattering functions). Neither is there any simple way of obtaining the requisite transport equations directly from the equation of transfer for radiance (in contradistinction to case for $H(z,\pm)$). In the present paragraph we derive the exact transport equations for $h(z,\pm)$ by a simultaneous use of (a): the connections between these functions and $H(z,\pm)$, provided by the distribution functions $D(z,\pm)$ and (b): the exact transport equations for $H(z,\pm)$.

We begin with the derivation of the transport equation for $h(z,-)$. By definition of $D(z,-)$,

$$h(z,-) = D(z,-)H(z,-) \ . \tag{9}$$

Taking the derivative of each side with respect to z :

$$\frac{dh(z,-)}{dz} = D(z,-)\frac{dH(z,-)}{dz} + H(z,-)\frac{dD(z,-)}{dz} \ .$$

By means of (5), this may be written

$$\frac{dh(z,-)}{dz} = D(z,-)\left\{\left[a(z,-)+b(z,-)\right]H(z,-) + b(z,+)H(z,+)\right\}$$

$$+ H(z,-)\,\frac{dD(z,-)}{dz}\,.$$

Using the definitions of $D(z,-)$ and $D(z,+)$ $\left(= h(z,+)/H(z,+)\right)$ and denoting the derivatives with respect to z by a prime (which will be used interchangeably with $\frac{d}{dz}$ in all that follows), the preceding equation may be written:

$$h'(z,-) = \left\{-\left[a(z,-)+b(z,-)\right] + \frac{D'(z,-)}{D(z,-)}\right\} h(z,-) + \frac{D(z,-)}{D(z,+)}\, b(z,+)\, h(z,+), \tag{10}$$

which is the general transport equation for $h(z,-)$.

Now, as in the case of $N(z,\theta,\phi)$ and $H(z,\pm)$ we may associate with $h(z,-)$ an equilibrium function $h_q(z,-)$ whose definition is:

$$h_q(z,-) = \frac{\frac{D(z,-)}{D(z,+)}\, b(z,+)\, h(z,+)}{\left[a(z,-)+b(z,-)\right] - \frac{D'(z,-)}{D(z,-)}} \tag{11}$$

or

$$h_q(z,-) = \frac{D^2(z,-)\, b(z,+)\, h(z,+)}{D(z,+)D(z,-)\left[a(z,-)+b(z,-)\right] - D'(z,-)\, D(z,+)},$$

With this definition of $h_g(z,-)$, the transport equation (10) may be written:

$$\frac{dh(z,-)}{dz} = \left[-\left(a(z,-)+b(z,-)\right) + \frac{D'(z,-)}{D(z,-)}\right]\left[h(z,-) - h_g(z,-)\right]. \quad (12)$$

Equation (12) is the reformulation of (11) which is of central interest in the present study, and as before we call special attention to the two functions:

$$\text{(i)} \quad +\left[a(z,-)+b(z,-)\right] - \frac{D'(z,-)}{D(z,-)}$$

$$\text{(ii)} \quad h_g(z,-) \quad (13)$$

The function (i) is the <u>attenuation function</u> for $h(z,-)$.
The function (ii) is the <u>equilibrium function</u> for $h(z,-)$.

The derivation of the transport equation for $h(z,+)$ proceeds in a similar manner to that leading to (12) and (13) in the case of $h(z,-)$. Therefore, the reader may easily verify that:

$$-\frac{dh(z,+)}{dz} = -\left\{\left[a(z,+)+b(z,+)\right] + \frac{D'(z,+)}{D(z,+)}\right\} h(z,+) \quad (14)$$

$$+ \frac{D(z,+)}{D(z,-)} b(z,-) h(z,-) .$$

Now if we set

$$h_q(z,+) = \frac{\frac{D(z,+)}{D(z,-)}\, b(z,-)\, h(z,-)}{\left[a(z,+) + b(z,+)\right] + \frac{D'(z,+)}{D(z,+)}} \tag{15}$$

$$= \frac{D^2(z,+)\, b(z,-)\, h(z,-)}{D(z,+)\, D(z,-)\left[a(z,+) + b(z,+)\right] + D'(z,+)\, D(z,-)},$$

then (14) may be written

$$-\frac{dh(z,+)}{dz} = -\left[\left(a(z,+) + b(z,+)\right) + \frac{D'(z,+)}{D(z,+)}\right]\left[h(z,+) - h_q(z,+)\right], \tag{16}$$

which is the desired reformulation of (14). We draw special attention to the functions:

$$\text{(i)} \quad -\left[a(z,+) + b(z,+)\right] - \frac{D'(z,+)}{D(z,+)}$$

$$\text{(ii)} \quad h_q(z,+) \tag{17}$$

The function (i) is the <u>attenuation function</u> for $h(z,+)$.
The function (ii) is the <u>equilibrium function</u> for $h(z,+)$.

We pause to observe the similarity of the functions in (8)(the set for $H(z,\pm)$) and with those in (13) and (17) (the set for $h(z,\pm)$).

These sets coincide when $D'(z,\pm)=0$, i.e., when $H(z,\pm)$ and $h(z,\pm)$ differ multiplicatively by a constant factor. That is, under this condition, (i) of (8) reduces to (i) of (13) and (17), and

$$\frac{H_q(z,\pm)}{h_q(z,\pm)} = D(z,\pm) = D(\pm) \qquad \text{for all } z .$$

Transport Equation for Scalar Irradiance

To obtain the transport equation for the scalar irradiance function $h(z)$, we begin by decomposing $h(z)$ into its up and downwelling components:

$$h(z) = h(z,+) + h(z,-) .$$

Then by using the definitions of the distribution functions:

$$D(z,\pm) = \frac{h(z,\pm)}{H(z,\pm)} ,$$

$h(z)$ may be represented in terms of $D(z,\pm)$ and $H(z,\pm)$:

$$h(z) = D(z,-)H(z,-) + D(z,+)H(z,+) .$$

Taking the derivative of $h(z)$, we have

$$\frac{dh(z)}{dz} = D(z,-)\frac{dH(z,-)}{dz} + H(z,-)\frac{dD(z,-)}{dz}$$

$$+ D(z,+)\frac{dH(z,+)}{dz} + H(z,+)\frac{dD(z,+)}{dz} .$$

We now make use of the exact transport equations for $H(z,\pm)$:

$$\frac{dh(z)}{dz} = D(z,-)\left\{-\left[a(z,-)+b(z,-)\right]H(z,-) + b(z,+)H(z,+)\right\}$$

$$+ H(z,-)D'(z,-) + H(z,+)D'(z,+)$$

$$+ D(z,+)\left\{\left[a(z,+)+b(z,+)\right]H(z,+) - b(z,-)H(z,-)\right\}.$$

The next step is to convert the products $D(z,\pm)H(z,\pm)$ into the equivalent functions $h(z,\pm)$ and write $h'(z)$ as a linear combination of $h(z,+)$, $h(z,-)$

$$\frac{dh(z)}{dz} = -\left[a(z,-)+b(z,-)\right]h(z,-) + \frac{D(z,-)}{D(z,+)}b(z,+)\,h(z,+)$$

$$+ \frac{D'(z,-)}{D(z,-)}h(z,-) + \frac{D'(z,+)}{D(z,+)}$$

$$+ \left[a(z,+)+b(z,+)\right]h(z,+) - \frac{D(z,+)}{D(z,-)}b(z,-)\,h(z,-).$$

Collecting coefficients of $h(z,\pm)$:

$$\frac{dh(z)}{dz} = A_-(z)\,h(z,-) + A_+(z)\,h(z,+), \tag{18}$$

where

$$A_-(z) = -\left[a(z,-) + b(z,-)\right] + \frac{D'(z,-) - D(z,+)\,b(z,-)}{D(z,-)}$$

and

$$A_+(z) = \left[a(z,+) + b(z,+)\right] + \frac{D'(z,+) + D(z,-)\,b(z,+)}{D(z,+)} .$$

Evidently (18) is unchanged if we write

$$\begin{aligned}\frac{dh(z)}{dz} &= A_-(z)\,h(z,-) + A_-(z)\,h(z,+) \\ &+ A_+(z)\,h(z,+) + A_+(z)\,h(z,-) \\ &- \left[A_-(z)\,h(z,+) + A_+(z)\,h(z,-)\right] .\end{aligned}$$

But then this equation may be reduced to:

$$\frac{dh(z)}{dz} = \left[A_-(z) + A_+(z)\right] h(z) - \left[A_-(z)\,h(z,+) + A_+(z)\,h(z,-)\right], \tag{19}$$

which is the transport equation for $h(z)$.

By defining

$$h_q(z) = \frac{A_-(z)\,h(z,+) + A_+(z)\,h(z,-)}{A_-(z) + A_+(z)} ,$$

Equation (19) is expressible as:

$$\frac{dh(z)}{dz} = \left[A_-(z) + A_+(z)\right]\left[h(z) - h_q(z)\right]. \tag{20}$$

For our present purposes, Equation (20) is of central interest, and we mark for future reference:

$$\begin{aligned} &(i) \quad -\left[A_-(z) + A_+(z)\right] \\ &(ii) \quad h_q(z) \end{aligned} \tag{21}$$

Expression (i) is the <u>attenuation function</u> for $h(z)$.
Expression (ii) is the <u>equilibrium function</u> for $h(z)$.

Preliminary Unification and Preliminary Statement of the Equilibrium Principle

We have now reached a point in our discussion where we must consolidate the results obtained so far. The consolidation will serve two purposes: it will yield a preliminary view of the structure of the universal transport equation, and secondly, it will prepare the way for a discussion of the transport equations for the apparent optical properties which takes place in the next section.

We turn now to the transport equations discussed so far, in particular the equations (3), (7), (12), (16) and (20). These six equations have a common mathematical structure, and the various components of the structure are associated with physical concepts common to the respective radiometric concepts. Specifically, let the general symbol $\mathcal{P}(z)$ stand for any one of the following six radiometric concepts:

$$\mathcal{P}(z) \; : \begin{cases} N(z, \theta, \phi) \\ H(z, \pm) \\ h(z, \pm) \\ h(z) \end{cases}$$

Furthermore, let $\mathcal{P}_{\alpha}(z)$ stand for the associated attenuation function for $\mathcal{P}(z)$. Finally, let $\mathcal{P}_{q}(z)$ stand for the associated equilibrium function for $\mathcal{P}(z)$. Then each of the six transport equations developed above is precisely of the form:

$$\frac{d\mathcal{P}(z)}{dz} = -\mathcal{P}_{\alpha}(z)\left[\mathcal{P}(z) - \mathcal{P}_{q}(z)\right] . \tag{22}$$

We now may make a key observation on the dynamic behavior of the five radiometric concepts which are associated with a general direction of flow ($h(z)$ is the only one of the preceding concepts which, by definition, is not associated with any particular directed pencil of radiation or general hemispherical flow). If $\mathcal{P}(z)$ stands for any one of these five concepts: $N(z,\theta,\phi)$, $H(z,\pm)$, $h(z,\pm)$, then it is easy to verify that:

$$\text{If } \mathcal{P}(z) > \mathcal{P}_q(z) \quad \text{we have from (22)} \quad \frac{d\mathcal{P}(z)}{d|z|} < 0, \tag{23}$$

and

$$\text{If } \mathcal{P}(z) < \mathcal{P}_q(z) \quad \text{we have from (22)} \quad \frac{d\mathcal{P}(z)}{d|z|} > 0,$$

where the symbol $d\mathcal{P}(z)/d|z|$ is defined as follows:

$$\frac{d\mathcal{P}(z)}{d|z|} = \frac{d\mathcal{P}(z)}{dz}$$

if $\mathcal{P}(z)$ is associated with the direction of increasing z (downwelling direction)

$$\frac{d\mathcal{P}(z)}{d|z|} = \frac{d\mathcal{P}(z)}{d(-z)}$$

if $\mathcal{P}(z)$ is associated with the direction of decreasing z (upwelling direction).

In other words, the equations (23) simply state that as the radiation represented by $\mathcal{P}(z)$ travels <u>in its assigned direction</u>, the magnitude of $\mathcal{P}(z)$ always changes in such a way that it tends to approach the magnitude of its equilibrium function $\mathcal{P}_q(z)$. This observation forms the core of the general equilibrium principle formulated below.

THE APPARENT OPTICAL PROPERTIES

The notion of "apparent optical property" has already been discussed in detail elsewhere (see Table 2 of Ref. 8). In particular, the present list of the more important apparent optical properties consists of the following seven quantities:

$$\begin{cases} K(z,\pm) \\ R(z,\pm) \\ D(z,\pm) \\ k(z) \end{cases}$$

For our present purposes, we may extend this list to include:

$$k(z,\pm) = \frac{-1}{h(z,\pm)} \frac{dh(z,\pm)}{dz}, \quad K(z,\theta,\phi) = \frac{-1}{N(z,\theta,\phi)} \frac{dN(z,\theta,\phi)}{dz}$$

We shall show below that a transport equation may be assigned to each of the above K-functions. We can also assign a transport equation to $R(z,\pm)$ and $D(z,\pm)$, and in fact will exhibit the transport equation for $R(z,-)$ and go on to deduce, by means of this equation, an interesting property about the depth behavior of $R(z,-)$. We will not, however, exhibit the transport equation for $R(z,+)$ and $D(z,\pm)$ for the following reasons: By definition $R(z,+) = 1/R(z,-)$, so that once a transport equation is obtained for $R(z,-)$, one for $R(z,+)$ would be superfluous. The reason for not obtaining transport equations for the optical properties $D(z,\pm)$ is more subtle and may be inferred from the preceding formulations: the preceding transport equations for $H(z,\pm)$, $h(z,\pm)$ make implicit or explicit use of the distribution functions. If we were to deduce the transport equations for $D(z,\pm)$ we would see that the quantities $H(z,\pm)$ or $h(z,\pm)$ would be explicitly involved in them. Therefore, a logical circularity would creep into the final set of transport equations if we insisted on obtaining transport equations for $D(z,\pm)$ in addition to those of $H(z,\pm)$ and $h(z,\pm)$. In order to avoid such a circularity we must decide on the elimination of one of the three sets of quantities: $H(z,\pm)$, $h(z,\pm)$, $D(z,\pm)$. Such a decision is easy to reach after we note that $H(z,\pm)$ and $h(z,\pm)$ are the fundamental observables in natural light fields, and that the $D(z,\pm)$ simply act as analytical liasons between these quantities. Therefore, we will agree that $D(z,\pm)$ are to continue to act as the connecting links between the irradiance and scalar irradiance concepts, and that they are to enter into the calculations solely in the capacity of dimensionless mathematical parameters. Their usual physical interpretation will, of course, be retained (namely that they are measures of the directional variation of the radiance distribution at a general depth z).

TRANSPORT EQUATIONS FOR APPARENT OPTICAL PROPERTIES

In accordance with the preceding discussion we will now obtain the transport equations for the following apparent optical properties: $K(z,\theta,\phi)$, $K(z,\pm)$, $k(z,\pm)$, $k(z)$ and $R(z,-)$.

Canonical Forms of Transport Equations for K-functions

The procedure for obtaining the transport equation for the six K-functions is facilitated by the preceding results, in particular by means of the six transport equations for $N(z,\theta,\phi)$, $H(z,\pm)$, $h(z,\pm)$ and $h(z)$. If $\mathcal{P}(z)$ stands for any of these six functions, then the corresponding K-function $K(\mathcal{P})$ is defined as:

$$K(\mathcal{P}) = \frac{-1}{\mathcal{P}(z)} \frac{d\mathcal{P}(z)}{dz} . \tag{24}$$

Using the generic equation (22) and the definition (24), we have:

$$-\mathcal{P}(z)\, K(\mathcal{P}) = -\mathcal{P}_\alpha \left[\mathcal{P}(z) - \mathcal{P}_q(z) \right] .$$

Solving this for $\mathcal{P}(z)$, we obtain the <u>canonical form</u>[3] of the transport equation for $\mathcal{P}(z)$:

$$\mathcal{P}(z) = \frac{\mathcal{P}_q(z)}{1 - \dfrac{K(\mathcal{P})}{\mathcal{P}_\alpha(z)}} . \tag{25}$$

This canonical form of the transport equation serves as the common starting point for the derivation of the equations governing individual K -functions. Thus, by taking the formal logarithmic derivative of each side of (25):

$$\frac{d \ln \rho(z)}{dz} = \frac{d \ln \rho_q(z)}{dz} - \frac{d}{dz} \ln \left[1 - \frac{K(\rho)}{\rho_\alpha(z)} \right],$$

and defining, in analogy to (24):

$$K_q(\rho) = \frac{-1}{\rho_q(z)} \frac{d\rho_q(z)}{dz}, \tag{26}$$

we have

$$-K(\rho) = -K_q(\rho) + \frac{\frac{d}{dz}\left[\frac{K(\rho)}{\rho_\alpha(z)}\right]}{1 - \frac{K(\rho)}{\rho_\alpha(z)}},$$

whence

$$\frac{d}{dz}\left[\frac{K(\rho)}{\rho_\alpha(z)}\right] = \left[\frac{K(\rho)}{\rho_\alpha(z)} - 1\right]\left[K(\rho) - K_q(\rho)\right]. \tag{27}$$

Dimensionless Transport Equation for $K(\mathcal{P})$

At this point we have two alternative routes open to a universal transport equation: one route starts with the adoption of a generalized notion of <u>optical depth</u>:

$$\tau = \int_0^z \mathcal{P}_\alpha(z')\,dz',$$

along with a <u>relativization</u> of $K(\mathcal{P})$ and $K_q(\mathcal{P})$ with respect to $\mathcal{P}_\alpha(z)$, thus:

$$\widetilde{K}(\mathcal{P}) \equiv \frac{K(\mathcal{P})}{\mathcal{P}_\alpha};$$

$$\widetilde{K}_q(\mathcal{P}) \equiv \frac{K_q(\mathcal{P})}{\mathcal{P}_\alpha}.$$

Then (27) may be written in the <u>dimensionless form</u>

$$\frac{d\widetilde{K}(\mathcal{P})}{d\tau} = \left[\widetilde{K}(\mathcal{P}) - 1\right]\left[\widetilde{K}(\mathcal{P}) - \widetilde{K}_q(\mathcal{P})\right]. \tag{28}$$

Equation (28) has the advantage of simplicity of structure and is therefore ideal for formal work (see, for example, the dimensionless form of (28) for the case $\mathcal{P}(z) = N(z,\theta,\phi)$, which was used in the proof of the asymptotic radiance hypothesis[4]). However, Equation (28) has the disadvantage of not showing the explicit effects on the associated K-functions produced by inhomogeneities of the medium, nor of the way in which the K-functions vary with <u>geometrical depth</u>, the natural measure of depth used in experimental work. Therefore, we will actually take the second route which consists in adopting geometrical depth and unrelativized K -functions. This results in a mathematically more cumbersome transport equation, but is actually of greater use in practical applications. By adopting the alternative route, we are now obliged to consider each of the K-functions in turn. The common starting point is Equation (25) in which the explicit forms of $\mathcal{P}_\alpha(z)$ and $\mathcal{P}_q(z)$ for the various concepts have been substituted.

Transport Equation for $K(z,\theta,\phi)$

From (25) we have

$$N(z,\theta,\phi) = \frac{N_q(z,\theta,\phi)}{1 + \sec\theta \, \dfrac{K(z,\theta,\phi)}{\alpha(z)}} , \tag{29}$$

in which we have set $\mathcal{P}_\alpha(z) = -\alpha(z)/\cos\theta$, $\mathcal{P}_q(z) = N_q(z,\theta,\phi)$, so that $K(\mathcal{P}) = K(z,\theta,\phi)$ and $K_q(\mathcal{P}) = K_q(z,\theta,\phi)$, using the definitions in (4). Taking the logarithmic derivative of each side of (29), and solving for $dK(z,\theta,\phi)/dz$:

$$\frac{dK(z,\theta,\phi)}{dz} = K^2(z,\theta,\phi) + \left[\frac{\alpha(z)}{\cos\theta} - K_q(z,\theta,\phi) + \frac{1}{\alpha(z)}\frac{d\alpha(z)}{dz}\right] K(z,\theta,\phi) - K_q(z,\theta,\phi)\frac{\alpha(z)}{\cos\theta} .$$

The right-hand side of this equation may be factored into the product of two functions yielding the desired form of the transport equation for $K(z,\theta,\phi)$:

$$\frac{dK(z,\theta,\phi)}{dz} = \left[K(z,\theta,\phi) - \mathcal{K}_\alpha(z,\theta,\phi)\right]\left[K(z,\theta,\phi) - \mathcal{K}_q(z,\theta,\phi)\right], \tag{30}$$

where

$$\begin{aligned} \mathcal{K}_\alpha(z,\theta,\phi) + \mathcal{K}_q(z,\theta,\phi) &= -\frac{\alpha(z)}{\cos\theta} + K_q(z,\theta,\phi) - \frac{1}{\alpha(z)}\frac{d\alpha(z)}{dz} \\ \mathcal{K}_\alpha(z,\theta,\phi)\,\mathcal{K}_q(z,\theta,\phi) &= -\frac{\alpha(z)}{\cos\theta} K_q(z,\theta,\phi) \end{aligned} \tag{31}$$

The functions $\mathcal{K}_\alpha(z,\theta,\phi)$ and $\mathcal{K}_q(z,\theta,\phi)$ appearing in (30) are the attenuation and equilibrium functions for $K(z,\theta,\phi)$. They are defined by the pair of simultaneous equations in (31) whose solutions are:

$$\left.\begin{matrix} 2\mathcal{K}_q \\ \\ 2\mathcal{K}_\alpha \end{matrix}\right\} = -\left[\frac{\alpha}{\cos\theta} - K_q + (\ln\alpha)'\right] \pm \left[\left(\frac{\alpha}{\cos\theta} - K_q + (\ln\alpha)'\right)^2 + 4K_q\frac{\alpha}{\cos\theta}\right]^{\frac{1}{2}}$$

The quantities K_q and $\mathcal{K}_q$ should not be confused with each other. K_q is the logarithmic derivative of N_q (see definition (26)) whereas $\mathcal{K}_q$ is the sought-for equilibrium function for $K(z,\theta,\phi)$ in the general context. Observe, however, that if the medium were homogeneous, then

$$\mathcal{K}_\alpha(z,\theta,\phi) = \frac{-\alpha}{\cos\theta} ,$$

$$\mathcal{K}_q(z,\theta,\phi) = K_q(z,\theta,\phi) .$$

More generally, in eventually homogeneous media (i.e., media in which $\alpha'(z) \to 0$ as $z \to \infty$)

$$\mathcal{K}_\alpha(z,\theta,\phi) \longrightarrow \frac{-\alpha}{\cos\theta} ,$$

$$\mathcal{K}_q(z,\theta,\phi) \longrightarrow K_q(z,\theta,\phi) \longrightarrow k_\infty .$$

This follows from the asymptotic radiance theorem and its various consequences.[9]

Transport Equations for $K(z,\pm)$

The appropriate form of (25) in the present context is obtained by substituting the attenuation and equilibrium functions for $H(z,\pm)$ in (25):

$$H(z,\pm) = \frac{H_q(z,\pm)}{1 \pm \dfrac{K(z,\pm)}{[a(z,\pm) + b(z,\mp)]}} .$$

Taking the logarithmic derivatives of each side, solving for $K'(z,\pm)$, and factoring the quadratic in $K(z,\pm)$, we have

$$\frac{dK(z,-)}{dz} = \left[K(z,-) - \mathcal{K}_\alpha(z,-)\right]\left[K(z,-) - \mathcal{K}_q(z,-)\right], \tag{32}$$

$$\frac{dK(z,+)}{dz} = \left[K(z,+) - \mathcal{K}_\alpha(z,+)\right]\left[K(z,+) - \mathcal{K}_q(z,+)\right]. \tag{33}$$

For $K(z,+)$:

$$\mathcal{K}_\alpha(z,+) + \mathcal{K}_q(z,+) = \left[a(z,-) + b(z,-)\right] + K_q(z,-) - \left(\ln\left[a(z,-) + b(z,-)\right]\right)'$$

$$\mathcal{K}_\alpha(z,+)\,\mathcal{K}_q(z,+) = \left[a(z,-) + b(z,-)\right] K_q(z,-)$$

For $K(z,-)$:

$$\mathcal{K}_\alpha(z,-) + \mathcal{K}_q(z,-) = -\left[a(z,+) + b(z,+)\right] + K_q(z,+) - \left(\ln\left[a(z,+) + b(z,+)\right]\right)'$$

$$\mathcal{K}_\alpha(z,-)\,\mathcal{K}_q(z,+) = -\left[a(z,+) + b(z,+)\right] K_q(z,+) ,$$

These simultaneous equations may be solved to obtain explicit expressions for the respective $\mathcal{K}_\alpha$'s and $\mathcal{K}_q$'s. We will not do this here, but rather take the space to point out that in all eventually homogeneous media that as $z \to \infty$,

$$\mathcal{K}_\alpha(z,\pm) \longrightarrow \mp\left[a(z,\pm)+b(z,\pm)\right],$$

$$\mathcal{K}_q(z,\pm) \longrightarrow K_q(z,\pm) \longrightarrow k_\infty .$$

This follows from the asymptotic radiance theorem and its various consequences.[9] As in the case of $K_q(z,\theta,\phi)$ and $\mathcal{K}_q(z,\theta,\phi)$, care should be taken so as not to confuse $K_q(z,\pm)$ with $\mathcal{K}_q(z,\pm)$. The former is defined in (26), the latter by the preceding simultaneous equations.

Transport Equations for $k(z,\pm)$ and $k(z)$

Starting with the general canonical equation (25) we have for $h(z)$:

$$h(z) = \frac{h_q(z)}{1+\dfrac{k(z)}{A_+(z)+A_-(z)}} .$$

Similarly, for $h(z,\pm)$:

$$h(z,\pm) = \frac{h_q(z,\pm)}{1 \pm \dfrac{k(z,\pm)}{\left[a(z,\pm)+b(z,\pm)\right] - \dfrac{D'(z,\pm)}{D(z,\pm)}}} .$$

The existence of these canonical equations for $h(z,\pm)$ and $h(z)$ is sufficient to prove the existence of the appropriate transport equations for $k(z,\pm)$ and $k(z)$ by following the procedure illustrated in the preceding two paragraphs. The results are

$$\frac{dk(z,\pm)}{dz} = \left[k(z,\pm) - k_\alpha(z,\pm)\right]\left[k(z,\pm) - k_\gamma(z,\pm)\right] , \tag{34}$$

$$\frac{dk(z)}{dz} = \left[k(z) - k_\alpha(z)\right]\left[k(z) - k_\gamma(z)\right] . \tag{35}$$

The exact forms for the respective $\mathcal{K}_\alpha$'s and $\mathcal{K}_\gamma$'s will not be worked out; this may be left as an exercise for the interested reader. The important point to observe is that we have now proved that for all six K-functions, the generic transport equation is:

$$\frac{dK(\mathcal{P})}{dz} = \left[K(\mathcal{P}) - \mathcal{K}_\alpha(\mathcal{P})\right]\left[K(\mathcal{P}) - \mathcal{K}_\gamma(\mathcal{P})\right] \tag{36}$$

Equations (22) and (36) form the two major sets of transport equations considered in this paper. These two equations cover all twelve transport equations for $\mathcal{P}$ and $K(\mathcal{P})$ considered so far.

As in the case of (22), it is easy to verify that if

$$K(\mathcal{P}) > \mathcal{K}_q(\mathcal{P}) \qquad \text{then from (36):} \quad \frac{dK(\mathcal{P})}{d|z|} < 0 ,$$

and if (37)

$$K(\mathcal{P}) < \mathcal{K}_q(\mathcal{P}) \qquad \text{then from (36):} \quad \frac{dK(\mathcal{P})}{d|z|} > 0 ,$$

which show that $K(\mathcal{P})$ always tends toward* its equilibrium function $\mathcal{K}_q(\mathcal{P})$.

We now turn to consider the last of the standard transport equations, namely that for $R(z,-)$.

Transport Equation for $R(z,-)$

By definition of $R(z,-)$:

$$R(z,-) = \frac{H(z,+)}{H(z,-)} .$$

Taking the logarithmic derivative of each side, and applying the definitions of $K(z,+)$ and $K(z,-)$, we have

$$\frac{dR(z,-)}{dz} = R(z,-)\left[K(z,-) - K(z,+)\right] .$$

*The term "tends toward" has a precise meaning here: if f_1 and f_2 are two real-valued functions defined on some common domain $\mathcal{D}$ of the reals then f_1 <u>tends toward</u> f_2 at $x \in \mathcal{D}$ if $\operatorname{sgn}[f_1(x) - f_2(x)] = \operatorname{sgn} f_1'(x)$ where " sgn " means "sign of."

Using the following representations[7] of $K(z,\pm)$:

$$K(z,\pm) = \mp\left[a(z,\pm)+b(z,\pm)\right] \pm b(z,\mp)\,R(z,\mp),$$

the derivative of $R(z,-)$ may be cast into the form

$$\frac{dR(z,-)}{dz} = -b(z,+)\,R^2(z,-) +$$

$$+\left[a(z,-)+a(z,+)+b(z,-)+b(z,+)\right]R(z,-) - b(z,-).$$

The right-hand side, which is a quadratic in $R(z,-)$, may be factored:

$$\frac{dR(z,-)}{dz} = -b(z,+)\left[R(z,-) - R_\alpha(z,-)\right]\left[R(z,-) - R_\gamma(z,-)\right], \quad (38)$$

Equation (38) is the required transport equation for $R(z,-)$, in which $R_\alpha(z,-)$ is the <u>attenuation function</u> for $R(z,-)$ and $R_\gamma(z,-)$ is the <u>equilibrium function</u> for $R(z,-)$. These functions are defined by the following system of simultaneous equations:

$$R_\alpha(z,-) + R_\gamma(z,-) = \frac{a(z,-)+a(z,+)+b(z,-)+b(z,+)}{b(z,+)}, \quad (39)$$

$$R_\alpha(z,-)\,R_\gamma(z,-) = \frac{b(z,-)}{b(z,+)}.$$

As in the case of the K-functions, these may be solved for $R_\alpha(z,-)$ and $R_\gamma(z,-)$:

$$\left.\begin{array}{l} 2R_\alpha(z,-) \\ \\ 2R_\gamma(z,-) \end{array}\right\} = \left\{ R(z,-) + \frac{1}{R(z,-)}\frac{b(z,-)}{b(z,+)} - \frac{1}{b(z,+)}\left[K(z,-) - K(z,+)\right]\right\} \pm \pm \left[\{\}^2 - 4\,\frac{b(z,-)}{b(z,+)}\right]^{\frac{1}{2}} . \tag{40}$$

R_α goes with the plus sign, R_γ with the minus sign.

We observe that, in eventually homogeneous media, as $z \to \infty$,

$$R_\alpha(z,-) \longrightarrow \frac{1}{R(z,-)}\,\frac{b(z,-)}{b(z,+)} \longrightarrow \frac{1}{R_\infty}\,\frac{b(-)}{b(+)} \tag{41}$$

$$R_\gamma(z,-) \longrightarrow R(z,-) \longrightarrow R_\infty .$$

These facts follow from (40) and the asymptotic radiance theorem.[9]

UNIVERSAL RADIATIVE TRANSPORT EQUATION AND THE EQUILIBRIUM PRINCIPLE

For the purposes of this section, let us refer to the thirteen quantities studied so far as the standard concepts (namely $N(z,\theta,\phi)$, $H(z,\pm)$, $h(z,\pm)$, $h(z)$, $K(z,\theta,\phi)$, $K(z,\pm)$, $k(z,\pm)$, $k(z)$, and $R(z,-)$. A directed standard concept is any of the preceding standard concepts except $h(z)$ and $k(z)$.

The evidence gathered in the preceding discussions may now be assembled in the form of

The Equilibrium Principle. Let X be an arbitrarily stratified source-free plane parallel medium with arbitrary incident lighting conditions. Let $\mathcal{C}(z)$ represent any of the standard concepts. Then associated with $\mathcal{C}(z)$ are two functions $\mathcal{C}_\alpha(z)$ and $\mathcal{C}_\gamma(z)$, the attenuation and equilibrium functions for $\mathcal{C}(z)$ respectively. The standard concept $\mathcal{C}(z)$ together with $\mathcal{C}_\alpha(z)$ and $\mathcal{C}_\gamma(z)$ satisfy the functional relation:

$$\frac{d\,\mathcal{C}(z)}{dz} = \mu(z)\left[\delta\,\mathcal{C}(z) - \mathcal{C}_\alpha(z)\right]\left[\mathcal{C}(z) - \mathcal{C}_\gamma(z)\right], \quad (42)$$

where $\mu(z)$ and δ are known parameters depending on $\mathcal{C}(z)$. The relation (42) is the universal radiative transport equation.

If $\mathcal{C}(z)$ is a directed standard concept and $\mu(z) > 0$, then

$$\frac{d\,\mathcal{C}(z)}{d|z|} \gtrless 0 \quad \text{if} \quad \mathcal{C}(z) \lesseqgtr \mathcal{C}_\gamma(z)\,; \tag{43}$$

and if $\mathcal{C}(z)$ is any standard concept, and X is eventually homogeneous then

$$\begin{aligned} \mathcal{C}_\alpha(\infty) &= \lim_{z\to\infty} \mathcal{C}_\alpha(z) \quad \text{exists,} \\ \mathcal{C}_\gamma(\infty) &= \lim_{z\to\infty} \mathcal{C}_\gamma(z) \quad \text{exists,} \end{aligned} \tag{44}$$

and

$$\lim_{z\to\infty} \mathcal{C}(z) = \mathcal{C}_\gamma(\infty)\,. \tag{45}$$

The proof of the statements (42), (43), (44) and (45) have essentially been covered in the preceding discussions either directly (as in the case of (42)), or indirectly by references to the appropriate sources in the bibliography (as in the case of (43) - (45)). Table I below gives the explicit forms of $\mu(z)$ and δ for the thirteen standard concepts: An examination of Table I shows that if $R(z,-)$ is removed from the list of standard concepts, a considerable simplification is effected in the form of (42). However, in the interests of completeness we have included $R(z,-)$.

TABLE I

STANDARD CASES OF THE UNIVERSAL RADIATIVE TRANSPORT EQUATION

STANDARD CONCEPT	VALUES OF μ, δ
$N(z,\theta,\phi)$ $H(z,\pm)$ $h(z,\pm)$ $h(z)$	$\mu(z) = 1, \ \forall z \geq 0$ $\delta = 0$
$K(z,\theta,\phi)$ $K(z,\pm)$ $k(z,\pm)$ $k(z)$	$\mu(z) = 1 \quad \forall z \geq 0$ $\delta = 1$
$R(z,-)$	$\mu(z) = b(z,+), \forall z \geq 0$ $\delta = 1$

Discussion of Some Properties of $R(z,-)$

We begin with some observations on how the relative magnitudes of the two quantities $R_\alpha(z,-)$, $R_\gamma(z,-)$ govern the local behavior of $R(z,-)$. In the transport equation (38) for $R(z,-)$ we note that $b(z,+) > 0$ in all scattering media. Furthermore, from the defining equations for $R_\alpha(z,-)$ and $R_\gamma(z,-)$, we conclude that in general,

$$R_\alpha(z,-) > R_\gamma(z,-) \tag{46}$$

in all scattering media. Thus, if $R(z,-)$ is such that

$$R_\alpha(z,-) > R(z,-) > R_\gamma(z,-), \tag{47}$$

then from (38), we conclude that

$$\frac{dR(z,-)}{dz} > 0 . \tag{48}$$

This follows from the transport equation and the facts

$$R(z,-) - R_\alpha(z,-) < 0 ,$$
$$R(z,-) - R_\gamma(z,-) > 0 .$$

Hence the product of the two corresponding factors on the right of (30) has a negative sign. Since $b(z,+) > 0$, the derivative on the left of (38) must have a positive sign, indicating that $R(z,-)$ is increasing at depth z .

Now in eventually homogeneous media that exhibid both scattering and absorption at all depths z , it is easy to see that $R(z,-) < 1$ for all z , and furthermore, that

$$\lim_{z\to\infty} R_\infty(z,-) = \frac{1}{R_\infty}\frac{b(-)}{b(+)} > 1 ,$$

which follows from (44), and the exact relation:

$$\frac{1}{R(z,-)}\frac{b(z,-)}{b(z,+)} = \frac{\int_{\Xi_-} \sigma_-(z;\xi')\,N(z,\xi')\,d\Omega(\xi')}{\int_{\Xi_+} \sigma_-(z;\xi')\,N(z,\xi')\,d\Omega(\xi')} ,$$

where

$$\sigma_-(z;\xi') = \int_{\Xi_\pm} \sigma(z;\xi;\xi')\,d\Omega(\xi) ,$$

in which the integration is taken over Ξ_+ or Ξ_- , depending on whether

$$\xi' \varepsilon\, \Xi_+ \quad \text{or} \quad \xi' \varepsilon\, \Xi_- .$$

Hence (see Figures 1 and 2) there is always a depth z_o in such media such that whenever $z > z_o$,

$$R_\infty(z,-) > 1 > R(z,-) .$$

Therefore, the left-hand side of (47) holds for sufficiently large z in all eventually homogeneous scattering-absorbing media. The criterion (47) for the eventual increase of $R(z,-)$ may then be replaced by the single inequality

$$R(z,-) > R_q(z,-) \tag{49}$$

and the criterion for the eventual decrease of $R(z,-)$ may be written as

$$R(z,-) < R_q(z,-) \tag{50}$$

Figure 1 shows the situation summarized by (49): $R(z,-)$ is eventually sandwiched between $R_\alpha(z,-)$ and $R_q(z,-)$ and is therefore constrained to rise monotonically toward its ultimate limit R_∞. $R(z,-)$ cannot be below $R_q(z,-)$ whenever the latter rises toward its limit R_∞, for if this is the case, then $R'(z,-) < 0$ by the transport equation (48), and $R(z,-)$ would not then attain its limit R_∞.

Figure 2 shows the depth-behavior of $R(z,-)$ summarized by (50): $R(z,-)$ is eventually below $R_q(z,-)$, and thus, by (48), is constrained to decrease monotonically toward its limit R_∞. $R(z,-)$ cannot be above $R_q(z,-)$ whenever the latter decreases toward its limit R_∞; for if this is the case, then $R(z,-)$ cannot attain its limit R_∞.

$R_q(z,-)$ thus acts as a directrix for $R(z,-)$ in the sense that if $R_q(z,-)$ decreases monotically toward its limit R_∞, then we have

$$R_q(z,-) > R(z,-) > R_\infty$$

for all z. On the other hand, if $R_q(z,-)$ increases monotically toward its limit R_∞, then

$$R_\infty > R(z,-) > R_q(z,-)$$

for all z.

The eventual monotonic behavior of $R_\alpha(z,-)$ and $R_q(z,-)$ in eventually homogeneous media may be established from an examination of (39). This observation, together with those of the preceding two paragraphs, may supply a proof of the conjecture that $R(z,-)$ tends monotically toward its limit R_∞. Whether $R(z,-)$ eventually decreases toward R_∞, or eventually increases toward R_∞ depends on the eventual values of the inherent optical properties of the medium.

SOME ADDITIONAL TRANSPORT EQUATIONS SUBSUMED BY THE UNIVERSAL TRANSPORT EQUATION

The standard transport equations enumerated in TABLE I constitute the most frequently used equations in general radiative transfer theory. This list, however, by no means exhausts the various ramifications of the universal transport equation as given by (42). An additional set of transport equations which fall under the domain of the universal transport equation will now be mentioned. This set is associated with less frequently used--but no less important--radiometric concepts than those of the standard type. We will consider in particular the following radiometric quantities:

(i)	n-ary radiance	N^n
(ii)	n-ary radiant energy	U^n
(iii)	path function	N_*
(iv)	vector irradiance	$\underset{\sim}{H}$

(i). <u>The transport equation governing N^n</u> is:

$$-\cos\theta \frac{dN^n(z,\theta,\phi)}{dz} = -\alpha(z)N^n(z,\theta,\phi) + N_*^n(z,\theta,\phi) \quad (51)$$

where

$$N_*^n(z,\theta,\phi) = \int_{\Xi} \sigma(z;\theta,\phi;\theta',\phi')\, N^{n-1}(z,\theta',\phi')\, d\Omega . \quad (52)$$

Here N^n, $n = 1, 2, \cdots$, is the n-ary scattered radiance,[2] i.e., radiance consisting of photons having been scattered precisely n-times with respect to those comprising N^0. In any particular problem, it is assumed that $N^0(z,\theta,\phi)$ is given. From this, $N^1_*(z,\theta,\phi)$ is obtainable by means of (52). Then $N^1_*(z,\theta,\phi)$ is known, and (51) becomes a differential equation in $N^1(z,\theta,\phi)$ which is easily solved in principle. Numerical solutions of $N^1(z,\theta,\phi)$ may be readily obtained by means of a large scale computer programmed for (51). Once $N^1(z,\theta,\phi)$ is known for all z and (θ,ϕ), (52) yields $N^2_*(z,\theta,\phi)$ and (51) may be solved for $N^2(z,\theta,\phi)$. By repeating this process, we are led to obtain $N^n(z,\theta,\phi)$ knowing $N^{n-1}(z,\theta,\phi)$. The total (observable) radiance $N(z,\theta,\phi)$ is defined as

$$N(z,\theta,\phi) = \sum_{n=0}^{\infty} N^n(z,\theta,\phi) \quad .$$

For our present purposes we write

$$N^n_q(z,\theta,\phi) = \frac{N^n_*(z,\theta,\phi)}{\alpha(z)} \quad ,$$

so that (51) may be written

$$\frac{dN^n(z,\theta,\phi)}{dz} = \frac{\alpha(z)}{\cos\theta}\left[N^n(z,\theta,\phi) - N^n_q(z,\theta,\phi)\right] . \tag{53}$$

When written in this form, Equation (53) closely parallels the form of Equation (3), so that we conclude, as in (4):

(a) $\dfrac{-\alpha(z)}{\cos\theta}$ is the attenuation function for $N^n(z,\theta,\phi)$

(b) $N^n_q(z,\theta,\phi)$ is the equilibrium function for $N^n(z,\theta,\phi)$

and thus the transport equation for $N^n(z,\theta,\phi)$ is subsumed by (42) in which $\mu(z) = 1$, $\delta = 0$.

(ii). <u>The transport equation governing U^n</u> is usually written in terms of a time parameter t instead of a space parameter z :[11]

$$\frac{dU^n(t)}{dt} = -\frac{U^n(t)}{T_\alpha} + \frac{U^{n-1}(t)}{T_s} \tag{54}$$

where $T_\alpha = 1/v\alpha$, $T_s = 1/vs$. However, we may introduce a new variable

$$r = vt$$

so that (54) becomes:

$$\frac{dU^n(r)}{dr} = -\alpha\, U^n(r) + s\, U^{n-1}(r) \, . \tag{55}$$

The symbol $U^n(r)$ represents the n-ary radiant energy content of a sphere of radius r about a point source (in a space X) which emits radiant flux in some prescribed manner starting from time $t = 0$. The space is assumed homogeneous ($\alpha(z) = \alpha$ for all z in the space).

$\mathcal{V}$ is the speed of light in X. By letting

$$U_*^n(r) = s\,U^{n-1}(r)$$

and

$$U_q^n(r) = \frac{U_*^n(r)}{\alpha} = \omega_o\, U^{n-1};$$

where $\omega_o = s/\alpha$, Equation (55) may be then written

$$\frac{dU^n(r)}{dr} = -\alpha\left[U^n(r) - U_q^n(r)\right]. \tag{56}$$

Hence

(a) α is the <u>attenuation function</u> for U^n

(b) U_q^n is the <u>equilibrium function</u> for U^n

and (56) is subsumed by (42).

(iii). <u>The transport equation governing N_*</u> has the form:

$$-\cos\theta\,\frac{dN_*(z,\theta,\phi)}{dz} = -\alpha N_*(z,\theta,\phi) + N_{**}(z,\theta,\phi). \tag{57}$$

where

$$N_{**}(z,\theta,\phi) = \int_{\Xi} \sigma(\theta,\phi;\theta',\phi')\, N_*(z,\theta',\phi')\, d\Omega . \tag{58}$$

Equation (57) holds in all homogeneous, generally scattering-anistropic media (thus the reason for explicitly dropping z -notation in α and σ). If we set

$$N_{*q}(z,\theta,\phi) = \frac{N_{**}(z,\theta,\phi)}{\alpha}$$

then

$$\frac{dN_{*}(z,\theta,\phi)}{dz} = \frac{\alpha}{\cos\theta}\left[N_{*}(z,\theta,\phi) - N_{*q}(z,\theta,\phi)\right]; \quad (59)$$

therefore

(a) $\frac{-\alpha}{\cos\theta}$ is the <u>attenuation function</u> for N_{*} ,

(b) N_{*q} is the <u>equilibrium function</u> for N_{*} .

(iv). <u>The transport equation for vector irradiance</u> $\underset{\sim}{H}$ has the form[3]

$$\frac{d}{dz}\bar{H}(z,\underset{\sim}{\Omega},\Xi_0) = -\left[a(z,\underset{\sim}{\Omega},\Xi_0) + b(z,\underset{\sim}{\Omega},\Xi_0)\right]\bar{H}(z,\underset{\sim}{\Omega},\Xi_0) \quad (60)$$

$$+ b(z,\underset{\sim}{\Omega},\Xi_0')\,\bar{H}(z,\underset{\sim}{\Omega},\Xi_0')$$

Here $\bar{H}(z,\underset{\sim}{\Omega},\Xi_0) = \underset{\sim}{\Omega}\cdot\underset{\sim}{H}(z,\Xi_0)$ is the component of $\underset{\sim}{H}(z,\Xi_0)$ along the direction of the unit inward normal $\underset{\sim}{\Omega}$ to a unit area at depth z . $\underset{\sim}{H}(z,\Xi_0)$ is the vector irradiance generated by radiant flux at z arriving from the general subregion Ξ_0 of the unit sphere Ξ . If $\Xi_0 = \Xi$, then $\underset{\sim}{H}(z, \Xi_0) = \underset{\sim}{H}(z)$ the usual vector irradiance at z . The quantity $\bar{H}(z, \underset{\sim}{\Omega}, \Xi_0')$ is the associated (net) irradiance on the same unit area contributed by the complement Ξ_0' of Ξ_0 with respect to Ξ . Because of the assumed stratification, $\underset{\sim}{H}(z, \Xi_0)$ (and hence all its components) depends only on z . By setting

$$\bar{H}_q(z, \underset{\sim}{\Omega}, \Xi_0) = \frac{b(z, \underset{\sim}{\Omega}, \Xi_0')\, \bar{H}(z, \underset{\sim}{\Omega}, \Xi_0')}{a(z, \underset{\sim}{\Omega}, \Xi_0) + b(z, \underset{\sim}{\Omega}, \Xi_0)}, \tag{61}$$

we may write (60) as

$$\frac{d\bar{H}(z, \underset{\sim}{\Omega}, \Xi_0)}{dz} = -\left[a(z, \underset{\sim}{\Omega}, \Xi_0) + b(z, \underset{\sim}{\Omega}, \Xi_0)\right]\left[\bar{H}(z, \underset{\sim}{\Omega}, \Xi_0) - \bar{H}_q(z, \underset{\sim}{\Omega}, \Xi_0)\right], \tag{62}$$

so that

(a) $a(z, \underset{\sim}{\Omega}, \Xi_0) + b(z, \underset{\sim}{\Omega}, \Xi_0)$ is the <u>attenuation function</u> for $\bar{H}(z, \underset{\sim}{\Omega}, \Xi_0)$,

(b) $\bar{H}_q(z, \underset{\sim}{\Omega}, \Xi_0)$ is the <u>equilibrium function</u> for $\bar{H}(z, \underset{\sim}{\Omega}, \Xi_0)$.

The transport equation (60) is a generalization of the standard two flow equations (5) for $H(z,+)$ and $H(z,-)$. (In the latter case, for example, $\Xi_0 = \Xi_-$, the downwelling hemisphere, and $\underset{\sim}{\Omega} = -\underset{\sim}{k}$, where $-\underset{\sim}{k}$ is the unit inward normal to the plane-parallel medium.)

Each of the four preceding transport equations may be cast into a <u>canonical form</u> (see (25)) by introducing the appropriate K-function for the associated radiometric quantity (see general definition (24)). Therefore, a transport equation for each of these K-function exists, and is of the form (42). The equilibrium principle holds for N^n, N_*, $\underset{\sim}{H}$ and U^n.

Therefore, the domain of applicability of the universal transport equation is quite wide. In fact, its domain covers the totality of radiometric functions used and known to date in radiative transfer theory (the 17 distinct types of radiometric concepts and their corresponding K-functions discussed above--34 concepts in all). By means of it, the general mathematical structure of the light field can be contained in a single unifying framework, and the necessity of invoking individual discussions and principles for each of the many radiometric quantities is now obviated. Thus:

"Frustra fit per plura quod potest fieri per pauciora" ——
William of Ockham (c. 1300-1347)

(It will be futile to employ many principles when it is possible to employ fewer.)

RWP/mja
4 January 1959

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The latter text was preceded by an earlier study which gave the first systematic mathematical classification of certain special transport problems covered by the equation of transfer for radiance in the astrophysical context:

Hopf, E., Mathematical Problems of Radiative Equilibrium Cambridge Tracts in Math. and Physics, No. 31 (1934).

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Figure 1

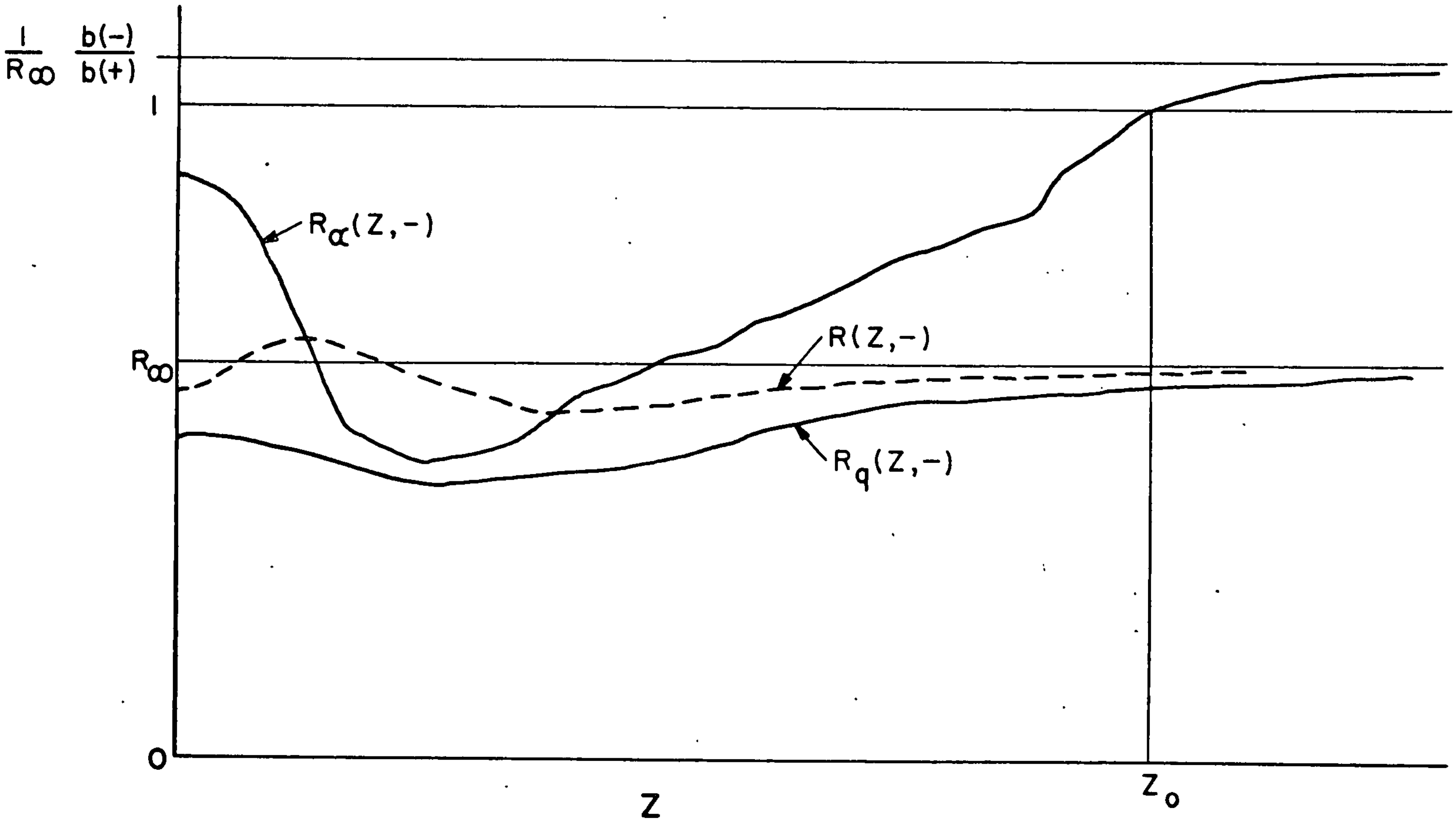

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Figure 2