GENERAL ANALYTICAL REPRESENTATIONS OF THE

OBSERVABLE REFLECTANCE FUNCTION

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The observable reflectance function, sometimes called the "up-down ratio," for natural light in the sea is nearly independent of depth within any uniform water-mass or stratum. Even small changes, however, may affect the range at which a downward looking observer can sight an object of very low contrast. This report presents a theoretical study of the variation with depth of the observable reflectance function for natural light in the sea.

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INTRODUCTION

The object studied in this paper is the observable reflectance function $R(\cdot,-)$ whose value at a depth z in an arbitrarily stratified plane-parallel optical medium is defined as:

$$R(z,-) = \frac{H(z,+)}{H(z,-)} ,$$

where, as usual, the quantities $H(z,\pm)$ are the observed upwelling $(+)$ and downwelling $(-)$ irradiances at depth z in the medium. Several representations of the function $R(\cdot,-)$ are established which will, (a) Explicitly exhibit in terms of differential equations and definite integrals the dependence of $R(\cdot,-)$ on the <u>inherent</u> optical properties of the medium, as far as this is possible; (b) Illustrate the dynamic equilibrium-seeking tendency of $R(\cdot,-)$ which appears to hold in all plane-parallel media; and finally, (c) Suggest some novel methods of solving the problem of predicting the depth-structure of $R(\cdot,-)$ in general media. To place the present contribution in its proper perspective for the general reader, we append the following observations.

The reflectance function $R(\cdot,-)$ is one of a set of seven apparent optical properties introduced in an earlier study.[1] This set consists of the functions $R(\cdot,\pm)$, $K(\cdot,\pm)$, $D(\cdot,\pm)$, and k, and is defined in terms of the four directly observable radiometric functions: $H(\cdot,\pm)$, $h(\cdot,\pm)$, where $h(\cdot,\pm)$ are the upwelling (+) and downwelling (-) scalar irradiance functions. The theory of the measurement of these radiometric quantities and a discussion of the salient physical characteristics of this extremely useful set of apparent optical properties may be found in references 1 and 2. Reference 2 contains a careful analysis of the optical properties of an optical medium into the classes of inherent and apparent optical properties, and the necessary distinctions that must be made between them, in both experimental and theoretical procedures.

The set of inherent optical properties of any scattering-absorbing optical medium consists of the functions α, σ, the volume attenuation and volume scattering function, respectively. These functions are by definition independent of the ambient light field. The apparent optical properties, however, depend jointly on the inherent optical properties and the ambient light field. Specifically, the apparent optical properties depend on α, σ and the radiance distributions $N(z,\cdot)$ in the medium.

Despite this dependence of the R, K, D, and k functions on ephemeral lighting conditions, they exhibit a behavior in both space and time of such a strikingly regular and generally predictable kind, that each is dignified with the appellation: "optical property" . However, we point out the fact that this is a matter of first appearances only, and that, under incisive analytical and experimental scrutiny, their regularities are seen to be at the mercy of variable boundary lighting conditions and the internal distribution of the values of α and σ in an optical medium. To emphasize this fact, the qualification "apparent" is put before "optical property".

Detailed examples of the regular behavior of the apparent optical properties are given in references 1, 2, 3, and 4. Some useful connections between K , and D are given in reference 5. The present study adds to this store of knowledge of the apparent optical properties by developing in detail the exact differential and integral representations of the reflectance function $R(\cdot, -)$ and drawing some theoretical and practical conclusions from them.

THE DIFFERENTIAL EQUATIONS FOR $R(\cdot,-)$

Unfactored Form

The physical setting for the derivations that follow is a plane-parallel source-free optical medium whose inherent optical properties spatially depend only on depth, and this depth dependence is assumed arbitrary. Further, the medium may have either finite or infinite optical depth with arbitrary boundary reflectance properties and arbitrary incident lighting conditions on its upper and lower boundaries.

Assume that an irradiance probe sweeps through a range of depths between z' and z'', where $0 \leq z' < z'' \leq \infty$, and that at each of the depths in this range readings $H(z,\pm)$ are taken, $z' \leq z \leq z''$. Then the reflectance $R(z,-)$ at each depth is determined by:

$$R(z,-) = \frac{H(z,+)}{H(z,-)} ,$$

and its depth rate of change $dR(z,-)/dz$ for downward motion or for upward motion is easily found. Thus, for a downward sweep, we have

$$\frac{dR(z,-)}{dz} = \frac{H(z,-)\dfrac{dH(z,+)}{dz} - H(z,+)\dfrac{dH(z,-)}{dz}}{H^2(z,-)} .$$

By definition[1] of $K(z,\pm)$ we have,

$$K(z,\pm) = \frac{-1}{H(z,\pm)} \frac{dH(z,\pm)}{dz} .$$

Hence the derivative of $R(\cdot,-)$ evaluated at depth z is:

$$\frac{dR(z,-)}{dz} = R(z,-)\left[K(z,-) - K(z,+)\right].$$

We may now introduce the exact representations of $K(z,\pm)$ established earlier:

$$\mp K(z,\pm) = \left[a(z,\pm) + b(z,\pm)\right] - b(z,\mp)\,R(z,\mp),$$

where $a(z,\pm) = a(z)\,D(z,\pm)$, and $b(z,\pm)$ are the values of the absorption and backward scattering functions at depth z for the upwelling (+) and downwelling (-) streams of radiant flux.[6] Substituting these representations of $K(z,\pm)$ in the above derivative, the result is:

$$\boxed{\begin{aligned} -\frac{dR(z,-)}{dz} &= b(z,+)\,R^2(z,-) - c(z)\,R(z,-) + b(z,-); \\ \text{where } c(z) &= a(z,-) + a(z,+) + b(z,-) + b(z,+). \end{aligned}} \quad (1)$$

This is the desired differential equation for $R(\cdot,-)$ in unfactored form, and is the basic differential equation governing the observable reflectance function. It is an exact equation within the presently chosen general physical setting, and forms the basis for all our subsequent deductions of the properties of $R(\cdot,-)$. The mathematical structure of (1) is that of a general Riccati equation[7], which generally has non elementary solutions.

Factored Form

The basic differential (1) may be factored by observing that its right-hand side is quadratic in $R(z,-)$ for each depth z. Thus for a given z, the roots of the quadratic equation:

$$b(z,+)t^2 - c(z)t + b(z,-) = 0$$

are

$$R_\alpha(z,-) = \frac{c(z) + \left[c^2(z) - 4b(z,-)b(z,+)\right]^{\frac{1}{2}}}{2b(z,+)},$$

$$R_q(z,-) = \frac{c(z) - \left[c^2(z) - 4b(z,-)b(z,+)\right]^{\frac{1}{2}}}{2b(z,+)}.$$

Hence (1) may be written,

$$\boxed{-\frac{dR(z,-)}{dz} = b(z,+)\left[R(z,-) - R_\alpha(z,-)\right]\left[R(z,-) - R_q(z,-)\right].} \quad (2)$$

Equation (2) is the <u>factored form</u> of the differential equation for $R(\cdot,-)$. The function $R_q(\cdot,-)$ is the <u>equilibrium function</u> for $R(\cdot,-)$, and $R_\alpha(\cdot,-)$ is the <u>attenuation function</u> for $R(\cdot,-)$.

Second-Order Form

We now introduce a new function Q defined on the depth interval of interest, and having the property:

$$\frac{Q'(z)}{Q(z)} = b(z,+)R(z,-),$$

where the prime denotes differentiation with respect to z .

It follows that Q satisfies a linear homogeneous second order differential equation whose coefficients are functions of $c(z)$ and $b(z,\pm)$. To see this, we differentiate each side of the above relation. The result is:

$$\frac{Q(z)\,Q''(z) - [Q'(z)]^2}{Q^2(z)} = b(z,+)R'(z,-) + b'(z,+)R(z,-) .$$

Then using (1) for the expression equivalent to $R'(z,-)$:

$$\frac{Q''(z)}{Q(z)} - \left[\frac{Q'(z)}{Q(z)}\right]^2 =$$

$$= b(z,+)\left[-b(z,+)R^2(z,-) + c(z)R(z,-) - b(z,-)\right] +$$

$$+ b'(z,+)R(z,-) ;$$

so that:

$$\frac{Q''(z)}{Q(z)} = b^2(z,+)R^2(z,-)$$

$$+ \left[-b^2(z,+)R^2(z,-) + c(z)R(z,-)b(z,+) - b(z,-)b(z,+)\right]$$

$$+ b'(z,+)R(z,-) .$$

Hence,

$$\frac{Q''(z)}{Q(z)} = \left[c(z)b(z,+) + b'(z,+)\right]R(z,-) - b(z,-)b(z,+) .$$

Using the defining relation for Q onee again, this becomes

$$\boxed{Q''(z) - \left[c(z) + \frac{b'(z,+)}{b(z,+)}\right]Q'(z) + b(z,-)b(z,+)Q(z) = 0} \quad (3)$$

Equation (3) is the desired homogeneous second order differential equation. Upon obtaining its solution the defining relation for Q is used to obtain the expression for $R(\cdot,-)$.

THE EQUILIBRIUM-SEEKING THEOREM FOR $R(\cdot, -)$

Preliminary Observations

Equilibrium theorems abound in the theory of radiative transfer. Perhaps the earliest explicit and unmistakable instance of an equilibrium theorem was given by means of Koschmieder's equation which describes the apparent radiance N_r of an object of inherent radiance N_0 as seen along an horizontal path of length r along which both the inherent optical properties of the medium and the ambient lighting conditions are constant:

$$N_r = N_0 e^{-\alpha r} + \frac{N_*}{\alpha}\left[1 - e^{-\alpha r}\right] .$$

Here the quantity N_* (the path function) is the (constant) radiance per unit length of path (at every point of the path) generated by scattering of the ambient light. The quantity α is the fixed value of the volume attenuation function for the medium along the path of sight. Assuming that this type of path may be extended indefinitely in the direction away from the object, the equation states that, for any medium with $\alpha > 0$,

$$\lim_{r \to \infty} N_r = \frac{N_*}{\alpha} .$$

The quantity N_*/α , usually denoted by N_q , is the equilibrium radiance of the path of sight. It is dependent on both the inherent optical properties of the medium (α, σ) and the lighting conditions along the path (N_*) . The term equilibrium radiance is understood in the following sense: For any initial choice of N_0 , N_r tends toward and eventually attains the value N_q . Thus if N_0 exceeds N_q , then N_r decreases from N_0 to N_q as r goes from 0 to ∞ . On the other hand, if N_0 is less than N_q , then N_r increases from N_0 to N_q as r goes from 0 to ∞ .

This phenomenon of the equilibrium-seeking tendency of the apparent radiance actually holds for an arbitrary path of sight in an arbitrary optical medium along which there are no sources and $\alpha > 0$. This may be seen by taking the general transfer equation for radiance:

$$\frac{dN}{dr} = -\alpha N + N_*$$

and writing it in the form:

$$\frac{dN}{dr} = -\alpha \left[N - N_q \right] ,$$

where we have set $N_q \equiv N_*/\alpha$ in general.

It must be emphasized that this equation is completely general; hence α may change from point to point along a path, N_* (and hence N_q) may depend on direction about a fixed point, and the angular dependence of N_* at two different points may be quite distinct. Now select any path of sight in the medium, and choose an initial point of the path. At this point suppose the value of N is N_0. If then $N_0 > N_q$, the above equation immediately shows that $dN/dr < 0$, so that N tends toward the value of N_q at this point as r increases. On the other hand, if $N_0 < N_q$, then $dN/dr > 0$, and N tends toward N_q once again as r increases. Now it is quite possible that N_q may change along the path. But the important fact to observe is that at every point of the path, regardless of the relative sizes of N and N_q, the tendency of N <u>at that point</u> is to change its value so as to <u>decrease</u> the absolute value of the difference $N - N_q$ at that point.

The phenomenon of the equilibrium-seeking tendency of radiance in an arbitrary medium gives rise to a host of equilibrium theorems for various other radiometric quantities and even for the apparent optical properties.

These equilibrium theorems were explored in detail in an earlier work[8] where it was shown that no less than 34 radiometric and related concepts were subject to a single equilibrium principle.

The following discussion of the equilibrium theorem for $R(\cdot,-)$ is patterned after that exhibited for N above. Furthermore, this special discussion will pave the way for some interesting observations of the properties of the reflectance function as seen in the light of the principles of invariance. These observations will be given in a report entitled, "The Principles of Invariance For Directly Observable Irradiances in Plane Parallel Media".

The Equilibrium-Seeking Theorem

To establish the present equilibrium theorem for $R(\cdot,-)$, consider an arbitrarily stratified source-free plane-parallel medium over the depth interval $0 \leq z' < z'' \leq \infty$ in which $\sigma \neq 0$. The medium thus may be optically shallow or deep; its boundary reflectances are arbitrary, as are the boundary lighting conditions. The present setting, therefore, is of maximum generality. Imagine a reflectance meter at depth z in the medium. The reading $R(z,-)$ gives the reflectance of the material between the level z and the lower boundary, inclusively. This number is a complex combination of the effects of the "inherent" reflectance of the medium in that depth interval and the angular structure of the downwelling incident flux at level z. The angular structure of the downwelling flux at level z in turn depends on the inherent optical properties of the medium above level z.

However, despite this complex situation, there exists at every level z along with $R(z,-)$, the values $R_\alpha(z,-)$ and $R_q(z,-)$ of the attentuation function and equilibrium function associated with $R(\cdot,-)$. These numbers are defined in (2) above.

It is now essential to our immediate purpose to establish the fact that $R_\alpha(z,-) \geq 1$ for all z. The proof of this fact, incidentally, supplies a sharper estimate of the magnitude $R_\alpha(z,-)$ originally obtained in reference 8.

We deduce the fact that $R_\alpha(z,-) \geq 1$ for all z on strictly analytical grounds starting from the defining equation:

$$R_\alpha(z,-) = \frac{c(z) + [c^2(z) - 4b(z,-)b(z,+)]^{\frac{1}{2}}}{2b(z,+)} .$$

Observe first that the value $R_\alpha(z,-)$, when considered as determined solely by the magnitude of $c(z)$, monotonically increases with $c(z)$. (That is, holding $b(z,\pm)$ fixed and letting $c(z)$ increase.) Thus in particular if for some special value c_o of $c(z)$ we can show that $R_\alpha(z,-) \geq 1$, then for all $c(z) \geq c_o$, we will certainly have $R_\alpha(z,-) \geq 1$. Now $c(z) = a(z,-) + a(z,+) + b(z,-) + b(z,+)$. Hence (since all a's and b's are nonnegative):

$$c(z) \geq b(z,+) + b(z,-) \equiv c_o ,$$

in fact the strict inequality holds in all real media.

But,
$$\frac{c_o + [c_o^2 - 4b(z,-)\, b(z,+)]^{\frac{1}{2}}}{2\, b(z,+)} = 1 .$$

It follows that $R_\alpha(z,-) \geq 1$ for $0 \leq z \leq z''$, in every plane-parallel optical medium.

We recall at this point the fact that [1],

$$R(z,-) \leq 1$$

in all real optical media (in fact the strict inequality holds in such media). From these inequalities we deduce the fact that the difference $R(z,-) - R_\alpha(z,-)$ in all real media is negative for all z .

Continuing with the development of the theorem, suppose that we now measure $R(z,-)$, $z > 0$, and then move the reflectance meter a small distance in the upward direction (maintaining, of course, its horizontal collection-orientation throughout the move.) What we are in effect doing by such a move is increasing by a small amount the material of the medium below the level occupied by the meter. This upward motion is the natural direction of motion one should go in order to discern the equilibrium-seeking behavior of $R(\cdot,-)$, just as the natural direction of motion of the observer in the equilibrium theorem for N was such that it increased the material between the observer and the initial point of the path.

(In this connection see the discussion of the contravariation of $K(z,+)$ and $D(z,+)$ presented in reference 5.) Therefore to analytically describe the result of this motion the derivative term of equation (2) is now read as: $dR(z,-)/d(-z)$.

Suppose that $R(z,-) < R_q(z,-)$ at the depth under consideration. (See Figures 1 and 2.) Hence, $R(z,-) - R_q(z,-)$ is negative. By the preceding observations, it is known that $R(z,-) - R_\alpha(z,-)$ is invariably negative in all real media. Thus the derivative $dR(z,-)/d(-z)$ is positive, indicating that $R(z,-)$ tends toward the value of the equilibrium reflectance $R_q(z,-)$ at this depth, as z decreases. On the other hand, if $R(z,-) > R_q(z,-)$, then $R(z,-) - R_q(z,-)$ is positive, and since $R(z,-) - R_\alpha(z,-)$ is invariably negative, it follows that in this case $dR(z,-)/d(-z) < 0$, so that once again $R(z,-)$ tends toward $R_q(z,-)$. This completes the proof of the theorem.

We summarize the Equilibrium-Seeking theorem symbolically as follows:

$$\boxed{\operatorname{sign}\left\{ \frac{dR(z,-)}{d(-z)} \right\} = \operatorname{sign}\left\{ R_q(z,-) - R(z,-) \right\}} \quad (4)$$

We may now make several observations on this equilibrium-seeking property of $R(\cdot,-)$.

Observations

1. At first sight the above conclusion, namely that $R(z,-)$ seeks $R_q(z,-)$ as z decreases, appears to contradict the conclusion reached in reference 8; namely that $\lim_{z\to\infty} R(z,-) = \lim_{z\to\infty} R_q(z,-) = R_\infty$. The two conclusions, however, are not contradictory; they are simply answers to two entirely different questions. In reference 8 we were concerned with the application of the ideas of a special type of theorem, namely the asymptotic radiance theorem, to $R(\cdot,-)$; the medium was optically infinitely deep and of fixed structure and size. The trends of the values $R(z,-)$ $R_\alpha(z,-)$ and $R_q(z,-)$ as $z \to \infty$ were then examined, assuming the asymptotic radiance theorem was applicable. The question was: under these conditions, what happens when $z \to \infty$? The answers were that $\lim_{z\to\infty} R(z,-) = \lim_{z\to\infty} R_q(z,-) = R_\infty$, when the latter limit exists, and that $\lim_{z\to\infty} R_\alpha(z,-) = [1/R_\infty][b(-)/b(+)] > 1$. In the discussion of the present paper we ask: what is the sign of $dR(z,-)/dz$ at every depth z as z decreases toward zero?

The answer is that: sign $\left\{dR(z,-)/d(-z)\right\}$ = sign $\left\{R_q(z,-) - R(z,-)\right\}$. The latter conclusion holds universally. The asymptotic theorem holds only for special media. It is still possible, however, for both theorems to hold in a given medium.

The only disparity between the present results and those of reference 8 arises in certain parts of Figures 1 and 2 of that reference. Those Figures were based on a less sharp estimate of the size of $R_\alpha(z,-)$ for <u>shallow-depth</u> regions than that given in the present report. We have shown in the present report that $R_\alpha(z,-) \geq 1$ <u>for all z in all media.</u> Hence those parts of Figures 1 and 2 of reference 8 which show $R_\alpha(z,-) < 1$ should now be modified in the light of the present sharper estimate. New figures which make use of the present sharper estimate of $R_\alpha(z,-)$ are given in Figures 1 and 2 below which, in addition, illustrate the equilibrium-seeking tendency of $R(\cdot,-)$.

2. By returning to the basic premises of the present discussion, we observe that the condition $\sigma \not\equiv 0$ was imposed. This condition has both physical and mathematical relevance to the conclusion (4). Mathematically, $R_\alpha(z,-)$ and $R_q(z,-)$ are <u>prima facie</u> undefined for the case $\sigma \equiv 0$. Physically, the reflectance of a purely absorbing medium or a vacuum is trivially zero.

To lift the veil of mathematical indeterminacy of $R(z,-)$ in the case of $\sigma \equiv 0$, we return to equation (1). Under the present conditions (1) reduces to:

$$\frac{dR(z,-)}{d(-z)} = -c(z)R(z,-),$$

where in this case $c(z) = a(z,-) + a(z,+) \geq 0$. Hence if the initial value of $R(\cdot,-)$ at z'' is $R(z'',-) \geq 0$, then clearly

$$R(z,-) = R(z'',-)\exp\left\{-\int_z^{z''} c(z')dz'\right\}.$$

By letting $b(z,\pm) \to 0$ in $R_q(z,-)$ we see that $R_q(z,-) = 0$.

The preceding formula for $R(z,-)$ shows that, for media with $a(z) > 0$ for all z, and $\sigma \equiv 0$ on $[z',z'']$, $\lim_{[z''-z]\to\infty} R(z,-) = 0$. Hence the equilibrium-seeking tendency of $R(\cdot,-)$ is borne out for this case also.

3. What about the opposite case to that in 2 above? Namely that $\sigma(z;\xi';\xi) \neq 0$ for all z, ξ', ξ, and $a \equiv 0$? It follows that $R_\alpha(z,-) = 1$ and that $R_q(z,-) = b(z,-)/b(z,+)$ for each z. Now suppose that $R(z,-) < 1$ at some depth z in the interval $[z',z'']$. This implies that $H(z,+) < H(z,-)$ and hence that the radiance distribution $N(z,\cdot)$ over Ξ_+ (the upward directions) is on the average less than the radiance distribution over

Ξ_- (the downward directions). Now from reference 8 (page 37) we see that in general,

$$\frac{1}{R(z,-)}\,\frac{b(z,-)}{b(z,+)} = \frac{\int_{\Xi_-} \sigma_-(z;\xi')\,N(z,\xi')\,d\Omega(\xi')}{\int_{\Xi_+} \sigma_-(z;\xi')\,N(z,\xi')\,d\Omega(\xi')}$$

But if the preceding supposition, namely that $R(z,-) < 1$ holds, then it follows that:

$$\frac{1}{R(z,-)}\,\frac{b(z,-)}{b(z,+)} > 1,$$

or in other words:

$$R(z,-) < \frac{b(z,-)}{b(z,+)} = R_q(z,-)\,.$$

By the Equilibrium-Seeking theorem, it follows that $dR(z,-)/d(-z)$ is positive. Hence as depth is decreased in the purely scattering medium, the values $R(z,-)$ increase monotonically. But since the preceding supposition namely $R(z,-) < 1$, was for an arbitrary $R(z,-)$-magnitude less than unity, it follows that for every z,

$$\lim_{(z''-z)\to\infty} R(z,-) = 1.$$

4. Thus we have in paragraphs (2) and (3) above proved (or outlined proofs) by rigorous mathematical reasoning some outstanding folklore about the elementary properties of $R(\cdot,-)$ in plane-parallel media. These proofs were arrived at by reasoning strictly from the various exact differential equations governing $R(\cdot,-)$. In this way we hope to illustrate the power inherent in that approach to radiative transfer problems which discards particular mathematical models and which concentrates on the study of directly observable quantities of the light filed. It is to be emphasized that the reasoning in this approach proceeds directly from the exact forms of the equations of transfer.

THE INTEGRAL REPRESENTATIONS OF $R(z,-)$

Starting with the factored form of the transport equation for $R(\cdot,-)$ we make use of the separation of variables that exists within it, and we write:

$$\frac{dR(z,-)}{\left[R(z,-)-R_\alpha(z,-)\right]\left[R(z,-)-R_q(z,-)\right]} = b(z,+)\, d(-z) \,.$$

If we formally integrate each side we obtain the desired integral representation by letting z range from some depth z_1, to $z_2 > z_1$. Let the corresponding values of $R(\cdot,-)$ at these depths be distinct: $R(z_1,-) \neq R(z_2,-)$, otherwise the problem of $R(\cdot,-)$ is trivial over this depth range. The range of integration can be subdivided into intervals so that over each there is a one to one correspondence between the values of $R(\cdot,-)$ and the points of the interval. With these observations we may then write:

$$\boxed{\int_{R(z_1,-)}^{R(z_2,-)} \frac{dt}{[t - R_\alpha(t,-)][t - R_q(t,-)]} = \int_{z_1}^{z_2} b(t,+)\,dt\,,} \tag{5}$$

which is the desired <u>integral representation of $R(\cdot,-)$</u>. The symbol t in the integrals acts as a dummy variable of integration.

An alternate integral representation of $R(\cdot,-)$ may be obtained from (1) in which the variables are also conveniently separated. The same general arguments used to establish (5) may now be directed to the equation (1). The result is:

$$\boxed{\int_{R(z_1,-)}^{R(z_2,-)} \frac{dt}{b(t,+)t^2 - c(t)t^2 + b(t,-)} = (z_2 - z_1)\,.} \tag{6}$$

APPLICATIONS

In this section we discuss two methods of evaluating $R(\cdot,-)$ by means of its differential and integral equation representations presented above. We illustrate the use of (5) for a very simple case, which is a useful approximation to reality, namely the case in which $R_\alpha(\cdot,-)$ and $R_q(\cdot,-)$ are constant functions. The second method is based directly on (1) or (3) and promises to yield a means of determining $R(\cdot,-)$ under realistic conditions.

Special Closed Form Solution

Suppose the functions $a(\cdot,\pm)$, $b(\cdot,\pm)$ and hence the functions $R_\alpha(\cdot,-)$ and $R_q(\cdot,-)$ are constant functions over some arbitrary depth interval $[z_1,z_2]$, $z_2 > z_1$, in a plane-parallel medium. Then (5) is immediately integrable, and definite integrals take the forms:

$$\frac{1}{R_q - R_\alpha}\left[\ln \frac{t - R_q}{t - R_\alpha}\right]_{R(z_1,-)}^{R(z_2,-)} = b(+)\,(z_2 - z_1)\ .$$

From the definitions of R_q and R_α, we see that $R_\alpha - R_q > 0$ and in fact

$$R_\alpha - R_q = \frac{[c^2 - 4b(-)b(+)]^{\frac{1}{2}}}{b(+)},$$

where

$$c = a(-) + a(+) + b(-) + b(+),$$

and where $a(\pm)$, $b(\pm)$ are the assumed constant values of the functions $a(\cdot,\pm)$, $b(\cdot,\pm)$ over the depth range $[z_1, z_2]$. In the present method all four of these quantities may be distinct.

Applying the limits to the left integral, we have:

$$\ln\left[\frac{R(z_2,-) - R_q}{R(z_2,-) - R_\alpha}\right] = \ln\left[\frac{R(z_1,-) - R_q}{R(z_1,-) - R_\alpha}\right] - [c^2 - 4b(-)b(+)]^{\frac{1}{2}}(z_2 - z_1).$$

Hence,

$$\left[\frac{R(z_2,-) - R_q}{R(z_2,-) - R_\alpha}\right] = \left[\frac{R(z_1,-) - R_q}{R(z_1,-) - R_\alpha}\right] \exp\left\{-[c^2 - 4b(-)b(+)]^{\frac{1}{2}}(z_2 - z_1)\right\}.$$

If we set $$C(z_1,-) = \frac{R(z_1,-) - R_q}{R(z_1,-) - R_\alpha},$$

then

$$\boxed{R(z_2,-) = \frac{R_q - R_\alpha C(z_1,-) \exp\left\{-[c^2 - 4b(-)b(+)]^{\frac{1}{2}}(z_2 - z_1)\right\}}{1 - C(z_1,-) \exp\left\{-[c^2 - 4b(-)b(+)]^{\frac{1}{2}}(z_2 - z_1)\right\}}} \quad (7)$$

Hence if the four constants $a(\pm)$ and $b(\pm)$ are known or estimable over an interval $[z_1, z_2]$ and $R(z_1,-)$ is known, then $R(z_2,-)$ is determinable. Observe that if we let $(z_2 - z_1) \to \infty$ thereby simulating an infinitely deep layer, then $R(z_2,-) \to R_q$. Hence R_q in this instance is the R_∞- quantity of the classical theory. The points of contact with the classical theory may be increased by observing that if we set $a(+) = a(-) = a^*$, and $b(+) = b(-) = b^*$, then

$$[c^2 - 4b(-)b(+)]^{\frac{1}{2}} = 2[a^*(a^* + 2b^*)]^{\frac{1}{2}}$$
$$= 2k,$$

where k is the diffuse absorption coefficient of the classical Schuster two-flow theory (see, e.g., equation (2) of reference 1).

By partitioning an inhomogeneous medium into essentially homogeneous layers, successive applications of (7) will yield a useful practical formula for the reflectance of the entire medium. The solution (7) automatically includes the effects of interreflections. Thus suppose the medium, which extends over an interval $[z_0, z_n]$, is partitioned into n homogeneous layers defined by the depths: $[z_0, z_1]$, $[z_1, z_2]$, . . . , $[z_{n-1}, z_n]$. If $R(z_n,-)$ is known (this may be the reflectance of the bottom boundary of the layer $[z_{n-1}, z_n]$) , then by (7) we find $R(z_{n-1},-)$. Another application of (7) with $R(z_{n-1},-)$ as the initial reflectance then yields $R(z_{n-2},-)$, and so on to $R(z_0,-)$ which then is the reflectance associated with the medium over the depth interval $[z_0, z_n]$.

Differential Analyzer Solutions

Equations (1) and (3) as they stand, are suitable for determinations of $R(\cdot,-)$ by means of differential analyzer techniques especially when the functions $a(\cdot,\pm)$ and $b(\cdot,\pm)$ vary extensively over the medium.

Series Solutions

By means of series solution techniques, equations (1) and (3) may also be used to solve the difficult problem of determining $R(\cdot,-)$ over some interval $[z_1, z_2]$ when $a(\cdot,\pm)$ and $b(\cdot,\pm)$ are non-constant and known over this interval. By expanding the coefficients of $R^2(z,-)$, $R(z,-)$, and the $b(z,-)$ term in (1) in terms of infinite series in z , recursion formulas may be obtained for the coefficients in the infinite series expansion of $R(z,-)$ over $[z_1, z_2]$.

SUMMARY

This report develops several differential and integral formulae governing the observable reflectance function $R(\cdot,-)$. Methods of using the formulae are outlined. Thus $R(\cdot,-)$ may be obtained directly without first solving for the irradiance functions $H(\cdot,\pm)$ as has been necessary previously. The methods discussed are general enough to allow the determination of $R(\cdot,-)$ if the absorption and backward scattering functions $a(\cdot,\pm)$, $b(\cdot,\pm)$ respectively for each stream in an arbitrarily inhomogeneous stratified medium are known. A general Equilibrium-Seeking theorem for $R(\cdot,-)$ is also demonstrated, the substance of which is the fact that the derivative of $R(\cdot,-)$ at each depth z invariably has an algebraic sign so as to decrease the absolute magnitude of the difference $R(z,-) - R_q(z,-)$ between $R(z,-)$ and the value $R_q(z,-)$ of the equilibrium reflectance function.

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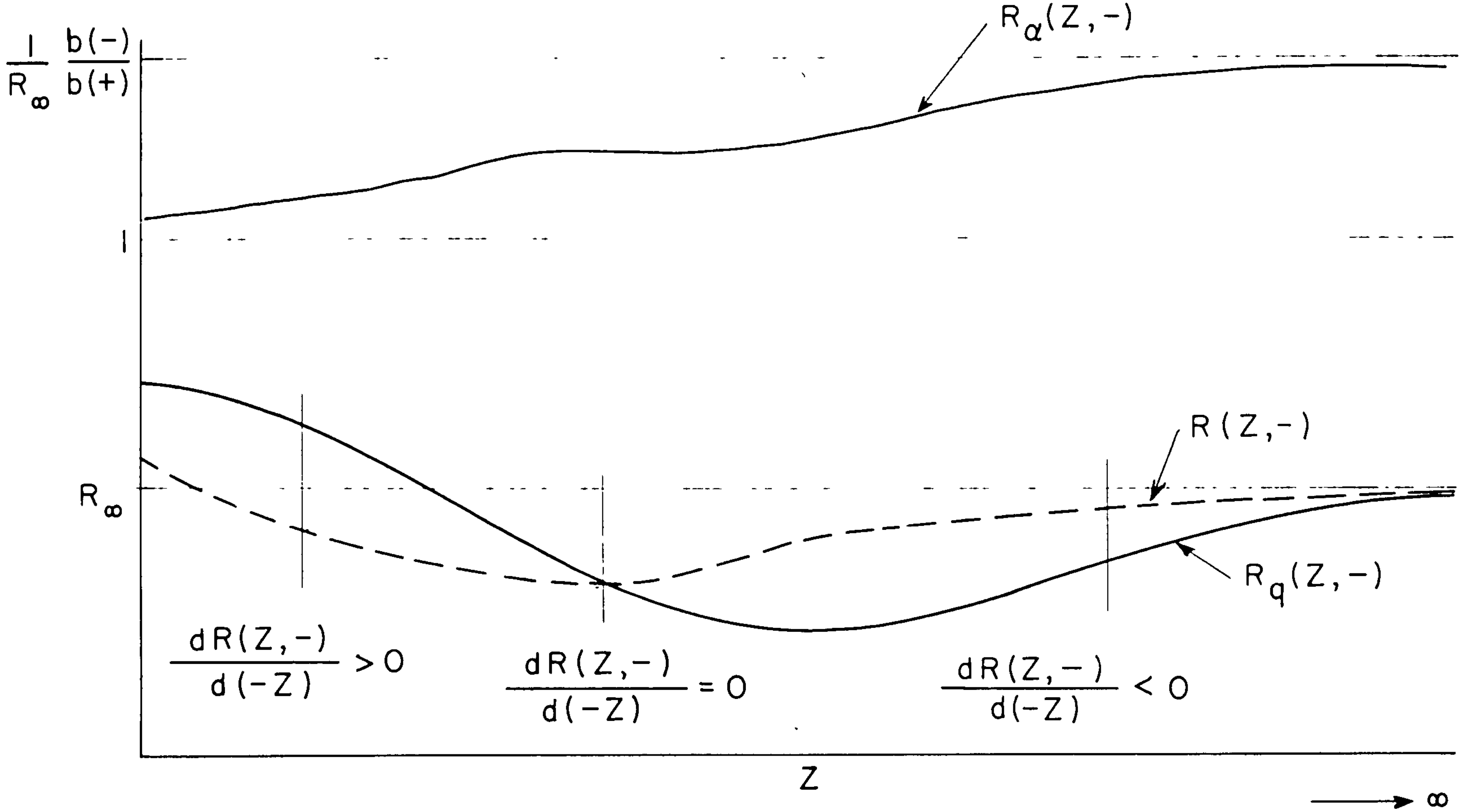

Rudolph W. Preisendorfer
Figure 1

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Figure 2