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DISTRIBUTION FUNCTIONS IN PLANE PARALLEL MEDIA

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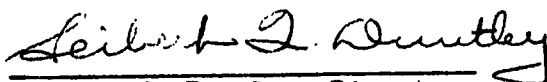
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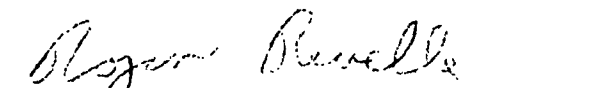
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The Covariation of the Diffuse Attenuation and Distribution Functions
in Plane Parallel Media

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INTRODUCTION

The purpose of this paper is to establish the theorem that at arbitrary fixed depths z the diffuse attenuation function value $K(z, -)$ and the distribution function value $D(z, -)$ vary directly (but not necessarily linearly) one with the other, in all steady state real plane-parallel media which are predominantly forward scattering. In this way we establish a useful criterion for the behavior of $K(z, -)$ in terms of the intuitively simpler concept $D(z, -)$. The theorem is expected to find its greatest use in natural hydrosols. By way of background to these results we now discuss the physical significance of $K(z, -)$ and $D(z, -)$.

It is a well known fact in the science of hydrologic optics that the amount of light in a natural hydrosol such as an ocean or deep lake decreases essentially in an exponential manner with depth from the surface of the hydrosol. This fact may be expressed symbolically as:

$$H_z = H_0 e^{-Kz} \quad (1)$$

where H_z is the amount of radiant flux of a given wavelength falling on a unit horizontal area (i.e., the irradiance) at depth z , and K is a constant determined by measuring the slope of a semi-log plot of measured values of H_z versus depth z . The quantity K has dimensions: per unit length, and is usually called the diffuse attenuation coefficient for the medium under study. Like H_z the quantity K depends implicitly on a specific wavelength λ of the radiant energy penetrating the hydrosol. For many engineering questions, questions of underwater photography, television and visibility, and for many purposes of marine biologists, the story of H_z and K may appropriately end with equation (1). However, for the more demanding purposes of geophysicists charged with the tasks of determining the fundamental optical constants of natural hydrosols, and for those who must make sense out of the experimental data leading to the numerical determination of the fundamental constants, the story of H_z and K has only begun to be told with equation (1).

Confronting these latter investigators in their quest for precisely measured radiometric quantities which must be interrelated by consistent rules of calculation, is a wealth of intricate and non-linear detail in the depth behavior of H_z . The simple exponential behavior of H_z as summarized in equation (1) must

now as a matter of experimental expediency be discarded and in its stead be made to appear a more detailed formula exhibiting the same general outlines of (1), but containing now all the potential variations which may be uncovered in a careful documentation of the light field in real hydrosols. This more detailed formula takes the form:

$$H(z, -) = H(0, -) \exp\left\{-\int_0^z K(z', -) dz'\right\}. \quad (2)$$

Equation (2) is the physicists' generalization of the mathematician's simple model of the light field expressed in equation (1). Let us examine (2) in detail and thereby uncover its similarities and dissimilarities with (1). First, $H(z, -)$ represents the measured irradiance at depth $z \geq 0$ in the medium produced by downwelling radiant flux on a unit horizontal area. Hence $H(0, -)$ is the downwelling irradiance of such flux on such a surface at depth $z = 0$ measured just below the air-water film. Second, suppose we plot the general equation (2) on semi-log paper with depth z as abscissa and $H(z, -)$ as ordinate. Equation (2) gives the value of $H(z, -)$ at a general depth z ; but equation (2) may then also be used to give the value of $H(z + \Delta z, -)$, i.e., the downwelling irradiance at a depth $z + \Delta z$, where Δz is any finite positive increment in depth. The appropriate formula for is:

$$H(z + \Delta z, -) = H(0, -) \exp\left\{-\int_0^{z + \Delta z} K(z', -) dz'\right\}. \quad (2)$$

Now, as in elementary calculus we may approximate the slope of the semilog plot of $H(z, -)$ at depth z by letting z grow by an increment Δz , finding the new value $H(z+\Delta z, -)$ and then performing the operation:

$$\frac{\ln H(z+\Delta z, -) - \ln H(z, -)}{\Delta z} \cong \left(\begin{array}{c} \text{slope of } \ln H(z, -) \\ \text{at depth } z \end{array} \right) \quad (3)$$

The smaller the magnitude of Δz , the more accurate is the estimate of the slope of the curve by this operation. Let us now perform the operation on the values of $H(z, -)$ and $H(z+\Delta z, -)$ as given by (1) and (2). From (1):

$$\ln H(z, -) = \ln H(0, -) - \int_0^z K(z', -) dz' \quad (4)$$

and from (2):

$$\ln H(z+\Delta z, -) = \ln H(0, -) - \int_0^{z+\Delta z} K(z', -) dz'. \quad (5)$$

Inserting these values in (3) we have:

$$\left(\begin{array}{c} \text{slope of } \ln H(z, -) \\ \text{at depth } z \end{array} \right) \cong \frac{\int_0^z K(z', -) dz' - \int_0^{z+\Delta z} K(z', -) dz'}{\Delta z}. \quad (6)$$

We now recall two elementary facts from integral calculus, the first is:

$$\int_0^{a+\Delta} f dz = \int_0^a f dz + \int_a^{a+\Delta} f dz, \quad (7)$$

which shows how the range $(0, a + \Delta)$ of integration may be broken into two parts: $(0, a)$ and $(a, a + \Delta)$; and the second fact that:

$$\int_a^{a+\Delta} f dz = \Delta f(a), \quad (8)$$

whenever Δ is small and f is continuous at a .

Applying these two facts to (6) where $K(\cdot, -)$ now takes the place of f in (7) and (8), we have, on application of (7) to (6):

$$\left(\begin{array}{l} \text{Slope of } \ln H(z, -) \\ \text{at depth } z \end{array} \right) \cong \frac{\int_z^{z+\Delta z} K(z', -) dz'}{\Delta z} \quad (9)$$

Then, applying fact (8) to (9) and letting $\Delta z \rightarrow 0$ we have:

$$-\left(\begin{array}{l} \text{Slope of } \ln H(z, -) \\ \text{at depth } z \end{array} \right) = K(z, -). \quad (10)$$

Equation (10) tells us that $K(z, -)$ is the negative of the slope of the semilog plot of $H(z, -)$ vs depth z .

Another way of obtaining (10) is to directly differentiate (2), the result being

$$\frac{dH(z, -)}{dz} = -K(z, -) H(z, -),$$

and then solve for $K(z, -)$:

$$\begin{aligned} K(z, -) &= \frac{-1}{H(z, -)} \frac{dH(z, -)}{dz} \\ &= - \frac{d \ln H(z, -)}{dz}. \end{aligned} \quad (11)$$

This latter method is more elegant than the preceding, and brings out more succinctly the geometric meaning of $K(z, -)$.

We may perform the same operation on H_z as given in equation (1). Hence, either by the Δz -method or the simple derivative scheme shown above, (1) yields:

$$K = - \left(\begin{array}{l} \text{slope of } \ln H_z \\ \text{at depth } z \end{array} \right) = \frac{-1}{H_z} \frac{dH_z}{dz}. \quad (12)$$

Thus both K and $K(Z, -)$ are the negative slopes of the semilog plots of H_z or $H(Z, -)$ versus depth Z . In this way they are similar. But where they differ is in the facts that K is independent of depth, and that $K(\cdot, -)$ is not independent of depth. And in this difference lies precisely the difference between mathematical fiction and physical reality. Is there a simple explanation for this gap in terms of the accepted concepts of hydrologic optics? We now consider in detail an explanation in terms of the currently accepted concepts of radiative transfer theory as applied to hydrological optics.

A careful examination of the experimental evidence leading to H -determinations shows that there are actually two mechanisms which may give rise to the gap between the simple classical K and the modern $K(Z, -)$. We have up to this point slanted the discussion to bring out only one of these mechanisms, the one which we may term the natural (or dynamical) mechanism of the variation of $K(\cdot, -)$. Thus, in the preceding discussion we centered attention on the depth behavior of $H(Z, -)$, a behavior which is discerned only as an irradiance probe moves downward into a natural hydrosol and continuously records the magnitude of $H(Z, -)$ as Z continuously increases. Now if the external lighting conditions on the upper boundary of the medium are arbitrary but fixed, in time, that is stationary, and if the hydrosol's inherent optical properties (namely the volume attenuation α and volume scattering function σ) are arbitrary but stationary, then the depth dependence

of $K(\cdot, -)$ is an indicator of the natural interaction of the penetrating photons with the material of the medium; it provides a running account of the depth-rate of attenuation of the number of the downwelling photons streaming past the level z . This dynamical aspect of the variation of $K(\cdot, -)$ has been studied in an elementary but detailed manner in an earlier work.¹ We may thus consider the natural (or dynamical) mechanism behind the depth behavior of $K(\cdot, -)$ to be fairly well understood. There remains, however, the problem of the so-called quasi-stationary mechanism which also gives rise to variations of $K(\cdot, -)$ to which we now turn, and which forms the central problem under study in the present work.

If, instead of continuously moving the irradiance probe vertically downward in a natural hydrosol, we hold the depth z fixed and let $H(z, -)$ be recorded as a function of time, we would expect in general, because of the continuously changing external lighting conditions above any natural hydrosol, a time dependence of $H(z, -)$ for the fixed depth z . For example, if the probe is set at a depth of five meters at 0600 hours local time in a certain lake, we would expect $H(z, -)$ generally to increase as the sun rises to reach a maximum around 1200 hours, and then to descend at 1800 to a reading comparable to the 0600 reading. On this basically regular diurnal variation of $H(z, -)$ there is superimposed a relatively more rapid variation in $H(z, -)$ induced for example by the movement of clouds or cloud layers between the sun

and the hydrosol's surface. Even though the time rates of these latter changes in the external lighting conditions are possibly thousands of times greater than those associated with the more stately diurnal changes, they nevertheless are far too small to cause any true transients in the natural light field. For this reason all these changes in the light field are of a quasi-stationary character.

Now, suppose a probe were to be sent quickly downward to record the stationary light field while the sun is covered by a cloud and then, when the cloud has just passed away from the sun and allows it to shine with full strength on the hydrosol, another quick probe uncovers the irradiance's depth profile under this sunny condition, we would discern, upon examination of the two semilog plots of $H(z, -)$, that $H(z, -)$ at a particular depth on the overcast plot would differ from the value at that same depth on the sunny plot.

Actually, experimenters need not go out of their way to uncover this phenomenon; it crops up with exasperating inevitability in any painstaking (and thus time-consuming) mapping of the irradiance profile in natural hydrosols. For in a particularly deep hydrosol the determination of the vertical profile may take on the order of a half hour; and when the probe ascends past one of the relatively shallower depths in a check re-run, the sun may have moved as much as five to ten degrees, cloud covers may have changed, thus producing a measurable change in the structure of the light field in the interim.

When plots of $H(z, -)$ are made of each of these runs the logarithmic slope $K(z, -)$ on each plot may be noticeably non-constant with depth; furthermore, and this is the crux of the matter at hand, the $K(z, -)$ value at a fixed depth z for the downward run may differ from that for the upward run. The experimenter would, at this juncture, if he considers this difference in $K(z, -)$ from one plot to another with care, soon realize that the difference may be the result of a superposition of two basically different physical mechanisms: On the one hand there is of course the possibility that the inherent optical properties of the medium may have changed during the interval between the times that the probe has visited the given depth z ; on the other hand there is the possibility that the external lighting conditions have changed during this period and that this change has in some way become manifest in the difference in the K -values for the given depth z .

If the investigator had taken care to record during the same period, and over the same depth interval, the inherent optical properties of the hydrosol, he may be able to quantitatively ascertain, using the methods of reference 1, those parts of the differences in the $K(z, -)$ values which are traceable to these causes. Thus, once again we are in possession of sufficient knowledge to understand and cope with the natural or dynamic aspects of the behavior of $K(z, -)$ even though we are now dealing with a quasi-stationary context. There remains, however, that component of the change of $K(z, -)$ at a given depth z which is traceable to the change of the external lighting conditions.

In order to relate the way in which $K(z, -)$ changes with external lighting conditions we must have some means of specifying in a precise manner the concept of "lighting condition." Clearly, in choosing a precise characterization of this concept the absolute amount of the incident radiant flux is of no essential importance. Of critical importance, however, is the relative amounts of radiant flux which arrive on the upper boundary of the medium, or on some internal horizontal plane, from the infinite number of possible directions in the hemisphere of incidence. One obvious, and incidentally the most complete, characterization would be by means of the downwelling radiance distribution $N(z, \cdot)$ at depth z . While this means may be of considerable use in other contexts, it requires of the experiments a prodigious auxiliary effort to provide the necessary measuring and recording apparatus to obtain this large number of readings. In the interests of experimental expediency, what is needed is a characterization of the relative values of $N(z, \cdot)$ without having to measure each of the infinitude of values $N(z, \xi)$, $\xi \in \Xi$ where Ξ is set of downward unit vectors. Since relative values are of primary interest, the ideal characterization would then involve not more than two readings of some simple kind.

In searching for two simple radiometric measurements which would be capable of characterizing the relative values of the downwelling radiance distribution it would be of some convenience to the experimenter if he could make use of his existing data, namely the values of $H(z, -)$. To see what possible choices remain if $H(z, -)$

is adopted as one of the two radiometric measurements which will characterize the relative magnitudes of $N(\bar{z}, \cdot)$, let us express $H(\bar{z}, -)$ in terms of these values. By definition, we have:

$$H(\bar{z}, -) = \int_{\underline{\Omega}} N(\bar{z}, \xi) \xi \cdot n \, d\Omega(\xi),$$

where n is the unit inward normal to the hydrosol. In practice, $N(\bar{z}, \cdot)$ is usually determined by a suitably chosen finite set $\{\xi_1, \dots, \xi_n\}$ of downward directions, and $H(\bar{z}, -)$ is then computed by the rule:

$$H(\bar{z}, -) = \sum_{i=1}^n N(\bar{z}, \xi_i) \xi_i \cdot n \, \Delta\Omega_i,$$

where $\Delta\Omega_i$ is the solid angle associated with the direction ξ_i . The quantities $\Delta\Omega_i$ are subject to no specific restrictions except that they be small enough so that $N(\bar{z}, -)$ is fairly uniform over each of their extents, and that of course $\sum \Delta\Omega_i = 2\pi$. Now the relative magnitudes of the quantities $N(\bar{z}, \xi_i)$, $i=1, \dots, n$ may be obtained by choosing any one of them, say $N(\bar{z}, \xi_1)$, and forming the quotients $N(\bar{z}, \xi_i)/N(\bar{z}, \xi_1) = g(\bar{z}, \xi_i)$. Then the quotient

$$\frac{N(\bar{z}, \xi_1)}{H(\bar{z}, -)} = \frac{\Delta\Omega_1}{\sum_{i=1}^n g(\bar{z}, \xi_i) \xi_i \cdot n \, \Delta\Omega_i}$$

would serve as a measure of the way in which the $g(\bar{x}, \xi_i)$ are distributed over the downwelling hemisphere Ξ_- . However, such a measure falls short of being satisfactory for several reasons: first, we have isolated a particular $N(\bar{x}, \xi_i)$ value and therefore have distinguished it with artificial importance; actually any one of the $n-1$ other values would serve just as well. Secondly, in order to measure $N(\bar{x}, \xi_i)$ we would require the services of a specially designed radiance meter, or bring into use for extremely restricted purposes the actual radiance distribution measuring apparatus -- which might as well then be used to determine $N(\bar{x}, \cdot)$ in toto. Finally, we would prefer to measure an amount of flux comparable in magnitude to $H(\bar{x}, -)$, for $N(\bar{x}, \xi_i)$ would generally be a far smaller number than $H(\bar{x}, -)$ which must then be divided into or divided by $N(\bar{x}, \xi_i)$, thereby setting the stage for the disruption of numerical accuracy in the data reduction tasks that follow.

All of these observations lead to the choice of the sum

$$h(\bar{x}, -) = \sum_{i=1}^n N(\bar{x}, \xi_i) \Delta\Omega_i,$$

as the most logical choice for the second radiometric measurement.

Its integral representation is:

$$h(\bar{x}, -) = \int_{\Xi_-} N(\bar{x}, \xi) d\Omega(\xi)$$

In this way we are led to consider the ratio

$$D(z, -) = \frac{h(z, -)}{H(z, -)}, \quad (13)$$

which we shall term the distribution function. In terms of the finite-summation representation of $h(z, -)$ and $H(z, -)$, $D(z, -)$ becomes:

$$D(z, -) = \frac{\sum_{i=1}^n N(z, \xi_i) \Delta\Omega_i}{\sum_{i=1}^n N(z, \xi_i) \xi_i \cdot n \Delta\Omega_i} \quad (14)$$

or, in terms of their integral representations

$$D(z, -) = \frac{\int_{\Xi} N(z, \xi) d\Omega}{\int_{\Xi} N(z, \xi) \xi \cdot n d\Omega} \quad (15)$$

The quantity $h(z, -)$ may be measured by simple devices, the basic design of which has been considered elsewhere.² To obtain an idea of how $D(z, -)$ serves to characterize the distribution of the irradiating flux let $n = 1$ in (14), i.e., we let the flux come from any single solid angle $\Delta\Omega$ in the general direction ξ ; the distribution function then is:

$$D(\bar{z}, -) = \frac{1}{\xi \cdot n} \quad (16)$$

Hence if the irradiation is incident vertically at depth \bar{z} , $\xi = n$ and $D(\bar{z}, -) = 1$. In general, the more obliquely incident the pencil of radiation, the smaller the dot product $\xi \cdot n$, and the larger is the distribution function. This conclusion, although obtained for a special case, is nevertheless valid for any number of pencils.

In what follows we will show in what manner the concepts $K(\bar{z}, -)$ and $D(\bar{z}, -)$ are qualitatively interdependent. In this way, by noting the behavior of the intuitively more simple and predictable quantity $D(\bar{z}, -)$ which, as we have just seen, serves as a convenient characterization of the lighting conditions -- we may gain insight into the dependence of the more complex quantity $K(\bar{z}, -)$ on these lighting conditions.

THE GENERAL LAW GOVERNING $K(\bar{z}, -)$ AND $D(\bar{z}, -)$

In this section we derive the general law which governs the exact interrelation of $K(\bar{z}, -)$ and $D(\bar{z}, -)$. We start with an established representation of $K(\bar{z}, -)$ which has been derived from the equation of transfer in an earlier work.³ This representation of $K(\bar{z}, -)$ is:

$$K(z, -) = \alpha(z, -) - \frac{\int_{\Xi} N_*(z, \xi) d\Omega}{H(z, -)} \quad (17)$$

The various terms in (17) are defined as follows:

- (i) $\alpha(z, -) = \alpha(z) D(z, -)$, where, of course, $\alpha(z)$ is the value of the volume attenuation function at depth z .
- (ii) $N_*(z, \xi)$ is the value of the path function at depth z for the direction ξ . Its analytical definition is:

$$N_*(z, \xi) = \int_{\Xi} N(z, \xi) \sigma(z; \xi'; \xi) d\Omega(\xi') \quad (18)$$

where $\sigma(z; \xi'; \xi)$ is the value of the volume scattering function at depth z for the incident direction ξ' and the scattered direction ξ .

In order to extract from (17) the desired explicit connection between $K(z, -)$ and $D(z, -)$, we must reformulate the second term of (17) in such a way as to cause $D(z, -)$ to appear explicitly in that term. Toward this end we first recall that $D(z, -)$ is by definition the ratio of $h(z, -)$ to $H(z, -)$. Now the appearance of the integral in the second term of (17) has all the external earmarks of an $h(z, -)$ -

type quantity; $N_*(z, \cdot)$ is a radiance function and it is integrated, like $N(z, \cdot)$, over Ξ_- , the set of downward directions. Thus in close analogy with $h(z, -)$ we define

$$h_*(z, -) = \int_{\Xi_-} N_*(z, \xi) d\Omega_r. \quad (19)$$

Physically, $h_*(z, -)$ is the downwelling scalar irradiance generated by the radiant flux scattered in a unit volume at depth z .

The analogy need not end with (19); in fact we can extend it quite naturally to include the *- counterparts to $h(z, +)$ and scalar irradiance $h(z)$ itself. Thus in analogy to $h(z, -)$ we define

$$h_*(z, +) = \int_{\Xi_+} N_*(z, \xi) d\Omega_r, \quad (20)$$

which is the upwelling scalar irradiance at depth z (and which forms the basis for defining $D(z, +) = h(z, +)/H(z, +)$ for the upwelling stream); we further define

$$h_*(z, +) = \int_{\Xi_+} N_*(z, \xi) d\Omega_r \quad (21)$$

which represents the upwelling scalar irradiance generated by the radiant flux scattered in a unit volume at depth \bar{z} . Finally, the scalar irradiance $h(\bar{z})$ at depth \bar{z} , being defined as:

$$h(\bar{z}) = \int_{\Xi} N(\bar{z}, \bar{s}) d\Omega \quad (22)$$

where Ξ is of course the set of all directions about depth \bar{z} , has its scattered analogy in the form:

$$h_*(\bar{z}) = \int_{\Xi} N(\bar{z}, \bar{s}) d\Omega . \quad (23)$$

Thus corresponding to

$$h(\bar{z}) = h(\bar{z}, -) + h(\bar{z}, +), \quad (24)$$

we have

$$h_*(\bar{z}) = h_*(\bar{z}, -) + h_*(\bar{z}, +). \quad (25)$$

An extremely useful and unexpectedly simple relation subsists between $h(z)$ and its scattered counterpart $h_*(z)$, namely that the ratio of $h_*(z)$ to $h(z)$ is precisely the value of the volume total scattering function Δ :

$$\Delta(z) = \frac{h_*(z)}{h(z)} . \quad (26)$$

The derivation of this relation along with some suggestions for its use in practical direct determinations of the values of Δ are given in reference 4.

We have now assembled all the required concepts needed for a complete formulation and discussion of the general law governing $K(z, -)$ and $D(z, -)$. Starting with (17) and using definition (19) we may write:

$$K(z, -) = \alpha(z, -) - \frac{h_*(z, -)}{H(z, -)} . \quad (27)$$

By means of the definition of $D(z, -)$ this may be written:

$$K(z, -) = \alpha(z, -) - \frac{h_*(z, -)}{h(z, -)} D(z, -) ,$$

and, finally, using the definition of $\alpha(z, -)$, we may recast this into the presently desired form of the law governing $K(z, -)$ and $D(z, -)$:

$$K(z, -) = \left[\alpha(z) - \frac{h_*(z, -)}{h(z, -)} \right] D(z, -) \quad (28)$$

THE ABSORPTION-LIKE CHARACTER OF $K(z, -)$

Before presenting the general proof of the covariation of $K(z, -)$ and $D(z, -)$ which will be based on some observations of the structure of (28), we pause to discuss the general radiative transfer nature of the function $K(\cdot, -)$. We will show by general arguments and also by means of a simple example that $K(z, -)$ is essentially an "absorption coefficient," i.e., it serves as an analytical book-keeping device for the depth-rate of absorption of the stream of downwelling photons as the stream passes a general depth z .

The heart of equation (28) resides in the difference of the two bracketed terms. The first term is a value of the volume attenuation function which shows that $K(z, -)$ first of all takes cognizance of the simultaneous loss of photons by means of both scattering and absorbing mechanisms. Thus, as a stream of photons crosses an hypothetical surface at depth z , the stream suffers a loss by having some of the photons scattered in all directions about the point of crossing and also by having some of its photons of (implicitly stated) wavelength λ converted into photons of (generally) longer wavelength or into non-radiant energy. Now the second term, involving $h_{\uparrow}(z, -)$ and being subtracted from $\alpha(z)$, in effect returns to the downwelling stream all photons which have been scattered in the "forward direction" - the direction of motion of the downwelling stream. The net loss to the downwelling stream is then represented by this difference; it

represents the amount of radiant flux per unit area that either has been scattered back into the upwelling stream, or which has suffered true absorption. These observations point up the dynamic interpretation of the magnitude of $K(\bar{z}, -)$. To see the predominantly absorption-like character of $K(\bar{z}, -)$ we now consider two extreme examples of optical media: one that is purely absorbing, the other purely scattering; we will show that in each of these two extremes the values of $K(\bar{z}, -)$ tend either immediately or eventually to become directly proportional to the given values $a(\bar{z})$ of the volume absorption function.

First, let $\mathcal{A}(\bar{z}) = 0$ for all \bar{z} , and let $a(\bar{z})$ be arbitrary. We then have the extreme case of a purely absorbing medium. By hypothesis, it follows that $N_*(\bar{z}, \xi) = 0$ for each \bar{z} and for all ξ . Furthermore, since $\alpha(\bar{z}) = a(\bar{z}) + \mathcal{A}(\bar{z})$ it follows that $\alpha(\bar{z}) = a(\bar{z})$. With these observations equation (28) reduces immediately to the simple form:

$$K(\bar{z}, -) = a(\bar{z}) D(\bar{z}, -) . \quad (29)$$

Here $D(\bar{z}, -)$ generally depends on depth along with $a(\bar{z})$. However if depth \bar{z} and $a(\bar{z})$ are held fixed and the external lighting conditions are varied so that $D(\bar{z}, -)$ is changed, it is quite clear that $K(\bar{z}, -)$ varies directly and linearly with these changes in $D(\bar{z}, -)$. This is the first and simplest instance

of the covariation of $K(z, -)$ and $D(z, -)$. The essentially absorption-like character may be seen by holding $D(z, -)$ fixed. Then $K(z, -)$ varies directly with the value $\alpha(z)$ of the absorption function.

Next, let $\alpha(z) = 0$ for all z in an optically infinitely deep medium in which $\mu(z) = \mu > 0$ for all z . We then have the other extreme case of a purely scattering medium. In such a medium the divergence $\nabla \cdot \underline{H}(z)$ of the net-irradiance vector vanishes at each depth z . In particular, in a plane-parallel medium such as that around which the present discussion is centered, this divergence relation takes the simple form:

$$\frac{d\bar{H}(z, -)}{dz} = 0$$

where $\bar{H}(z, -) = H(z, -) - H(z, +)$, $H(z, +)$ being the upwelling irradiance at depth z . It follows that

$$\bar{H}(z, -) = \text{constant}$$

for all z . Since the medium is in steady state and no photons which enter it are ever lost by absorption, we expect that the time rate of emergence $H(0, +)$ of photons at the surface just equals the time rate of incidence $H(0, -)$. Hence

$$H(z, -) = H(z, +)$$

for all z . Now the greater the depth z in the medium the more alike optically are the regions above and below the level z , and the more alike are their optical responses to the equal irradiances impinging on the common internal boundary at depth z . Thus we are led, by considerations of increasing symmetry above and below horizontal planes at great depths to conclude that $h(z,-) \rightarrow h(z,+)$ as $z \rightarrow \infty$. Hence, at great depths in the present medium, $D(z,-) \rightarrow D(z,+)$, so that the angular structure of the up and downwelling radiance distributions become equal, and in fact (by symmetry) uniform. An immediate consequence of these observations is the fact that

$$\frac{h_{*}(z,-)}{h(z,-)} \longrightarrow \frac{h_{*}(z,-) + h_{*}(z,+)}{h(z,-) + h(z,+)} = \frac{h_{*}(z)}{h(z)} = 1.$$

Hence, since equation (28) for the present case of $Q(z) = 0$ takes the form:

$$K(z,-) = \left[1 - \frac{h_{*}(z,-)}{h(z,-)} \right] D(z,-),$$

we conclude that

$$K(z,-) \longrightarrow 0 = Q(z), \quad (30)$$

as $z \rightarrow \infty$. Thus, in the both extreme types of media, where one is purely absorbing and the other purely scattering, the diffuse attenuation function was shown to tend toward values which are directly

proportional to the values of the volume absorption function for these media. It is in this sense that we understand the absorption-like character of $K(\bar{z}, -)$. Another instance (Equation (34)) supporting this interpretation of $K(\bar{z}, -)$ will be encountered as a matter of course in the concluding section of this work, which presents a practical rule of thumb based on the covariation of $K(\bar{z}, -)$ and $D(\bar{z}, -)$.

FORWARD SCATTERING MEDIA

A necessary prerequisite to the establishing of the covariation rule between $K(\bar{z}, -)$ and $D(\bar{z}, -)$ is the introduction of the notion of a forward scattering medium. Briefly, a forward scattering medium is one for which the volume scattering function has a predominant forward scattering lobe as compared to its backward scattering lobe.

For a precise definition, let \hat{n} as shown be the unit inward normal to the medium, and let \bar{s} be an arbitrary element of Ξ ; then we define

$$A(\bar{z}, \bar{s}) = \int_{\Xi_-} \sigma(\bar{z}; \bar{s}; \bar{s}') d\Omega_k(\bar{s}'), \quad (31)$$

where $\Xi_- = \{ \bar{s} : \bar{s} \cdot \hat{n} \geq 0 \}$.

Some general properties of $\mathcal{A}(z, \xi)$ are easily deduced. For example, if $\xi \cdot n = 0$, then $\mathcal{A}(z, \xi) = \mathcal{A}(z)/2$ since the axis of the scattering lobe would then lie in the plane at depth z and the region of integration would be over precisely half the scattering lobe. Furthermore, for every $\xi \in \Xi$

$$\mathcal{A}(z, \xi) + \mathcal{A}(z, -\xi) = \mathcal{A}(z)$$

Let us adopt the convention here that ξ is always in Ξ_- . Then since n is a special element of Ξ_- , the value $\mathcal{A}(z, \xi)$, $\xi \in \Xi_-$ has the interpretation of a forward scattering function for the direction ξ . Physically $\mathcal{A}(z, \xi)$ gives the fraction of flux of a beam of direction ξ that is scattered in all downward (forward, with respect to ξ) directions. We emphasize this fact by adding a "+" subscript to $\mathcal{A}_+(z, \xi)$, which also serves to remind us of the convention that $\xi \in \Xi_-$. If now we set $\Theta = \arccos(\xi \cdot n)$ we may write $\mathcal{A}_+(z, \xi)$ as $\mathcal{A}_+(z, \Theta)$ and we finally may define a forward scattering medium as one for which $\mathcal{A}_+(z, \Theta)$ decreases monotonically with increasing Θ in the range $0 \leq \Theta \leq \pi/2$.

For subsequent purposes we append the additional definition:

$$\mathcal{A}_-(z, \xi) = \int_{\Xi_+} \sigma(z; \xi; \xi') d\Omega(\xi'), \quad \xi \in \Xi_- \quad (32)$$

where $\Xi_+ = \{ \xi : \xi \cdot n \leq 0 \}$.

Clearly, (31) and (32) imply:

$$A_+(z, \xi) + A_-(z, \xi) = A(z)$$

for all $\xi \in \Xi_-$, and analogously to $A_+(z, \xi)$ we define $A_-(z, \xi)$ as the backward scattering function for the direction $\xi \in \Xi_-$. Since $A(z)$ is independent of ξ , we have an alternate means of characterizing a forward scattering medium now using $A_-(z, \Theta)$, which of necessity is monotonically increasing with Θ in any forward scattering medium.

THE COVARIATION RULE FOR $K(z, -)$ AND $D(z, -)$

We now may state the covariation rule:

LET X BE AN ARBITRARILY STRATIFIED PLANE PARALLEL FORWARD SCATTERING MEDIUM WITH FIXED INHERENT OPTICAL PROPERTIES. IF $z \geq 0$ IS ANY FIXED DEPTH IN X , THEN $K(z, -)$ AND $D(z, -)$ INCREASE, REMAIN CONSTANT, OR DECREASE TOGETHER. THUS IF IN PARTICULAR, OVER A CERTAIN TIME PERIOD $K(z, -)$ AND $D(z, -)$ EXHIBIT INCREMENTS IN THEIR VALUES OF MAGNITUDE $\Delta K(z, -)$ AND $\Delta D(z, -)$, THEN THESE INCREMENTS ARE SIMULTANEOUSLY POSITIVE, ZERO OR NEGATIVE.

The complete proof of the rule is tedious because it requires an analysis of the total light field. However, the following argument forms the essential outline of a proof and makes the rule plausible. Suppose that, under the conditions of the hypothesis above, $\Delta D(z, -)$ is positive and $\Delta D(z, +)$ is zero. This implies an increase in the magnitude of $D(z, -)$ which, by our previous discussion of

the properties of $D(z, -)$, implies that the preponderance of the radiation has now shifted to make a greater angle with the unit inward normal to an hypothetical surface at depth z . Now the term $h_{\mu}(z, -) / h(z, -)$ in (28) is simply a weighted mean of the volume scattering function values $\sigma(z; \xi'; \xi)$ wherein the weighting factors are the upwelling and downwelling radiance components. As observed above, the increase in $D(z, -)$ requires that the radiance components carrying the preponderance of the downwelling flux to be now associated with larger (H) values in the arguments of forward scattering function $\Delta_+(z, H)$; since the medium is forward scattering, this weighted mean then generally experiences a decrease in magnitude; the net result being an increase in the bracketed quantity in (28). The total change of $K(z, -)$ is the combination of the increase of the bracketed quantity and the increase in the factor $D(z, -)$; that is, $K(z, -)$ experiences an increase in magnitude. Summarizing: an increase in $D(z, -)$ is attended by an increase in $K(z, -)$ all other things remaining fixed in the forward scattering medium. A similar argument may be applied to the assumption that $D(z, -)$ exhibits a decrease. With these two facts established it is then a necessary consequence in all physical situations that $\Delta K = 0$ whenever $\Delta D = 0$.

ILLUSTRATIONS OF THE RULE

1. Most natural hydrosols are forward scattering media; in fact $\Delta_+(\bar{z}, 0)$ is on the order of ten to twenty times the magnitude of $\Delta_+(\bar{z}, \pi/2)$, over the visible spectrum. The values $\sigma(\bar{z}; \xi; \xi)$ and $\sigma(\bar{z}, \xi; \xi_{\perp})$, where $\xi \cdot \xi_{\perp} = 0$, often subtend a ratio of forward to side scattering on the order of 100 to 1, over the visible spectrum. The rule may safely be extended even to natural aerosols, these media being predominantly forward scattering; even the borderline case of Rayleigh atmospheres wherein $\Delta_+(\bar{z}, \theta) = \Delta(\bar{z})/2$ for θ in $[0, \pi/2]$, are subject to the rule since the square bracketed quantity in (28) does not generally change magnitude, with a change in $D(\bar{z}, -)$.

2. As a specific illustration, suppose the sky above a lake is completely overcast and that the downwelling distribution and diffuse attenuation functions at some relatively shallow depth \bar{z} have values D_0 and K_0 respectively under this overcast condition. Suddenly, the near-zenith sun breaks through the clouds. The resulting value D_1 , of the distribution function, is expected to be less than D_0 : $D_1 < D_0$, which follows from the fact that the predominant portion of the radiation now comes from generally less oblique directions. It follows that $\Delta D < 0$, that is the increment of $D(\bar{z}, -)$ is negative, and thus the covariation rule requires that $\Delta K(\bar{z}, -)$ is negative, so that the new value K_1 of the diffuse attenuation function is less than K_0 : $K_1 < K_0$.

3. As a final illustration, suppose that we fix attention on a relatively shallow depth in a natural hydrosol which is irradiated by a clear sunny sky for an entire afternoon. As the sun descends, $D(z, -)$ clearly increases because the direction of the predominant portion of the irradiating flux supplied by the sun increases its angle with the vertical (i.e., $1/\xi \cdot n$ increases). The covariation rule would then require that $K(z, -)$ exhibit a corresponding increase.

THE CONTRAVARIATION OF $K(z, +)$ AND $D(z, +)$

Up to this point in the discussion we have excluded any detailed mention of the upwelling irradiance $H(z, +)$. However, all that has been discussed for the downwelling stream of radiant energy may be applied mutatis mutandis to the upwelling stream, i.e., by replacing minus signs by plus signs, etc. Therefore, the discussion of the interdependence of $K(z, +)$ and $D(z, +)$ may be conducted quickly in a minimum amount of space by pointing up only the basic differences between the two cases. The theory governing $K(z, +)$ and $D(z, +)$ may be found in reference 3.

Now the first of these main differences between the upwelling and downwelling streams $H(z, +)$ and $H(z, -)$ lies in their magnitudes over the visible portion of the spectrum; the ratio of their magnitudes, $H(z, +)/H(z, -)$ known as the reflectance of the medium at depth z :

$$R(z, -) = \frac{H(z, +)}{H(z, -)}$$

is very nearly and almost universally in the neighborhood of 0.02.

Thus $H(z, -)$ is on the order of 50 times the size of $H(z, +)$.

Secondly, a fairly constant tie exists between the two streams by virtue of the ratio and sum of their distribution functions. It is found that almost universally over the visible portion of the spectrum:

$$\frac{D(z, +)}{D(z, -)} = 2 ,$$

$$D(z, -) + D(z, +) = 4 ,$$

which then requires that $D(z, -) = 4/3$, and $D(z, +) = 8/3$.

The (+) stream counterpart to (17) is:

$$-K(z, +) = \alpha(z, +) - \frac{\int_{\Xi+} N_*(z, \xi) d\Omega_r}{H(z, +)} ,$$

which may be reduced to a corresponding expression to (28):

$$-K(z, +) = \left[\alpha(z) - \frac{h_*(z, +)}{h(z, +)} \right] D(z, +) . \quad (33)$$

In much the same manner as $K(z, -)$, the dynamical and quasi-stationary mechanisms giving rise to the depth and temporal changes of $K(z, +)$ may be discussed in complete detail. The only precautionary observation that should be made here is that the dynamical mechanisms governing $K(z, +)$ should be examined as depth z decreases, this being the natural direction of flow of the upward stream. Finally, owing to the

negative sign in front of $K(z, +)$, the signs of the increments in $K(z, +)$ and $D(z, +)$ are opposite. Thus there is a contravariation in the magnitudes of $K(z, +)$ and $D(z, +)$. This is the final distinction that must be made between the two streams, for the present.

Whence does this striking difference in the relative variations of the magnitudes of $K(z, \pm)$ and $D(z, \pm)$ arise? In what light should this difference be viewed? The answer to the first question is that the difference arises in the definitions of $K(z, +)$ and $K(z, -)$; each is defined as:

$$K(z, \pm) = \frac{-1}{H(z, \pm)} \frac{dH(z, \pm)}{dz} .$$

Now a plane parallel medium representing a natural hydrosol, by its very physical nature and usual coordinate system, selects the downward direction as the direction of increasing z values. Thus the spatial evolution of quantities associated with the downwelling stream are treated in a natural way, i.e., so that the natural unfolding of radiant energy in a downward direction takes place in the direction of the natural unfolding of the coordinate system, i.e., along with increasing z coordinates. The upwelling stream on the other hand naturally evolves spatially in the direction of decreasing z values, hence the contravariation, or topsy turvy interdependence of $K(z, +)$ and $D(z, +)$. This contravariation can be erased and converted to a covariation if we re-interpret the derivative dH/dz as $dH/d(-z)$ when considering the upward-flowing case.

In answer to the second question, all that can reasonably be done is to view this state of affairs as an inessential perversity of standard coordinate systems, and to understand that it is the inevitable result of an attempt to depict an inherently three dimensional process by an artificial two-dimensional symbology designed by a basically one-dimensional thought process.

A COVARIATION RULE OF THUMB

The general law (28) governing the interdependence of $K(Z, -)$ and $D(Z, -)$, while of extreme importance in establishing the exact relationship between these two quantities, is somewhat unwieldy for use in quick estimates of the relative magnitudes of their increments. We conclude the present work with the derivation of a simple rule of thumb, based on experimental evidence, which relates in a linear manner the relative magnitudes of $K(Z, -)$ and $D(Z, -)$, and also their increments. We begin with the exact expression for $R(Z, -)$ in terms of $K(Z, \pm)$ and $a(Z, \pm)$ as developed in reference 5:

$$R(Z, -) = \frac{K(Z, -) - a(Z, -)}{K(Z, +) + a(Z, +)} .$$

Here $a(Z, \pm) = a(Z) D(Z, \pm)$. Now it is an experimental fact that the difference between the magnitudes of $K(Z, +)$ and $K(Z, -)$ over the visible spectrum is very small for many practical purposes, being on the order of five percent of magnitude of $K(Z, -)$

(See reference 1). Hence we may replace $K(z,+)$ by $K(z,-)$ in the preceding formula and solve for $K(z,-)$. The result is:

$$K(z,-) = a(z) \left[\frac{R(z,-)D(z,+) + D(z,-)}{1 - R(z,-)} \right] \quad (34)$$

Equation (34) may be taken, as it stands, as a rule of thumb connecting $K(z,-)$ with $a(z)$. This equation serves to underscore the conclusion reached earlier in this work that $K(z,-)$ is basically an absorption-like optical property of a medium.

To arrive at the desired rule of thumb we now make use of the experimental-numerical relations, cited earlier in the work, between $D(z,+)$ and $D(z,-)$ and also of the fact that $R(z,-)$ is of the order .02. The result is:

$$\begin{aligned} K(z,-) &= a(z) \left[\frac{R(z,-) \frac{D(z,+)}{D(z,-)} + 1}{1 - R(z,-)} \right] D(z,-) \\ &= a(z) \left[\frac{\frac{1}{50} \times 2 + 1}{1 - \frac{1}{50}} \right] D(z,-) \\ &= \frac{52}{49} a(z) D(z,-) = 1.06 a(z) D(z,-) \end{aligned}$$

Hence

$$\Delta K(z, -) = 1.06 a(z) \Delta D(z, -), \quad (35)$$

which is the desired rule of thumb relating the covariation of $K(z, -)$ and $D(z, -)$.

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