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PRINCIPLES OF INVARIANCE ON DISCRETE SPACES

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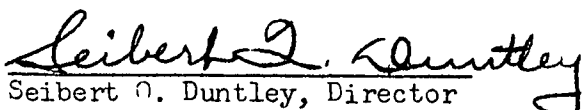
August 1959
Index Number NS 714-100

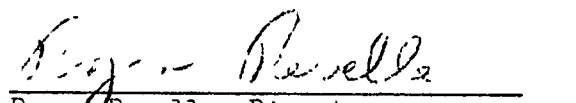
Bureau of Ships
Contract NObs-72092

SIO REFERENCE 59-55

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Principles of Invariance on Discrete Spaces

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IT WILL BE SHOWN THAT

The discrete-space approach to radiative transfer theory leads to a derivation of the principles of invariance. The principles of invariance which once were without any links to the main body of radiative transfer theory are shown to be derivable by elementary means from the local interaction principle. This latter principle was introduced earlier¹ and shown to possess sufficient detail to allow a formulation and solution of the basic transfer equations on every discrete space. The local interaction principle thus serves as a unifying principle from which the entire discrete theory of radiative transfer can be constructed. By suitably designed convergence arguments these results may be carried over to the continuous theory. Thus, in the final analysis, the principle of local interaction can form the single point of departure for the construction of all of radiative transfer theory.

The plan of the present note is as follows: First it is shown that the principle of local interaction, which describes how radiant

flux is scattered and absorbed by a single point of a discrete space X_n , can be rearranged so as to give the same description for an arbitrary subset of points of X_n . This is the so called divisibility property of the principle. The principle of invariant imbedding is then an immediate consequence, and from this follows the derivation of the principles of invariance. The paper closes with a discussion of seven outstanding problems of the continuous theory which may be studied using the techniques of discrete-space theory.

THE DIVISIBILITY PROPERTY OF THE LOCAL INTERACTION PRINCIPLE

We begin with a brief explication of:

The Notion of a Divisible Property

Although the notion of a divisible property is definable in quite general spaces, we may and shall restrict our discussion to spaces with a finite number of "points". Suppose, then, that X_n consists of a finite number n of points. (For concreteness, X_n may be envisioned as some well-defined bounded subset of euclidean three-space E_3 , such as a cube.) We now manufacture a new space from X_n as follows: Let P_r be a given partition of X_n . That is, P_r is a collection of pairwise disjoint subsets of X_n whose union is X_n . Thus, $P_r = \{X_{\rho_1}, X_{\rho_2}, \dots, X_{\rho_r}\}$ is a partition of X_n if X_{ρ_i} has ρ_i elements of X_n , X_{ρ_i} and X_{ρ_j} have no elements in common for each distinct X_{ρ_i} and X_{ρ_j} , and $\sum_{i=1}^r \rho_i = n$. We now consider P_r as a space whose elements or points are the X_{ρ_i} , $i = 1, \dots, r$. Thus, a point of P_r is a subset of the original set X_n . For reasons which are based on established topological terminology, we shall call P_r the quotient space generated by X_n .

Now suppose that X_n , the original space, has some property P . For example, P may be: "The number of elements of X_n within every sphere of radius r is less than r^3 ". Or, again, P may be: "Every pair of points of X_n may be connected by a line which does not contain any other members of X_n ," and so on. Suppose now that the property P is meaningful for a quotient space P_r generated

by X_n . (By suitably defining the notion of "sphere", "line" and "intersect" in the preceding examples, the two specific properties cited above can be shown to be meaningful for \mathcal{P}_r .) Then, if \mathcal{P} holds for \mathcal{A} when it holds for X_n we say that \mathcal{P} is a divisible property.* Thus a divisible property on a space X_n is one which may be inherited by the quotient space of X_n .

We now turn to the particular property of current interest. In particular, we will show that

The Local Interaction Property is Divisible

Suppose the principle of local interaction¹ is applicable to a discrete space X_n . For brevity we say that X_n has the local interaction property: \mathcal{P} . We shall now show that \mathcal{P} is a divisible property.

Let $\mathcal{A} = \{X_{\mathcal{A}_1}, X_{\mathcal{A}_2}, \dots, X_{\mathcal{A}_r}\}$ be a partition of a discrete space X_n . We may then (using the conventions and notation of reference 1) rearrange the analytic statement of the principle of local interaction:

* Mathematical readers may verify that such properties as compactness and separability supply examples of divisible properties for topological spaces and their quotient spaces.

$$\left\{ \begin{array}{l} N_{ij} = \sum_{k=1}^n N_{ki} \Sigma(x_i; \xi_{ki}; \xi_{ij}) + \sum_{k=1}^{\Delta_i} N_{ki}^0 \Sigma^0(x_i; \xi_{ki}^0; \xi_{ij}), \\ i = 1, \dots, n \quad ; \quad j = 1, \dots, n, \end{array} \right. \quad (1)$$

which is clearly a system of n^2 equations in the n^2 unknowns N_{ij} , into a new form which is governed by the given partition, as follows:

First, we observe that we may, without any loss of generality, renumber the points of X_n , if necessary, so that $X_{P_1} = \{x_1, \dots, x_{P_1}\}$, $X_{P_2} = \{x_{P_1+1}, \dots, x_{P_1+P_2}\}$, \dots , $X_{P_r} = \{x_{n-P_r+1}, \dots, x_n\}$.

Second, for every pair of (not necessarily distinct) elements X_{P_i}, X_{P_j} of P_r we define the $1 \times P_i P_j$ vector $N_+(P_i|P_j)$ as follows:

$$N_+(P_i|P_j) = [N_{K_1, l_1}, N_{K_2, l_2}, \dots, N_{K_1, l_{P_j}}, \dots, N_{K_{P_i}, l_1}, \dots, N_{K_{P_i}, l_{P_j}}] \quad (2)$$

where K_1 is the least x -subscript occurring in X_{P_i} , $K_2 = K_1 + 1$, $K_3 = K_2 + 1$, and $x_{l_i} \in X_{P_j}$, etc. Furthermore, we define the $1 \times P_i P_j$ vector:

$$N_-(P_i|P_j) = [N_{K_1, l_1}, N_{K_2, l_1}, \dots, N_{K_{P_i}, l_1}, \dots, N_{K_1, l_{P_j}}, \dots, N_{K_{P_i}, l_{P_j}}]. \quad (3)$$

The vectors (2) and (3) are clearly generalized forms of the $1 \times n^2$ vectors N_+ and N_- defined in reference 1. For by letting $r=1$, so that $P_i = P_j = n$, (2) and (3) reduce to N_+ and N_- respectively.

Third, we define the matrix $\Sigma(P_k | P_i | P_j)$, which is a diagonal $P_k P_i \times P_i P_j$ matrix of the form:

$$\Sigma(P_k | P_i | P_j) = \begin{pmatrix} \Sigma(P_k | x_{K_1} | P_j) & \circ & & & \\ \circ & \Sigma(P_k | x_{K_2} | P_j) & & & \\ & & \ddots & & \\ & & & \circ & \\ & & & & \Sigma(P_k | x_{K_{P_i}} | P_j) \end{pmatrix} \quad (4)$$

where, in turn, $\Sigma(P_k | x_{K_i} | P_j)$, $x_{K_i} \in X_{P_i}$, is a $P_k \times P_j$ matrix whose element in the u th row, and v th column is $\Sigma(x_{K_i}; \xi_{uK_i}; \xi_{K_i v})$, $x_u \in X_{P_k}$, $x_v \in X_{P_j}$.

Fourth, we designate by $N^o(P_i)$ the row vector of the form:

$$N^o(P_i) = [N^o_{1K_1}, N^o_{2K_1}, \dots, N^o_{\Delta_1 K_1}, \dots, N^o_{1K_{P_i}}, \dots, N^o_{\Delta_i K_{P_i}}] \quad (5)$$

where x_{K_1} is the first element of X_{P_i} , and Δ_1 is the number of source directions for x_{K_1} ; $x_{K_{P_i}}$ is the last element of X_{P_i} and Δ_i is the number of source directions for $x_{K_{P_i}}$. Clearly $N^o(P_i)$ has $(\sum_1^{\Delta_i} \Delta_i)$ elements, where the summation runs over the P_i index integers for X_{P_i} .

As the fifth and final step, we define the matrix $\Sigma^o(P_i | P_j)$, which is a $P_i \cdot (\sum_1^{P_i} \Delta_i) \times P_i P_j$ matrix of the form:

$$\Sigma^{\circ}(P_i|P_j) = \left(\begin{array}{cccc} \Sigma^{\circ}(x_{K_1}|P_j) & \circ & & \\ \circ & \Sigma^{\circ}(x_{K_2}|P_j) & & \\ & & \ddots & \\ & & & \circ \\ \circ & & & \Sigma^{\circ}(x_{K_{P_i}}|P_j) \end{array} \right) \quad (6)$$

where, in turn, $\Sigma^{\circ}(x_{K_i}|P_j)$, $x_{K_i} \in X_{P_i}$ is an $(\sum_1^{P_i} \Delta_i) \times P_j$ matrix (Δ_i is the number of source directions accessible to X_{K_i}) whose element in the u th row and v th column is: $\Sigma^{\circ}(x_{K_i}; \xi_{uK_i}^{\circ}; \xi_{K_i v})$, $\xi_{uK_i}^{\circ}$ is a source direction at $x_{K_i} \in X_{P_i}$, and $x_v \in X_{P_j}$.

With these formulations the principle of local interaction is readily cast into its appropriate form for the quotient space P_T :

$$N_+(P_i|P_j) = \sum_{K=1}^T N_-(P_K|P_i) \Sigma(P_K|P_i|P_j) + N^{\circ}(P_i) \Sigma^{\circ}(P_i|P_j) \quad (7)$$

$i = 1, \dots, T ; j = 1, \dots, T$

The formal identity of (1) and (7) is evident under the following pairings of concepts within each parenthesis: (X_n, P_T) , (x_j, X_{P_j}) , $(N_{ij}, N_+(P_i|P_j))$, $(\Sigma(x_i; \xi_{K_i}; \xi_{ij}), \Sigma(P_K|P_i|P_j))$, $(N_{K_i}^{\circ}, N^{\circ}(P_i))$, $(\Sigma^{\circ}(x_i; \xi_{K_i}^{\circ}; \xi_{ij}), \Sigma^{\circ}(P_i|P_j))$.

It follows that property P is divisible.

An immediate corollary of the divisibility of the local interaction property is that the system (7) of T^2 equations in the

r^2 unknowns $N_+(P_i|P_j)$, $i, j = 1, \dots, r$, is uniquely and explicitly solvable using the techniques and proofs developed in reference 1. We now develop some further corollaries of a rather interesting kind, corollaries which show how the divisibility property :

CAN BE USED IN HIERARCHIES OF DISCRETE SPACES

First of all, we shall understand here that a hierarchy $\mathcal{H}_\alpha(X_n)$ of discrete spaces is a finite sequence, of at least two terms, of successively constructed quotient spaces, whose initial member is some given discrete space X_n : $\mathcal{H}_\alpha = \{X_n, P_{r_1}, \dots, P_{r_\alpha}\}$. Thus P_{r_i} is the quotient space derived from its immediate predecessor in the sequence.

The discussion of the preceding section explicitly demonstrated the divisibility of the local interaction property for the simple hierarchy $\mathcal{H}_1(X_n) = \{X_n, P_{r_1}\}$. However, the arguments leading to the conclusion of the divisibility of \mathcal{P} were independent of the explicit magnitudes n and r_1 ; in fact n and r_1 were arbitrary, fixed integers. It follows that the general demonstration of the divisibility of \mathcal{P} is immediately applicable, mutatis mutandis, to the hierarchy $\{P_{r_1}, P_{r_2}\}$. In this way we have shown that: if \mathcal{P} holds for a given discrete space X_n , it holds for every member of a given hierarchy $\mathcal{H}_\alpha(X_n)$ constructed from X_n .

The consequences of this conclusion are manifold: they range from the practical level in which numerical answers to transfer problems are of primary interest up to the conceptual level concerned with the basic principles of radiative transfer theory. We shall however

reserve the practical consequences for separate and detailed discussion in subsequent notes, and concentrate at present only on general theoretical conclusions. In particular, we now discuss with the aid of a specific example the basic idea behind the principles of invariance as seen from the hierarchy point of view in discrete-space theory.

Imagine a relatively extensive lattice X_n of points in E_3 , which consists of points (x, y, z) ; x, y , and z are integers, with $-999 \leq x, y \leq 999$, $-99 \leq z \leq 0$. Here $n = (200)^2 \times 100 = 4 \times 10^6$. Suppose $X_{4,000,000}$ is irradiated at each point of its upper boundary ($z=0$). Now for most engineering purposes the space $X_{4,000,000}$ is infinite and the transfer problem is practically insuperable. However, from the mathematical point of view, $X_{4,000,000}$ is, of course, finite and thus amenable to the inherently exact finitary analysis. The light field generated within this lattice has many general and useful properties which are discoverable by finitary techniques. For example, suppose we generate the quotient space (X_{P_1}, X_{P_2}) where X_{P_1} is the subset of X_n consisting of all points with $z \geq -50$, and X_{P_2} consists of all points with $z < -50$. Hence $P_1 = P_2 = 2 \times 10^6$. Now (X_{P_1}, X_{P_2}) is viewable as a two-point system of radiometrically interacting points. The interrelations between the radiance quantities $N_+(P_1|P_2)$, $N_+(P_2|P_1)$ are therefore no more complex, in principle, than those between N_{12} and N_{21} used in the elementary introduction to reference 1, and in fact are given by the results of that earlier discussion. Thus, each of the two sets of $2,000,000$ radiometrically interacting points may be conceptually compressed into a single point which interacts with the other.

This then is the heart of the idea behind the principles of invariance: that an extensive and complex optical medium can be partitioned into radiometrically interacting subsets, each of which is considered as a point and is assigned a local interaction property, thus leading to an inherently simpler object to work with than the original space.

In the light of these observations, the word "invariance" in the name "principles of invariance" thus takes on in the present context a new* and deeper meaning, namely that it signifies the invariant character of the principle of local interaction under the transformation of a given arbitrary space to its associated quotient spaces (which is, as we have seen, a consequence of the divisibility property of P).

We now go on to show in detail that the divisibility of P can be used in hierarchies of discrete spaces:

TO DERIVE THE INVARIANT IMBEDDING RELATION

The invariant imbedding relation² is a compact symbolic statement which, as a special case, yields the four principles of invariance on a wide class of carrier spaces, e.g., on all the classical one-parameter carrier spaces of radiative transfer theory

* The original meaning of the term stems from the invariance of the reflected radiance distribution (emerging at the upper boundary of an infinitely deep homogeneous medium) under the operation of either adding to or removing from the medium layers of finite optical thickness.

(half-spaces, slabs, cubes, cylinders, spheres, etc.). Our purpose in the present section is to derive the invariant imbedding relation for discrete spaces from first principles, i.e., the principle of local interaction. In this way we make the first step toward the erasure of the abiogenetic character¹ of the principles of invariance. A preliminary detail that must be covered before this step is taken concerns the definition of:

The Permutation Matrices

First, recall that X_n has been partitioned into an arbitrary collection $\{X_{P_1}, X_{P_2}, \dots, X_{P_r}\}$ of disjoint subsets. The permutation matrix $M(P_j|P_i)$ serves to map the field radiance vector $N_-(P_i|P_j)$ into the specific radiance vector $N_+(P_i|P_j)$, and conversely, thereby reducing the number of radiance functions that must be carried through the formulations. Specifically, we require a $P_i P_j \times P_i P_j$ matrix $M(P_j|P_i)$ with the property:

$$N_+(P_i|P_j) = N_-(P_i|P_j) M(P_j|P_i) \quad (8)$$

Since $N_+(P_i|P_j)$ and $N_-(P_i|P_j)$ are vectors composed of the same components but in different orderings, a permutation matrix with the property shown in (8) must exist. In fact:

$$M(P_j|P_i) = \begin{pmatrix} F_{11} & F_{21} & \dots & F_{P_i 1} \\ F_{12} & F_{22} & \dots & F_{P_i 2} \\ \vdots & \vdots & \ddots & \vdots \\ F_{1P_j} & F_{2P_j} & \dots & F_{P_i P_j} \end{pmatrix}, \quad (9)$$

where

$$F_{uv} = \begin{pmatrix} \circ & \circ & \dots & \circ & \dots & \circ \\ \circ & \circ & \dots & \circ & \dots & \circ \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \circ & \circ & \dots & 1 & \dots & \circ \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \circ & \circ & \dots & \circ & \dots & \circ \end{pmatrix} \begin{matrix} \xrightarrow{P_j} \\ \xleftarrow{v} \\ \uparrow u \\ \downarrow P_i \\ \downarrow \end{matrix} \quad (10)$$

That is, F_{uv} is a $P_i \times P_j$ matrix all of whose entries are zero except for a unit entry in the u th row and v th column.

$M(P_j|P_i)$ thus consists of a $P_j \times P_i$ rectangular array of matrices each of P_i rows and P_j columns. From this definition of $M(P_j|P_i)$ we immediately deduce:

$$N_-(P_i|P_j) = N_+(P_i|P_j) M(P_i|P_j), \quad (11)$$

so that for all $i, j = 1, \dots, r$

$$M(P_i|P_j) M(P_j|P_i) = I, \quad (12)$$

where I is the $P_i P_j \times P_i P_j$ identity matrix. Hence

$$[M(P_i | P_j)]^{-1} = M(P_j | P_i) ; \quad (13)$$

furthermore

$$\text{transpose } [M(P_i | P_j)] = M(P_j | P_i) ; \quad (14)$$

and finally, by defining

$$|M(P_i | P_j)| = \text{determinant of } M(P_i | P_j) , \quad (15)$$

we find that

$$|M(P_i | P_j)| = 1 . \quad (16)$$

The matrix $M(P_i | P_j)$ is a general case of the permutation matrix M introduced in reference 1. For by letting $\nu=1$ (the trivial partition of X_n) so that $P_i = P_j = n$, we have

$$M(n|n) = M . \quad (17)$$

By means of properties (8) and (12) (or equivalently (11)) we can rewrite the local interaction principle (7) in the more symmetrical form:

$$N_+(P_i|P_i) = \sum_{k=1}^T N_+(P_k|P_i) M(P_k|P_i) \Sigma(P_k|P_i|P_j) + N^0(P_i) \Sigma^0(P_i|P_j) . \quad (18)$$

Thus in any formula involving $N_-(P_i|P_j)$ we may always replace the latter by $N_+(P_i|P_j)$, using (11). On this basis we may and shall use only specific radiance vectors $N_+(P_i|P_j)$. The need for the + subscript is now obviated, and henceforth will be dropped from the notation. By writing

$$\Sigma_M(P_k|P_i|P_j) = M(P_k|P_i) \Sigma(P_k|P_i|P_j) , \quad (19)$$

(7) is then reduced to its basic symmetric form:

$$N(P_i|P_j) = \sum_{k=1}^T N(P_k|P_i) \Sigma_M(P_k|P_i|P_j) + N^0(P_i) \Sigma^0(P_i|P_j) . \quad (20)$$

$i = 1, \dots, T ; j = 1, \dots, T$

We are now ready to establish:

The Internal Component of the Invariant Imbedding Relation

Let X_n be a given discrete space. Partition X_n into two subsets X_p and X_q ; hence $X_p \cap X_q = \phi$ (the empty set), $X_p \cup X_q = X_n$ and $p+q=n$. If the principle of local interaction holds on X_n , then by its divisibility property it holds on the quotient space $P_2 = (X_p, X_q)$ and the system (20) assumes a particularly simple structure in the form of the following set of four statements:

$$N(p|q) = N(q|p) \sum_M (q|p|q) + N(p|p) \sum_M (p|p|q) \quad (21)$$

$$+ N^0(p) \sum^0(p|q)$$

$$N(q|p) = N(p|q) \sum_M (p|q|p) + N(q|q) \sum_M (q|q|p) \quad (22)$$

$$+ N^0(q) \sum^0(q|p)$$

$$N(p|p) = N(p|p) \sum_M (p|p|p) + N(q|p) \sum_M (q|p|p) \quad (23)$$

$$+ N^0(p) \sum^0(p|p)$$

$$N(q|q) = N(q|q) \sum_M (q|q|q) + N(P|q) \sum_M (P|q|q) \quad (24) \\ + N^o(q) \sum^o(q|q)$$

The internal component of the invariant imbedding relation is by definition concerned with the analytical relation between the pair: $[N(P|P), N(q|P)]$ or the pair $[N(q|q), N(P|q)]$. In particular the relation requires the characterization of $N(P|P)$ (or $N(q|q)$) in terms of $N(q|P)$ (or $N(P|q)$). If the relation can be established between the members of either pair, then by the arbitrariness of the partition $P_2 = (X_P, X_q)$ it will be established also for the other pair. An examination of (21)-(24) shows that the required relation follows immediately from (23). For we may write:

$$N(P|P)(I - \sum_M (P|P|P)) = N(q|P) \sum_M (q|P|P) + N^o(P) \sum^o(P|P).$$

Since the linear transformation $\sum_M (P|P|P)$ is norm contracting (which follows from (16) and the relevant discussions in reference 1), the transformation $[I - \sum_M (P|P|P)]$ has an inverse. We define:

$$Q(q|P|P) = \sum_M (q|P|P) [I - \sum_M (P|P|P)]^{-1} \quad (25)$$

$$\mathcal{O}^{\circ}(P|P) = \Sigma^{\circ}(P|P) [\mathbf{I} - \Sigma_M(P|P|P)]^{-1}, \quad (26)$$

Whence:

$$N(P|P) = N(q|P) \mathcal{O}(q|P|P) + N^{\circ}(P) \mathcal{O}^{\circ}(P|P), \quad (27)$$

which is the desired form of the internal component of the invariant imbedding relation on a discrete space X_n with partition (X_p, X_q) .

The physical significance of (27) is clear: $N(P|P)$ represents the specific radiance distributions associated with points within the subspace X_p of X_n . $N(q|P)$ represents the specific radiance distributions associated with the remaining points of X_n such that the flux is directed from points of X_q to those of X_p . The mapping $\mathcal{O}(q|P|P)$ shows how the impinging radiation from X_q onto X_p generates the radiance distribution within X_p . If X_p has no sources, then (27) reduces to the simple linear relation:

$$N(P|P) = N(q|P) \mathcal{O}(q|P|P). \quad (28)$$

An examination of the set (21)-(24) shows that in deriving (27) (and its counterpart involving $N(Q|Q)$), we have made use of only half of the relations in that set namely (23) and (24). The remaining set of relations that can be derived are those between the ordered pairs $[N(P|Q), N(Q|P)]$, $[N(Q|P), N(P|Q)]$. Clearly, by the symmetry of the partition, establishing the required relation between the members of either of these pairs automatically does the same for the other. The relation that governs these pairs is known as:

The External Component of the Invariant Imbedding Relation

The external component may be established by starting with, say, (21) in which, by means of (23), $N(P|P)$ has been represented in terms of $N(Q|P)$. In other words we can make use of (21) and (27) to find:

$$N(P|Q) = N(Q|P) \sum_M (Q|P|Q) + [N(Q|P) \mathcal{Q}(Q|P|P) + N^{\circ}(P) \mathcal{Q}^{\circ}(P|P)] \times \\ \times \sum_M (P|P|Q) + N^{\circ}(P) \sum^{\circ}(P|Q)$$

We set

$$\mathcal{Q}(Q|P|Q) = \sum_M (Q|P|Q) + \mathcal{Q}(Q|P|P) \sum_M (P|P|Q), \quad (29)$$

$$\mathcal{Q}^{\circ}(P|Q) = \sum^{\circ}(P|Q) + \mathcal{Q}^{\circ}(P|P) \sum_M (P|P|Q), \quad (30)$$

so that we may write the desired external component as:

$$N(P|Q) = N(Q|P) Q(Q|P|Q) + N^o(P) Q^o(P|Q). \quad (31)$$

If no sources impinge directly on X_P we have the simple linear relation:

$$N(P|Q) = N(Q|P) Q(Q|P|Q). \quad (32)$$

Relations (27) and (31) together completely characterize the radiance distributions within X_P in terms of those in the complementary subset X_Q irradiating X_P .

The external component relation (31) allows a formal solution of $N(P|Q)$ in terms of the boundary source conditions on the members of the partition of X_n : for by interchanging P and Q in (31) we obtain the new relation:

$$N(Q|P) = N(P|Q) Q(P|Q|P) + N^o(Q) Q^o(Q|P), \quad (33)$$

which, together with (31) yields:

$$N(P|Q) = \frac{N^o(P) Q^o(P|Q) + N^o(Q) Q^o(Q|P) Q(Q|P|Q)}{[I - Q(P|Q|P) Q(Q|P|Q)]} \quad (34)$$

The existence of the inverse of the denominator-transformation is guaranteed by the general existence of uniqueness of N_+ established in reference 1.

We now may assemble the two preceding components; the result will then be:

The Invariant Imbedding Relation

Starting with the partition (X_p, X_q) of X_n and the principle of local interaction, presumed to hold on X_n , we have derived the internal and external components of the invariant imbedding relation which are summarized in Equation (27) and (31) respectively. It is now a simple matter to assemble these components to form the desired invariant imbedding relation. We begin by defining the radiance vector $N(P|Q)$:

$$N(P|Q) = [N(P|P), N(P|Q)] \quad (35)$$

In order to weld statements (27) and (31) into a single statement, we now follow the lead of (35) and define the new linear operators in matrix form:

$$\mathcal{M}(q|P|n) = \begin{pmatrix} \mathcal{P}(q|P|P) & 0 \\ 0 & \mathcal{Q}(q|P|q) \end{pmatrix} \quad (36)$$

$$\mathcal{M}^{\circ}(P|n) = \dots [\mathcal{P}^{\circ}(P|P), \mathcal{Q}^{\circ}(P|q)] \quad (36a)$$

Then it follows from (35), (36) and the two components (27) and (31) that:

$$N(P|n) = N(q|P) \mathcal{M}(q|P|n) + N^{\circ}(P) \mathcal{M}^{\circ}(P|n) \quad (37)$$

which is the desired invariant imbedding relation.

AND THE PRINCIPLES OF INVARIANCE

In the interests of brevity and concreteness we choose to derive the principles of invariance in the following special setting:

Let X_n be a cubic lattice, i.e.,

$$X_n = \left\{ (i, j, k) : |i|, |j| \leq c, a \leq k \leq b \right\},$$

where a, b, c and i, j, k take on only finite integral values.

Hence $n = (2c+1)^2(b-a+1)$. We now may reproduce, formally, in the discrete-space context, the statement of the invariant imbedding relation given in reference 2.

Partition X_n into two subsets X_p and X_q such that:

$$X_p = \left\{ (i, j, k) : |i|, |j| \leq c, x \leq k \leq z, [x, z] \subset [a, b] \right\}.$$

Hence $p = (2c+1)^2(z-x+1)$, where x, z are integers. Define $X_q = X_n - X_p$.

Thus, X_p is a slab imbedded in X_n ---like the meat in a sandwich--- with X_q acting as the two slices of bread.

Now select some level y in X_p such that* $a \leq x \leq y \leq z \leq b$. The remainder of the discussion will be a formalization of the

* We use $x y z$ notation at this stage instead of $i j k$ notation in order to allow a close notational contact between the present work and the continuous theory of reference 2.

following simple idea: For a given level y in X_p break $N(P|n)$ into two parts $N_+(y)$ and $N_-(y)$, where $N_+(y)$ directs radiance into all levels of X_n on or above level y and $N_-(y)$ directs radiance into all levels of X_n below y . Also break $N(Q|P)$ into two parts: $N_+(z)$ and $N_-(z)$ which represent, respectively, radiance incident on X_p from all levels of X_n below X_p , and all levels of X_n above X_p . By breaking up the vectors $N(P|n)$ and $N(Q|P)$ into two parts in the above manner, we automatically determine a cleavage of $\mathcal{M}(Q|P|n)$ into four pieces whose dimensions are governed by the various lengths of the four newly manufactured radiance components. The four fragments of the cleavage of $\mathcal{M}(Q|P|n)$ turn out to be none other than the discrete counterparts to the complete reflectance and transmittance operators introduced ad hoc in reference 2. We now turn to the formal details.

Step 1: The vector $N(P|n)$ clearly has ρn components ($= \rho^2 + \rho q$) of the form N_{ij} , $x_i \in X_p$, $x_j \in X_n$. We now partition X_n once again. This time let $X_n(y, +)$ be the set of points of X_n whose \bar{x} -coordinates are $\leq y$. Let $X_n(y, -)$ be the remaining points of X_n , i.e., those points whose \bar{x} -coordinates are $> y$. Rearrange the components of $N(P|n)$ into two groups so that the first group, consisting of, say, k components, are of the form N_{ij} , where $x_j \in X_n(y, +)$; and the remaining l components are of the form N_{ij} , $x_j \in X_n(y, -)$ so that $k + l = \rho n$. The exact order of occurrence of these components in each of these two groups is immaterial. The important point to observe is that there exists a $\rho n \times \rho n$ permutation matrix $Q_{\rho n}$ which operates on $N(P|n)$ to

yield a vector $V(P|n)$ which is partitioned in the desired way:

$$V(P|n) = N(P|n) Q_{Pn} .$$

That is,

$$V(P|n) = \left[\underbrace{N_{i_1 j_1}, \dots, N_{i_k j_k}}_{\alpha_{j_i} \in X_n(y, +)} , \underbrace{N_{\alpha_1 \beta_1}, \dots, N_{\alpha_\ell \beta_\ell}}_{\alpha_{\beta_i} \in X_n(y, -)} \right] . \quad (38)$$

It is convenient to set

$$N_+(y) = \left[N_{i_1 j_1}, \dots, N_{i_k j_k} \right] , \quad (39)$$

$$N_-(y) = \left[N_{\alpha_1 \beta_1}, \dots, N_{\alpha_\ell \beta_\ell} \right] , \quad (40)$$

so that

$$V(P|n) = \left[N_+(y), N_-(y) \right] . \quad (41)$$

Step 2: We now split up $N(q|p)$. Let $X_q(\bar{z}, +)$ be the set of points of X_q whose \bar{z} -coordinates are $> \bar{z}$, and let $X_q(x, -)$ be the remaining points of X_q (hence, those whose \bar{z} -coordinates are $< x$). Rearrange the pq components of $N(q|p)$ so that the first r elements of the rearrangement are of the form N_{ij} , $x_i \in X_q(\bar{z}, +)$, and the remaining Δ elements are of the form N_{ij} , $x_i \in X_q(x, -)$ where $r + \Delta = pq$. The order of the N_{ij} in each of the two groups is immaterial. The important point to observe is that there exists a $pq \times pq$ permutation matrix Q_{pq} which operates on $N(q|p)$ to yield a vector $V(q|p)$ which is partitioned in the desired way:

$$\begin{aligned}
 V(q|p) &= N(q|p) Q_{pq} = \\
 &= \left[\underbrace{N_{a_1 b_1}, \dots, N_{a_r b_r}}_{x_{a_i} \in X_q(\bar{z}, +)}, \underbrace{N_{c_1 d_1}, \dots, N_{c_\Delta d_\Delta}}_{x_{c_i} \in X_q(x, -)} \right]. \quad (42)
 \end{aligned}$$

It is convenient to set

$$N_+(\bar{z}) = \left[N_{a_1 b_1}, \dots, N_{a_r b_r} \right], \quad (43)$$

$$N_-(x) = \left[N_{c_1 d_1}, \dots, N_{c_\Delta d_\Delta} \right], \quad (44)$$

so that

$$V(q|P) = [N_+(z), N_-(x)] , \quad (45)$$

Step 3: Returning now to the invariant imbedding relation (37) we insert the $pq \times pq$ identity matrix, in the form $Q_{pq} Q_{pq}^{-1} = I$ between $N(q|P)$ and $\mathcal{M}(q|P|n)$, and operate on each side of (37) with Q_{pn} . The result is:

$$N(P|n) Q_{pn} = N(q|P) Q_{pq} Q_{pq}^{-1} \mathcal{M}(q|P|n) Q_{pn} + N^o(P) \mathcal{M}^o(P|n) Q_{pn} ,$$

which may be written as

$$[N_+(y), N_-(y)] = [N_+(z), N_-(x)] \mathcal{M}(x, y, z) + N^o(P) \mathcal{M}^o(y)$$

where we have set:

$$\mathcal{M}(x, y, z) = Q_{pq}^{-1} \mathcal{M}(q|P|n) Q_{pn} ,$$

$$\mathcal{M}^o(y) = \mathcal{M}^o(P|n) Q_{pn} .$$

The vector pairs $[N_+(y), N_-(y)]$ and $[N_+(z), N_-(x)]$ induce a natural cleavage of $\mathcal{M}(x, y, z)$ into four sub-blocks:

$$m(x, y, z) = \begin{pmatrix} \mathcal{T}(z, y, x) & \mathcal{Q}(z, y, x) \\ \mathcal{Q}(x, y, z) & \mathcal{T}(x, y, z) \end{pmatrix}. \quad (46)$$

The dimensions of $\mathcal{T}(z, y, x)$ are $r \times k$; those of $\mathcal{Q}(z, y, x)$ are $r \times l$; those of $\mathcal{Q}(x, y, z)$ are $l \times k$ and those of $\mathcal{T}(x, y, z)$ are $l \times l$. Hence the desired form of the invariant imbedding relation for the cubic lattice X_n is:

$$\begin{aligned} [N_+(y), N_-(y)] &= [N_+(z), N_-(z)] \begin{pmatrix} \mathcal{T}(z, y, x) & \mathcal{Q}(z, y, x) \\ \mathcal{Q}(x, y, z) & \mathcal{T}(x, y, z) \end{pmatrix} \\ &+ N^o(p) m^o(y). \end{aligned} \quad (47)$$

This completes the derivation of the special form of the invariant imbedding relation for a cubic lattice X_n .

We now have essentially reached our goal. The remaining steps leading to the particular forms of the principles of invariance have been given in detail in reference 2. Thus the operators $\mathcal{T}(x, y, z)$ and $\mathcal{Q}(x, y, z)$ are the complete transmittance and reflectance operators which yield the the appropriate standard R and T operators for every subset X_p of the discrete space X_n , where X_p is of the form defined above.

ET CETERA

The preceding discussion has shown in detail some consequences of the local interaction principle in discrete spaces. In particular it was shown that the local interaction principle possessed the divisibility property - i.e., that it held not only on a point level in a given discrete space, but also on the set level in that space. This divisibility property, therefore, opens the principle to applications on hierarchies of discrete spaces generated from some given discrete space, a fact which allowed the derivation of the important invariant imbedding relation for such spaces, and hence a derivation of the corresponding principles of invariance.

As far-reaching as the preceding simple set of consequences of a discrete-space approach to radiative transfer problems appears to be, these consequences barely begin to tap the rich store of theoretical and practical possibilities inherent in the approach. We list below some further possibilities which may be examined individually or in various combinations in the discrete-space context. A perusal of the list will make it quite clear that the discrete-space methods show promise in unearthing new and deeper insights into their relatively intractable continuous-space counterparts.

1. Time Dependent Formulations. For completeness, the steady-state formulations developed above should be rounded out by a time-dependent formulation of the principle of local interaction. This is a routine matter, for the only new wrinkle that appears is the explicit introduction of time into the $N_{\lambda j}$ components and the Σ function along with appropriate cognition of retardation effects induced by the finite speed of propagation of radiant energy:

$$N_{i j}(t) = \sum_{k=1}^n N_{k i}(t - \tau_{k i}) \Sigma(x_i, t; \xi_{k i}; \xi_{i j}) + \sum_{k=1}^{n_i} N_{k i}^{\circ}(t) \Sigma^{\circ}(x_i, t; \xi_{k i}^{\circ}; \xi_{i j}), \quad (48)$$

where $\tau_{k i} = |x_k - x_i|/v_{k i}$, $v_{k i}$ being the average speed of propagation along the path defined by the points x_k and x_i . What is not routine, however, but still more tractable than the general continuous case, is the solution procedure for the corresponding time-dependent radiance distribution $N_{+}(t)$ over the space X_n .

2. Eigenvalue Problems in Radiative Transfer Theory

Recent mathematical work in transport theory has turned up some surprises for physicists who, steeped in the eigenfunction lore of classical mechanics, expected quick and easy victories over transport problems by the straightforward application of these classical methods to the latter problems. The inapplicability of eigenfunction techniques within even the simplest of geometrical settings for transport phenomena blocked their theories from any sweeping assaults by such methods. Beyond the initial mathematical

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spade-work⁴ in eigenvalue studies of transport phenomena, and a few subsequent related studies^{5,6} no essential further progress has been made to date. This is because the mathematical difficulties in the continuous case for general geometrical and physical settings (e.g., bounded curvilinear media with non-isotropic scattering) are currently insuperable. By means of the easily formulated and solved discrete-space equations, however, these analytical barriers may be by-passed; and even though there is a considerable conceptual and methodological gap between the continuous and discrete formulations, this gap may yet not be so wide as to prevent some penetrating insight into the continuous-space problems which might give rise to a hint of the necessary approaches that must be developed to break the current barriers in the continuous case.

Thus in the time-dependent formulation of the local interaction principle, we may direct attention to solutions of (48) in the form

$$N_{ij}(t) = N_{ij} \phi(t)$$

where ϕ is some ring homomorphism (e.g., $\phi = \exp$) on the reals into the reals, so that we are led by means of (48) to consider systems of equations of the form:

$$N_{ij} = \sum_{k=1}^n N_{k\lambda} \phi(-\tau_{k\lambda}) \Sigma(x_i; t; \xi_{k\lambda}; \xi_{ij}) + \sum_{k=1}^{4\lambda} \left[\frac{N_{k\lambda}^0}{\phi(t)} \right] \Sigma^0(x_i; t; \xi_{k\lambda}^0; \xi_{ij}), \quad (49)$$

which, under suitable initial conditions $N_{\kappa i}^0 = 0$, $t > 0$ reduces to an eigenvalue problem of the following kind (using the notation of reference 1) where Φ is an "eigen matrix" involving the quantities $\phi(-\tau_{\kappa i})$:

$$N_+ (I - M\Phi\Sigma) = 0. \quad (50)$$

3. Polarized Radiation on Discrete Spaces. In order to round out the discrete-space theory to include a corresponding description of polarized light fields we must evidently start with the principle of local interaction and elevate it from the simple scalar level, used throughout the preceding discussions, to the appropriate vector level for the polarized discrete-space formulation. The development of the polarized version, even in the simplest form, requires more space than is conveniently available here, and is therefore reserved for subsequent study. We merely remark in passing that in such a development the components $N_{\kappa j}$ are replaced by their Stokes vector counterparts $\underline{N}_{\kappa j}$ and $\Sigma(\alpha_{\kappa i}; \xi_{\kappa i}; \zeta_{\kappa i})$ is replaced by a phase matrix which relates $\underline{N}_{\kappa i}$ to $\underline{N}_{\kappa j}$. The general linear structure of the principle of local interaction is unchanged in the transition from the scalar to the vector level. This program of development for the discrete space may be patterned after that in the corresponding continuous-space theory.³

4. Reciprocity Principles. An optical reciprocity principle is in a general sense a statement about the spatial or temporal reversibility of certain optical phenomena. In radiative transfer theory the reciprocity principles are concerned with such things as:

(a) the reversibility of the path of a beam of light in an optical medium, (b) the invariance of beam transmittance of a path under

reversal of the directional sense of the path, (c) the interchangeability of directions of incidence and reflection (or incidence and transmission) in formulae describing the response of the members of very general sets of optical media to optical signals.

By means of the discrete-space formulation of radiative transfer theory, for example by means of such relations as the partition principle (32), a detailed and exhaustive study of reciprocity principles can be made which could shed much light on the corresponding problems of the continuous theory. Even from the general form of the partition principle as it stands in (32), it is quite clear that the validity of a reciprocity principle is a very special and rare event in general media of arbitrary local and global structure. Since reciprocity principles are ultimately derivable from microstructure considerations using local interaction principles, the present discrete-space formulations are particularly suitable for such studies in the discrete context.

5. Equations of transfer and Nets of Discrete Spaces

The principle of local interaction is a set of linear equations, exact in its own context, which has been shown to be fruitful in the reproduction of the discrete-space counterparts of the principles of invariant imbedding and invariance. It should be noted that the local interaction principle can also give rise to the discrete space counterpart of the (integro differential) equation of transfer and its various derived forms. (Conversely, the equation of transfer can, by suitable modification, give rise to the local interaction principle.) By themselves the formulations of discrete versions of

the basic transfer equations may lead to no further numerical results than those obtained by the local interaction equations; however, these discrete forms of the transfer equations considered not on a single discrete space but rather a net of discrete spaces may lead to still further conceptual insight into the mathematical workings of the continuous formulations. A net* of discrete spaces is essentially a partially ordered set \mathfrak{X} of discrete spaces $X_j, j=1,2,\dots$, ordered, say, by set inclusion \subset , so that a relation of the form $X_m \subseteq X_n$ is meaningful for certain sets D of integers $m, n \in D$, and corresponding members X_m, X_n of \mathfrak{X} . In this way the notion of a convergent net of discrete spaces which converges to a continuous space X may be evolved with an attendant notion of the convergence of a "discrete-space theory on X_n " to a "continuous theory on X ". Thus any sets of solutions or sets of theorems holding for members of a net \mathfrak{X} may be examined for indications of convergence to their possible continuous analogs as $X_n \rightarrow X$.

6. The Equivalence of the Methodologies of Invariant Imbedding and the Principles of Invariance.

The principle of invariant imbedding on the one hand, as devised by Bellman and Kalaba⁷, is a rule of action which leads to the mathematical formulation of a wide class of transport problems. In essence the principle directs the theorist to imbed a given space X_0 (in which a transport problem is being considered for solution)

* The notion of a net occurs in general discussions of convergence theory. See, c.g., reference 8.

into a space X of a generally similar kind that includes X_0 as a proper subset. The analytical representation $\Psi(X_0)$ of a transport phenomenon (perhaps a reflectance or transmittance of X_0) which was of interest in X_0 is now considered in its new environment X , and is represented whenever possible as a functional relation on the sets X and $X-X_0$. The end result is some analytical connection $F(\Psi(X), \Psi(X-X_0))=0$ between the representations $\Psi(X)$ and $\Psi(X-X_0)$, which may be solved analytically for $\Psi(X)$; and hence (because of the generality of the imbedding) for $\Psi(X_0)$.

On the other hand, the principles of invariance as expounded by Chandrasekhar³ and others² are rules of action which also lead to the mathematical formulation of a wide class of transport problems. In essence the principles direct the theorist to partition X_0 into two generally non-trivial subsets X_p and X_q (hence $X_0 = X_p \cup X_q$ and $X_p \cap X_q = \phi$). The phenomenon of interest on X_0 is now considered on X_p and X_q and its analytical representations $\Psi(X_p)$, $\Psi(X_q)$ are related, according to the principles, by a well-defined analytical connection $G(\Psi(X_p), \Psi(X_q))=0$ which may lead to a solution of $\Psi(X_p)$ and $\Psi(X_q)$ and hence (because of the generality of the partition) for $\Psi(X_0)$.

The preceding descriptions of the two widely used current methodologies in general transport theory were rendered in their exhibited forms to point up their ostensibly parallel-structures. However, the parallelism goes even deeper when viewed from a strictly mathematical vantage point. In fact, from the latter vantage point the two methodologies can be shown to be equivalent. This would be accomplished in much the same way as the outer and inner direct

products of isomorphic spaces are shown to be isomorphic in the domain of modern algebra. In this procedure, the role of the "outer direct product" would be assigned to the invariant imbedding approach, and that of "inner direct product" to the principle of invariance approach.

Except for some minor details the ground work for the task of demonstrating the equivalence of the methodologies of invariant imbedding and the principles of invariance has been essentially accomplished on the discrete-space level in the preceding sections on the derivation of the invariant imbedding relation and the principles of invariance. The proof of the equivalence of the methodologies in continuous settings may be attempted by employing the notions of convergent nets of discrete spaces.

7. Transpectral Scattering. The law of monochromatic scattering as given by the volume scattering function σ refers strictly to scattering of a given incident monochromatic sample of flux without change in wave length. That is, σ gives, e.g., the angular distribution of the yellow light scattered from a yellow beam of incident light, etc. However, fluorescent phenomena, and other general absorption-reemission phenomena which we shall term collectively as transpectral scattering phenomena are also encountered in practical everyday work in radiative transfer. The phenomenon of transpectral scattering is not explicitly formulated in the present day theory. The existence of transpectral scattering is, however, given some recognition by means of the emission function N_η and the absorption function Q in the equation of transfer. When exhibited

this way in the equation of transfer, the phenomena of emission, absorption and scattering appear to spring from fundamentally different mechanisms. While this may at times be a convenient representation of these mechanisms, it gives the theory an artificial posture and, more importantly, closes it to further connections with the electromagnetic field theories of radiant energy.

This state of affairs can be remedied by introducing a transpectral volume scattering function Σ defined on $X_n \times (\Xi \times \Lambda) \times (\Xi \times \Lambda)$ where Λ is the non negative real line representing the spectrum. The value $\Sigma(x; \xi, \lambda'; \xi, \lambda)$ of Σ would be interpreted as follows: A point x is irradiated from the direction ξ' with radiant flux of wave length λ' . The function Σ then gives the fraction of the radiant flux of wavelength λ scattered by x in the direction ξ , and leads to the transpectral form of the local interaction principle; the details, however, are too long for presentation here, and are reserved for a subsequent note.

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