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RADIATIVE TRANSFER ON A LINEAR LATTICE

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November 1959
Index Number NS 714-100

Bureau of Ships
Contract NObs-72092

SIO REFERENCE 59-67

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SIO Ref. 59-67

Radiative Transfer on a Linear Lattice

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INTRODUCTION

In this report the two-flow problem in a plane-parallel optical medium is studied by means of the tools of discrete-space radiative transfer theory. The end result is an explicit solution in the discrete-space context of the general two-flow problem for an arbitrarily stratified source-free plane-parallel medium with arbitrary upper and lower boundary source conditions.

In carrying through the details of the solution of the general two-flow problem within the discrete-space context the following three purposes are accomplished: (a) To demonstrate the fact that the discrete-space theory leads to solutions of certain classes of transfer problems which are intractable in the continuous theory; (b) To exhibit a particularly simple and relatively concrete example of the abstract general concepts of the discrete space theory of radiative transfer presented earlier.[1,2,3] Finally, (c) to prepare the ground work for a computation program leading to the numerical tabulation of the light field in various linear lattices. Before embarking on the systematic realization of these purposes, it may be helpful to the reader to point out some of the details of their background.

Physical Aspects of the Two-Flow Problem

The physical origin of the two-flow radiative transfer problem may be described as follows. Consider any of the following three optical media: A planetary or stellar atmosphere, or a planetary hydrosphere. Imagine the medium to be partitioned into concentric spherical shells. At the common spherical boundary surface between any two contiguous shells, the radiation flow across the boundary is conceptually divided into two streams: an outward (or upwelling) flow and an inward (or downwelling) flow. The problem then is to find some analytical representation of these two flows which connects them with the given boundary lighting conditions, the inherent optical properties of the medium, and the location of the point on the surface across which the flows take place.

The highly localized character of radiative transfer activity and the lateral extensiveness of the above spherical media on which the activity takes place may be used to show that the effects of the sphericity of the media are negligible except in the most demanding radiometric measurements.[4]

Hence the relatively complex spherical-parallel partitioning of these media can be replaced by the mathematically more tractable and physically simpler plane-parallel partitioning.

It can be shown that the <u>exact</u> equations governing the two-flow problem in an arbitrarily stratified plane-parallel source-free medium are:[5]

$$\mp \frac{dH(z,\pm)}{dz} = -[a(z,\pm)+b(z,\pm)]H(z,\pm) + b(z,\mp)H(z,\mp),$$

where $H(z,\pm)$ are the irradiances (radiant flux per unit area) crossing a plane at depth z in the outward (+) (or upwelling) and inward (-) (or downwelling) directions, and $a(z,\pm)$, $b(z,\pm)$, are the values of the absorption and backward scattering functions at depth z for the various streams.

In the continuous theory, it is just as difficult to solve this set exactly as it is to solve the equation of transfer for radiance exactly, even though each of the radiometric quantities $H(z,\pm)$ at each depth z , generally requires two less variables in its description than the radiance quantity $N(z,\theta,\phi)$. This unpleasant state of affairs is traceable to the fact that the correct definitions of the functions $a(\cdot,\pm)$ and $b(\cdot,\pm)$ must take into account not only the possible spatial variations of the inherent optical properties of the medium, but also the spatial and angular variations of the radiance distributions within the medium.[5]

It has been found possible to ignore with a fair measure of impunity the second of these complicating factors in the two-flow equations, especially in the planetary hydrosphere problem. That is, to a first approximation, the effects of the change of the angular structure of the radiance distributions in natural waters may be neglected in the description of the absorption and backward scattering functions $a(\cdot,\pm)$, and $b(\cdot,\pm)$.[5] However, the effects of general inhomogeneities of the inherent optical properties of a medium cannot be completely erased from the functions $a(\cdot,\pm)$, and $b(\cdot,\pm)$, even if the clever ruse of changing geometrical depth z to optical depth τ is employed. It is precisely this latter complication which gives rise to the analytical intractability of the set of irradiance equations under more realistic physical settings, but which does not in the slightest impede the discrete-space formulations and solution of the two-flow problem. This important feature of the discrete-space formulation will be brought out in detail in the main discussion below.

Geometrical Aspects of the Two-Flow Problem

By adopting the plane-parallel geometrical setting, the general two-flow equations are reduced to a strictly one-dimensional formulation. The three dimensional aspects of the "plane-parallel" setting are therefore spurious in the sense that the lateral extensions of the parallel planes which separate the various layers of the medium play an inessential role in the formulation and solution of the transfer problem. Thus, as far as the continuous geometrical setting is concerned, it is possible to represent the medium as a vertical cylinder of unit horizontal cross section of arbitrary shape. The discrete space formulation also explicitly recognizes this possibility and, in its characteristically economical manner, further reduces the vertical cylinder to an equivalent (in the sense made clear below) discrete set of points equally spaced along a vertical line.

When the points of a discrete space are arrayed along a straight line, it is called a <u>linear lattice</u>. A linear lattice, then, is the natural discrete-space setting for the associated two-flow problem. Therefore, a linear lattice, in its simplest geometric guise, is a finite, bounded set of points on a straight line segment in Euclidean three-space. Such a discrete space is easily comprehended; and the analytical representation of the transfer process on it should be relatively free of symbolic traffic snarls. However, this special space is still complex enough to exhibit in non-trivial form the more important concepts of general discrete-space transfer theory.

When viewed in this way, the linear lattice formulation of the two-flow problem and, indeed, the discrete space formulation of the general radiative transfer problem, may appear overly simple and perhaps even shallow. However, it might be well to point out again that the hierarchical aspects of discrete spaces and their attendant concepts (such as the divisibility property of the local interaction principle) endow the primitive geometric notion of a discrete space with an unexpectedly great logical depth and its theory with a correspondingly wide domain of applicability.[2]

It therefore seems appropriate, especially in view of the deceptively simple appearance of the geometric aspects of the formulations, to choose the simplest of geometrical realizations of a discrete space, namely a linear lattice, and to unfold its unexpectedly rich and intricate inner structure layer by layer. The details of this process will now be considered.

THE LINEAR LATTICE

Consider the following subset X_n of E_3 (Figure 1):

$$X_n = \left\{ (x,y,z) : x = y = 0 \text{ , } 1 \leq z \leq n \text{ ; } x,y,z,n = \text{integers} \right\} .$$

Here the ordered triple of integers (x,y,z) represents a point in E_3. Therefore X_n consists of the first n integral points along the z-axis of E_3. X_n is a <u>linear lattice of length n</u>. If we let $x_j = (0,0,j)$, $j = 1, \dots, n$, then in the notation of the general theory $X_n = \{ x_j : j = 1, \dots, n \}$. The latter representation of X_n allows us to apply directly all the notations and conventions of the basic theory of discrete spaces developed in reference 1. In particular, we see that there are only two unit vectors in the local direction space of each point, namely $\xi_+ = k$ and $\xi_- = -k$ (Figure 1). It follows that the eclipse convention limits the specific radiance distribution at each x_j to two generally non-zero values: $N(x_j, \xi_+)$ and $N(x_j, \xi_-)$ which, in the interests of brevity, we will write as $N(j,+)$ and $N(j,-)$, respectively. The notation for the local scattering function Σ also becomes simpler on the linear lattice: $\Sigma(x_j ; \xi_+ ; \xi_-)$ can be written with less effort as $\Sigma(j ; + ; -)$. There are generally only three other possibilities at x_j, namely $\Sigma(j ; + ; +)$, $\Sigma(j ; - ; -)$, and $\Sigma(j ; - ; +)$.

THE LOCAL INTERACTION PRINCIPLE ON A LINEAR LATTICE

In this work we shall limit all sources on X_n to be incident on x_1 and x_n only, and that $N^0(x_1, \xi_-)$ and $N^0(x_n, \xi_+)$ only will be considered, which are arbitrary but fixed. In a subsequent work, the general problem of internal sources will be studied on an arbitrary extended cubic lattice. The local interaction principle now becomes:

$$\boxed{\begin{aligned} N(j, \pm) &= N(j-1, -)\cdot \Sigma(j\,; -\,; \pm) \\ &\quad + N(j+1, +)\cdot \Sigma(j\,; +\,; \pm)\,, \qquad j = 1, \ldots, n\,. \end{aligned}} \tag{1}$$

Expression (1) is actually two statements: the first is obtained by reading the upper signs, the second by reading the lower signs together. With the convention that $N(0,-)$ is identified with $N^0(x_1, \xi_-)$, and that $N(n+1,+)$ is identified with $N^0(x_n, \xi_+)$, (1) becomes a particularly simple and compact representation of the local interaction principle on the linear lattice along with the hypothesized boundary conditions.

HIERARCHIES OF LINEAR LATTICES

In an earlier work [2] we defined <u>hierarchy of discrete spaces</u> to be a finite sequence $\mathcal{H}_\alpha(X_n)$ of quotient spaces say Y_{r_i}, such that $Y_{r_{i+1}}$ is the quotient space of Y_{r_i} : $\mathcal{H}_\alpha(X_n) = \{X_n, Y_{r_1}, \ldots, Y_{r_\alpha}\}$. Recall that a quotient space is obtained from a given space X_n by first partitioning X_n into a set of mutually disjoint subsets $y_{11}, \ldots, y_{1r}$ whose union is X_n. Then, considering each subset y_{1i} as a "point", form the discrete space $Y_{r_1} = \{y_{11}, \cdots, y_{1r_1}\}$. Y_{r_2} is then formed from Y_{r_1} by repeating this process of partitioning and point identification.

As an example, consider the linear lattice X_n defined above. Suppose $n = 100$. Then let $y_{11} = (x_1, x_2)$, $y_{12} = (x_3, x_4)$, . . . , $y_{1,50} = \{x_{99}, x_{100}\}$. Hence r_1 in this case is equal to 50 and $Y_{50} = \{y_{11}, \cdots, y_{1,50}\}$. Then Y_{50} is a discrete space of fifty elements: y_{1i}, $i = 1, \cdots, 50$.

We can extend the preceding example by defining a new space $Y_{r_2} = Y_{25} = \{y_{21}, y_{22}, \cdots, y_{2,25}\}$ whose elements are: $y_{21} = \{y_{11}, y_{12}\}, \ldots, y_{2,25} = \{y_{1,49}, y_{1,50}\}$. Then Y_{25} is a discrete space whose i th element or "point" is of the form $y_{2,i} = \{y_{1,2i-1}, y_{1,2i}\}$, where $y_{1,i}$ is an element of $Y_{r_1} = Y_{50}$, which in turn is an ordered pair of points of X_n. In this way we have a second-order hierarchy $\mathcal{H}_2(X_{100}) = \{X_{100}, Y_{50}, Y_{25}\}$ constructed from the linear lattice X_{100}.

On the basis of the results of reference 2 we can assert that, if the local interaction principle holds on X_{100}, it holds on Y_{50}, and then in turn also on Y_{25}. As far as the local interaction principle is concerned, the function Σ could just as well take the form $\Sigma(x_j ; \xi_+ ; \xi_-)$ or $\Sigma(y_{1j} ; \xi_+ ; \xi_-)$, or $\Sigma(y_{2j} ; \xi_+ ; \xi_-)$, where $x_j \in X_{100}$, $y_{1j} \in Y_{50}$, $y_{2j} \in Y_{25}$. That is, a "point" to the local interaction principle is a relative concept; it has no absolute meaning. In one context x_j could be a point, in the usual sense, of "point of E_3". Then y_{1j} would actually be a pair of points of E_3, and y_{2j} would then be a pair of a pair of points of E_3. In still another context, x_j could be in actuality a slab in E_3, say the slab defined by $z: j - \epsilon < z < j + \epsilon$, $0 < \epsilon \leq 1$, $j = 1, \ldots, n$. Then y_{1j} could be various finite sets of such slabs x_j, and y_{2j} would be various finite sets of such sets y_{1j}, and so on. The generality inherent in the local interaction principle and the notion of a discrete space is therefore quite high.

The Concept of Polarity

Physicists are aware of the term <u>isotropy</u> which designates the invariance, under certain rotation groups, of given physical properties of matter or geometrical properties of space. In radiative transfer theory we say that an optical medium is <u>isotropic at a point</u> x if $\sigma(x; U\xi'; U\xi) = \sigma(x; \xi'; \xi)$ for every unitary transformation* U of the direction space Ξ.

As an example of isotropy, let us consider the linear lattice X_n. The four general values Σ may take on at x_j are: $\Sigma(x_j; \xi_+; \xi_-)$, $\Sigma(x_j; \xi_+; \xi_+)$, $\Sigma(x_j; \xi_-; \xi_+)$, $\Sigma(x_j; \xi_-; \xi_-)$. Observe that a unitary transformation on the location space of X_n (i.e., on the pair of vectors (ξ_+, ξ_-)) merely interchanges the two vectors ξ_+ and ξ_-. Hence $(U\xi_+)\cdot(U\xi_-) = \xi_-\cdot\xi_+ = -1 = \xi_+\cdot\xi_-$, and so on with (ξ_+, ξ_+), (ξ_-, ξ_+), (ξ_-, ξ_-). For X_n to be isotropic at x_j then requires that (in abbreviated notation):

$$\Sigma(j; +; +) = \Sigma(j; -; -),$$

$$\Sigma(j; +; -) = \Sigma(j; -; +). \qquad (2)$$

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* A <u>unitary transformation</u> U on a vector space V with an inner product defined on $V \times V$, which we shall denote by $v_1 \cdot v_2$, has the defining property: $(Uv_1)\cdot(Uv_2) = v_1 \cdot v_2$ i.e., U preserves inner products. It may be shown that U also preserves lengths.

Now, we have introduced the idea of a hierarchy of spaces and have observed that Σ in the principle of local interaction can describe equally well the scattering properties of <u>points</u> or <u>sets of points</u>. The most frequently occurring hierarchies are the first order hierarchies (X_n, Y_p). The question of isotropy arises often in connection with such hierarchies. More precisely, and in the terminology of quotient spaces [2], we inquire as to whether the property of isotropy in X_n is divisible. If it turns out that it is not divisible, i.e., if x_j in X_n <u>possesses isotropy</u> for all $x_j \in X_n$ but some y_j in Y_p <u>does not</u> possess isotropy, then we say that y_j on Y_p <u>possesses polarity</u>. Usually, in this event, we will also say that the whole space Y_p <u>possesses polarity</u>. Until recently the property of polarity has been virtually unexplored in radiative transfer theory. However, the recognition of the presence of polarity in generally inhomogeneous plane parallel slabs has lead recently to an extensive generalization of the classical principles of invariance.[6]

Thus, in brief, polarity has to do with isotropy on the quotient space level. As would be expected, polarity of Y_p is intimately influenced by the polarity of its hierarchical predecessor X_n. However, it is quite possible for the latter to be isotropic at all points and of symmetric geometric structure but yet endow Y_p with polarity. This phenomenon will be illustrated for the case of linear lattices later in the present discussion. An illustration drawn from the continuous theory may be found in reference 6.

TWO-FLOW EQUATIONS ON A LINEAR LATTICE

We now show the close connection between the principle of local interaction and the classical Schuster equations for the two-flow analysis of the light field in a plane-parallel slab. In fact, by introducing the notion of a convergent net [2] of linear lattices we may actually pass from the present results to a continuous limit and thereby attain a derivation of the Schuster equations. In a similar manner we may derive the general equation of transfer for radiance from the local interaction principle. However, all these tasks are far beyond our immediate goals. Nevertheless it turns out that we may partially substantiate these assertions for the case of the Schuster equations as we prepare the ground work for the discussions later in this work. To do this we need the linear lattice form of the local conservation property:[1]

$$\boxed{A(x_j, \xi) + \sum_{\xi_i \in \Xi_j} \Sigma(x_j; \xi : \xi_i) = 1, \quad \xi \in \Xi_j} \qquad (3)$$

Here Ξ_j is the local direction space of x_j. $A(x_j, \xi)$ is the value of the local absorption function at x_j for an incident direction ξ. The property (3) reduces to the following two statements in the case of a linear lattice X_n:

$$A(j,\pm) + \Sigma(j;\pm;+) + \Sigma(j;\pm;-) = 1, \quad j = 1, \ldots, n, \tag{4}$$

where the upper signs are read together and lower signs are read together.

Now returning to the local interaction principle (1) and explicitly writing the expression for $N(j,\pm)$:

$$N(j,+) = N(j-1,-)\,\Sigma(j,-;+) + N(j+1)\,\Sigma(j;+;+) . \tag{5}$$

Then from (4) we may replace $\Sigma(j;+;+)$ by

$$1 - A(j,+) - \Sigma(j;+;-),$$

so that with the definition

$$\Delta N(j,+) = N(j,+) - N(j+1,+),$$

(5) becomes:

$$\Delta N(j,+) = -\left[A(j,+) + \Sigma(j;+;-)\right] N(j+1,+) + \Sigma(j;-;+)\, N(j-1,-) . \tag{6}$$

Further, from (1) again:

$$N(j,-) = N(j-1,-)\,\Sigma(j;-;-) + N(j+1,+)\,\Sigma(j;+;-) ; \tag{7}$$

and from (4):

$$\Sigma(j;-;-) = 1 - A(j,-) - \Sigma(j;-;+) ,$$

so that with the definition

$$\Delta N(j,-) = N(j,-) - N(j-1,-) ,$$

(7) becomes:

$$\Delta N(j,-) = -\left[A(j,-) + \Sigma(j;-;+)\right] N(j-1,-) + \Sigma(j;+;-)\, N(j+1,+) . \tag{8}$$

Equations (6) and (8) are the desired two-flow equations for the linear lattice. The quantities $\Sigma(j;-;+)$ and $\Sigma(j;+;-)$ play the role of backward scattering functions. If $\Delta > 0$ is the distance between x_j and x_{j+1}, define:

$$\frac{A(j,\pm)}{\Delta} \equiv a(j,\pm) ,$$

$$\frac{\Sigma(j,\pm;\mp)}{\Delta} \equiv b(j,\pm) .$$

Then (6) and (8) may be written

$$\boxed{\begin{aligned}\frac{\Delta N(j,\pm)}{\Delta} = & -\left[a(j,\pm) + b(j,\pm)\right] N(j+\Delta,\pm) \\ & + b(j,\mp) N(j-\Delta,\mp) .\end{aligned}} \qquad (9)$$

Setting $j = z$, letting $\Delta \to 0$ and assuming various regularity properties, we have, formally:

$$\frac{\Delta N(j,\pm)}{\Delta} \longrightarrow \mp \frac{dN(z,\pm)}{dz}$$

$$N(j\pm\Delta,\pm) \longrightarrow N(z,\pm) \qquad , \text{etc.}$$

so that the formal limits of the relations in (9) are

$$\mp \frac{dN(z,\pm)}{dz} = -\left[a(z,\pm) + b(z,\pm)\right] N(z,\pm) + b(z,\mp)\, N(z,\mp), \qquad (10)$$

which are evidently the <u>general</u> Schuster two-flow equations,[5] where $N(z,\pm)$ now is interpreted as the up and downwelling <u>irradiances</u> at depth z .

THE PRINCIPLES OF INVARIANCE ON A LINEAR LATTICE

In reference 2, which dealt with the derivation of the principles of invariance on a general discrete space, we went so far as to derive for illustrative purposes the explicit invariant imbedding relation for a cubic lattice in E_3 (see equation (47) of reference 2). We will now go one step further and explicitly derive from this the principles of invariance for the X_n' of the present study. In this way we give still one further specific instance of the assertion that: the principle of local interaction is a fundamental principle in modern radiative transfer theory in the sense that from it can be derived all the equations of transfer and all the principles of invariance extant in the theory.

The Requisite Form of the Invariant Imbedding Relation

The remainder of this section will depend heavily on the contents of reference 2, especially the section entitled "AND THE PRINCIPLES OF INVARIANCE." In particular we will now consider a special cubic lattice X_n in which $a = 1$, $b = n$, $c = 0$. Thus, with these dimensions, the cubic lattice of reference 2 degenerates to the linear lattice X_n of the present study. We will also retain the general partitioning of X_n into X_p and X_q (See Figure 2). The main point to observe is that the invariant imbedding relation (47) of reference 2, of course, still holds in this more special context. For convenience, we reproduce the statement here:

$$[N_+(y), N_-(y)] = [N_+(z), N_-(x)] \begin{pmatrix} \mathcal{T}(z,y,x) & \mathcal{R}(z,y,x) \\ \mathcal{R}(x,y,z) & \mathcal{T}(x,y,z) \end{pmatrix} + N^\circ(P)\, \mathcal{M}^\circ(y), \quad x \leq y \leq z. \tag{11}$$

We now select for special consideration the component $N(y,+)$ of $N_+(y)$ (see Figure 2) and the component $N(y,-)$ of $N_-(y)$ where y is some fixed integral value designating a point in X_p. Furthermore, observe that, because of the eclipse convention and the defined form of the partition $\{X_p, X_q\}$ the only non zero component of $N_+(z)$ is $N(k+1,+)$ (in the notation of the present linear lattice) and the only non-zero component of $N_-(x)$ is $N(i-1,-)$. It follows that we can express $N(y,\pm)$ as linear combination of $N(k+1,+)$ and $N(i-1,-)$ by choosing the appropriate component of the products of $N(k+1,+)$ and $N(i-1,-)$ with the $\mathcal{R}$ and $\mathcal{T}$ matrices. This observation allows us to restrict attention in (11) to the pair of special components $[N(y,+), N(y,-)]$ and to the product of $[N(k+1,+), N(i-1,-)]$ with the matrix $\mathcal{M}$, where the matrix $\mathcal{M}$ now may be considered to consist of the four 1×1 matrices $\mathcal{R}$ and $\mathcal{T}$. (In the notation of (47), reference 2, these conclusions follow from the fact that we have set $r = s = k = l = 1$. Thus the R and T factors defined below are scalars--i.e., real numbers.)

Derivation of the Principles

Following the methodology of reference 6, we derive the two main statements of the principles of invariance from (11). First, we set $i = j - j$, so that (11) becomes:

$$[N(j,+), N(j,-)] = [N(k+1,+), N(i-1,-)] \times$$

$$\times \begin{pmatrix} \mathcal{T}(k+1, j, j-1) & \mathcal{R}(k+1, j, j-1) \\ \mathcal{R}(j-1, j, k+1) & \mathcal{T}(j-1, j, k+1) \end{pmatrix} . \qquad (12)$$

Still following the methods of reference 6 as closely as the discrete space setting will allow, we define

$$\begin{aligned} \mathcal{T}(k+1, j, j-1) &\equiv T(k+1, j) , \\ \mathcal{R}(j-1, j, k+1) &\equiv R(j, k+1) . \end{aligned} \qquad (13)$$

Then, reading off the first component in the vector equation (12), we have, with the definitions (13):

$$\boxed{\text{I.} \quad N(j,+) = N(k+1,+)\, T(k+1,j) + N(j-1,-)\, R(j,k+1) .} \qquad (14)$$

This is the first main statement of the invariance principle. The second main statement is obtained from (11) by setting $k = j$:

$$[N(j,+), N(j,-)] = [N(j+1,+), N(i-1,-)] \times$$

$$\times \begin{pmatrix} \mathcal{T}(j+1, j, i-1) & \mathcal{R}(j+1, j, i-1) \\ \mathcal{R}(i-1, j, j+1) & \mathcal{T}(i-1, j, j+1) \end{pmatrix} . \qquad (15)$$

Then reading off the second component of (15) and using the definitions

$$\mathcal{T}(i-1, j, j+1) \equiv T(i-1, j)$$

$$\mathcal{R}(j+1, j, i-1) \equiv R(j, i-1) , \qquad (16)$$

we have the desired second main statement of the invariance principle:

$$\text{II.} \quad N(j,-) = N(i-1,-)\,T(i-1, j) + N(j+1,+)\,R(j, i-1) . \qquad (17)$$

The remaining principles of invariance now follow automatically. In I, let $j=1$, $k=n$; and then let $j=1$, with k arbitrary:

$$\text{III} \quad N(1,+) = N(0,-)\,R(1,n+1) + N(n+1,+)\,T(n+1,1)$$
$$= N(0,-)\,R(1,k+1) + N(k+1,+)\,T(k+1,1) \qquad (18)$$

Finally, in II let $j=n$, $i=1$; and then $j=n$, i arbitrary:

$$\text{IV} \quad N(n,-) = N(0,-)\,T(0,n) + N(n+1,+)\,R(n,0)$$
$$= N(i-1,-)\,T(i-1,n) + N(n+1,+)\,R(n,i-1) \qquad (19)$$

The Standard Reflectance and Transmittance Factors

It is important to see how the factors R and T are associated notationally with the corresponding subset of X_n. Consider an arbitrary ordered connected subset $S(a,b)$ of X_n of the kind shown in Figure 3. $S(a,b)$ is the set of $b-a+1$ points of X_n beginning with a and ending with b, where $1 \leq a \leq b \leq n$. Observe that the R-factor which represents the reflectance of $S(a,b)$ when flux is incident at a is: $R(a,b+1)$ and that the T-factor which represents the transmittance of $S(a,b)$ is $T(a-1,b)$. Thus the R-notation uses the first point of $S(a,b)$ encountered by the flux and the point just beyond the terminal point of $S(a,b)$. On the other hand the T-factor uses the point of X_n just before the initial point of $S(a,b)$ and the terminal point b of $S(a,b)$. Observe how this rule applies also to the R and T factors associated with $S(b,a)$, the oppositely sensed subset of X_n (See Figure 3). Finally, observe that by taking the absolute magnitude of the difference of $a-1$ and b in $T(a-1,b)$, we can determine the number of points in $S(a,b)$. Similarly with the other factors.

This notation springs naturally and automatically from the discrete form of the invariant imbedding principle. The notation yields a maximum amount of information with an irreducible minimum of symbols. For example, suppose $a=b$, so that $S(a,a)$

consists of the single point x_a of X_n. Now the notation $R(a,a+1)$ and $T(a-1,a)$ show unambiguously that x_a is being irradiated from "above". The slight asymmetry exhibited by the differences in the domain of definition of R and T is a simple consequence of the discrete character of X_n and is more than offset by the informational content this notational system carries. For example, if $S(a,b)$ degenerates to the single point x_a, then, it is instantly clear that in the present linear lattice setting:

$$\begin{aligned} R(a,a+1) &= \Sigma(a;-;+) \\ R(a,a-1) &= \Sigma(a;+;-) \\ T(a-1,a) &= \Sigma(a;-;-) \\ T(a+1,a) &= \Sigma(a;+;+) \,. \end{aligned} \tag{20}$$

One further observation is that for each $x_a \in X_n$, the equalities:

$$\begin{aligned} R(a,a) &= 0 \\ T(a,a) &= 1 \,, \end{aligned} \tag{21}$$

follow <u>formally</u> from, and are consistent with, the principles I and II. (For example, in I, set $j = k+1$, and in II set $j = i-1$). R and T thus possess the zero and identity operation properties of R and T in the general theory. (See reference 6.).

EQUATIONS GOVERNING THE R AND T FACTORS

General Partition Relations For Downward Flux

Suppose X_n is partitioned into two connected subsets X_m and X_{n-m} of the form shown in Figure 4. That is, we take the first m elements of X_n : $1, 2, \dots, m$ where $1 \leq m \leq n$, and consider them as the elements of a linear lattice X_m . The remaining $n-m$ elements then comprise the linear lattice X_{n-m} . The object of this section is to obtain expressions for the R and T factors for downward flux associated with X_n in terms of the R and T factors associated with the arbitrary partition elements X_m and X_{n-m}.

To obtain the requisite relation for $R(1,n+1)$, start with principle III by setting $k = m$, $N(n+1,+) = 0$ and writing:

$$N(1,+) = N(0,-)\ R(1,n+1)$$

$$= N(m+1,+)\ T(m+1,1) + N(0,-)\ R(1,m+1) \ . \qquad (22)$$

Next, in I, set $j = m+1$, $k = n$, so that

$$N(m+1,+) = N(m,-)\, R(m+1, n+1) \, . \tag{23}$$

Finally, in II, set $j = m$, $i = 1$, whence:

$$N(m,-) = N(0,-)\, T(0,m) + N(m+1,+)\, R(m,0) \, . \tag{24}$$

Eliminating $N(m,-)$ from (23) and (24) we have:

$$\boxed{N(m+1,+) = \frac{N(0,-)\, T(0,m)\, R(m+1, n+1)}{1 - R(m+1,n+1)\, R(m,0)} ,} \tag{25}$$

which, when inserted in (22) and observing that $N(0,-)$ is arbitrary, yields the following desired formula for $R(1,n+1)$:

$$\boxed{R(1,n+1) = R(1,m+1) + \frac{T(0,m)\, R(m+1,n+1)\, T(m+1,1)}{1 - R(m+1,n+1)\, R(m,0)} \, . \qquad 1 \le m \le n} \tag{26}$$

A similar derivation exists for $T(0,n)$ in terms of the R and T factors of X_m and X_{n-m} : Starting with principle IV by setting $i-1=m$ we have

$$N(n,-) = N(0,-)\,T(0,n)$$
$$= N(m,-)\,T(m,n) \tag{27}$$

Eliminating $N(m+1,+)$ from (23) and (24) we have:

$$\boxed{N(m,-) = \frac{N(0,-)\,T(0,m)}{1 - R(m+1,n+1)\,R(m,0)},} \tag{28}$$

which, when inserted in (27) and observing that $N(0,-)$ is arbitrary, yields the following formula for $T(0,n)$:

$$\boxed{T(0,n) = \frac{T(0,m)\,T(m,n)}{1 - R(m+1,n+1)\,R(m,0)}.} \tag{29}$$

General Partition Relations For Upward Flux

Observe that (26) and (29) represent R and T for X_n in terms of <u>an arbitrary partition</u> $\{X_m, X_{n-m}\}$ of X_n. We may also obtain the formulas for $T(n+1,1)$ and $R(n,0)$, i.e., the transmittance and reflectance factors for upward flux, by a similar process. However, by studying the structure of (26) and (29) the required formulas for these factors may be written down immediately:

$$R(n,0) = R(n,m) + \frac{T(n+1,m+1)\, R(m,0)\, T(m,n)}{1 - R(m+1,n+1)\, R(m,0)}, \tag{30}$$

$$T(n+1,1) = \frac{T(n+1,m+1)\, T(m+1,1)}{1 - R(m+1,n+1)\, R(m,0)}. \tag{31}$$

General Recurrence Relations

By setting $m=1$ in the general partition relations (26) and (29), we obtain a particularly simple and useful numerical procedure for the evaluation of $R(1,n+1)$ and $T(0,n)$ for arbitrary n :

From (26), with $m=1$:

$$R(1,n+1) = R(1,2) + \frac{T(0,1)\,R(2,n+1)\,T(2,1)}{1 - R(2,n+1)\,R(1,0)}, \qquad (32)$$

and from (29) with $m=1$:

$$T(0,n) = \frac{T(0,1)\,T(1,n)}{1 - R(2,n+1)\,R(1,0)}. \qquad (33)$$

Observe that, in view of (20), we have

$$\begin{aligned} R(1,2) &= \Sigma(1,-;+) \\ R(1,0) &= \Sigma(1,+;-) \\ T(0,1) &= \Sigma(1;-,-) \\ T(2,1) &= \Sigma(1,+;+) \end{aligned} \qquad (34)$$

which are known. Thus if the R and T factors are known for a space $X_{n-1} = \{x_2, x_3, \ldots, x_n\}$, $n > 1$, then we can find them for the space $X_n = \{x_1, x_2, \ldots, x_n\}$, for every discrete space X_n, $n = 1, 2, \ldots$.

A similar set of recurrence relations can be obtained by setting, in (26) and (29), $m = n-1$. The results are:

$$R(1,n+1) = R(1,n) + \frac{T(0,n-1)\,R(n,n+1)\,T(n,1)}{1 - R(n,n+1)\,R(n-1,0)}, \qquad (35)$$

$$T(0,n) = \frac{T(0,n-1)\,T(n-1,n)}{1 - R(n,n+1)\,R(n-1,0)} \,. \qquad (36)$$

Observe, however, the following curious phenomenon: Equations (35) and (36) cannot, in general, be used to find $R(1,n+1)$ and $T(0,n)$ recursively since knowledge of $R(n-1,0)$ and $T(n,1)$ is needed. On the other hand system (32)-(33), when solved for $R(1,n+1)$ first and then for $T(0,n)$, is completely solvable. Furthermore the factors $T(n+1,1)$ and $R(n,0)$ may be found recursively by using the pair (30)-(31). In particular, set $m = n-1$, whence:

$$R(n,0) = R(n,n-1) + \frac{T(n+1,n)\,R(n-1,0)\,T(n-1,n)}{1 - R(n,n+1)\,R(n-1,0)} \qquad (37)$$

and

$$T(n+1,1) = \frac{T(n+1,n)\; T(n,1)}{1 - R(n,n+1)\, R(n-1,0)} , \tag{38}$$

where, of course,

$$\begin{aligned}
R(n,n+1) &= \Sigma(n; -; +) \\
R(n,n-1) &= \Sigma(n; +, -) \\
T(n-1,n) &= \Sigma(n; -, -) \\
T(n+1,n) &= \Sigma(n, +; +)
\end{aligned} \tag{39}$$

REMARKS ON THE POLARITY OF THE R AND T FACTORS

The Polarity Theorem

Consider the ostensibly simple discrete space $X_2 = \{x_1, x_2\}$ where we have set $n=2$. From (32) and (33):

$$R(1,3) = R(1,2) + \frac{T(0,1)\, R(2,3)\, T(2,1)}{1 - R(2,3)\, R(1,0)}$$

$$= \Sigma(1;-;+) + \frac{\Sigma(1;-;-)\, \Sigma(2;-,+)\, \Sigma(1,+;+)}{1 - \Sigma(2,-;+)\, \Sigma(1;+,-)}, \qquad (40)$$

and

$$T(0,2) = \frac{\Sigma(1,-;-)\, \Sigma(2;-;-)}{1 - \Sigma(2,-;+)\, \Sigma(1;+;-)}; \qquad (41)$$

and from (37), (38)

$$R(2,0) = \Sigma(2;+;-) + \frac{\Sigma(2;+;+)\,\Sigma(1;+;-)\,\Sigma(2;-,-)}{1 - \Sigma(2;-;+)\,\Sigma(1;+;-)}, \qquad (42)$$

and, finally,

$$T(3,1) = \frac{\Sigma(2;+;+)\,\Sigma(1;+;+)}{1 - \Sigma(2;-,+)\,\Sigma(1;+;-)} \qquad (43)$$

If one takes the trouble to examine in detail the set of relations (40)-(43) one can verify that they constitute the essential steps of a <u>complete proof</u> of the following theorem for <u>linear lattices</u> X_n :

<u>Polarity Theorem</u>: <u>If X_n is isotropic at every point, then T possesses no polarity. If X_n is isotropic at every point and is homogeneous*, then R possesses no polarity. If X_n is isotropic at every point, but not homogeneous, then R may possess polarity</u>.

That the set (40)-(43) constitutes a proof of the preceding theorem follows from the fact that the space $X_2 = \{x_1, x_2\}$, ostensibly a discrete space of two elements, may actually be interpreted as a general partition (quotient space) of some arbitrary X_n, $n > 1$. For simplicity in interpreting the meaning of the theorem one may assume that x_1 and x_2 are points of E_3, or two contiguous plane-parallel slabs in E_3. When X_n is not a linear lattice, but rather a cubic lattice, or some general discrete space, then the first statement of the preceding theorem no longer holds because T can be shown to possess polarity in such general geometric situations. The remaining two sentences then apply with equal force to R and T, individually. Thus R and T in isotropic, <u>inhomogeneous</u> discrete spaces generally possess polarity. This has been essentially proved in reference 6 for the continuous case. The basis for the proof for the cubic lattice case will occur in a subsequent paper. The details of the proof for a cubic lattice are essentially the same as those occurring in (40)-(43).

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* X_n is <u>homogeneous</u> if Σ is independent of $x_i \in X_n$.

The Polarity-Free Case

In view of the preceding theorem, we conclude that: if a linear lattice is <u>isotropic and homogeneous</u> the standard reflectance and transmittance factors do not possess polarity. Thus, for any subset $S(a,b)$ of X_n , $1 \leq a \leq b \leq n$, we have $R(a, b+1) = R(b, a-1) \equiv R_{b-a+1}$ and $T(a-1, b) = T(b+1, a) \equiv T_{b-a+1}$. With this notation, the partition relations, for example (26) and (29), become:

$$\left.\begin{aligned} R_n &= R_m + \frac{T_m^2 R_{n-m}}{1 - R_m R_{n-m}} \\ T_n &= \frac{T_m T_{n-m}}{1 - R_m R_{n-m}} \\ 1 &\leq m \leq n \end{aligned}\right\} \tag{44}$$

The associated recurrence equations (32) and (33) are:

$$\left.\begin{aligned} R_n &= R_1 + \frac{T_1^2 R_{n-1}}{1 - R_1 R_{n-1}} \\ T_n &= \frac{T_1 T_{n-1}}{1 - R_1 R_{n-1}} \\ n &\geq 1 \end{aligned}\right\} \tag{45}$$

The sets (44), (45) are the well-known[7] formulas for the reflectance of a pile of plates, or a collection of n optically identical contiguous systems, each characterized by the same single reflectance-transmittance pair (R_1, T_1). Thus the present theory contains this classical theory as a special case when we introduce isotropy and homogeneity into the optical structure of X_n.

SOLUTION OF THE TWO-FLOW PROBLEM

General Partition-Scheme

The solution of the two flow problem with an arbitrary source condition on the upper boundary follows immediately from (25) and (28) and knowledge of the R and T factors for all partitions of $X_n : \{X_m, X_{n-m}\}$. By superposition arguments, the solution for a source condition which is in effect at both boundaries, is readily formulated from this result.

There is an alternate solution procedure which makes direct use of the local interaction principle and knowledge of $R(1, n+1)$ only. This is the

Local Recurrence-Scheme

From From (1):

$$N(j+1, +) = \frac{N(j,+) - N(j-1,-)\ \Sigma(j\ ;\ -\ ;\ +)}{\Sigma(j\ ;\ +\ ;\ +)} \tag{46}$$

$j = 1, \ldots, n-1,$

and from (1) again, we have an expression for $N(j,-)$:

$$N(j,-) = N(j-1,-)\,\Sigma(j\ ;\ -\ ;\ -) + N(j+1,+)\,\Sigma(j\ ;\ +\ ;\ -) \tag{47}$$

$j = 1, \ldots, n$.

In order to start this particular type of numerical computation, one needs to know $N(1,+)$ which, by principle III, is simply $N(0,-)\,R(1, n+1)$.

THE PLANE-PARALLEL MEDIUM AND ITS ASSOCIATED LINEAR LATTICE

It was announced in the Introduction that one of the purposes of the present study was to prepare the ground-work for a computation program leading to the numerical tabulation of the light field in various linear lattices. The linear lattices used will be discrete-space representations of various continuous plane—parallel media. The purpose of this section is to develop the steps required in the transition from a given continuous plane-parallel medium to its discrete representative.

Construction of the Lattice

Let X be a slab of depth z_1 in E_3, i.e., $X = \{(x,y,z) : 0 \le z \le z_1\}$. (See Figure 5). Let X be stratified, i.e., the volume scattering function σ and volume attenuation function α depend only on z in X. Further, let the upper boundary $x_0 = \{(x,y,z) : z = 0\}$ have associated with it reflectance and transmittance factors with respect to incident irradiances: $\Sigma(0;-;+)$ and $\Sigma(0,-;-)$, which are respectively the reflectance and transmittance of x_0 for incident downwelling irradiance (in the direction of $-k$). Similarly $\Sigma(0;+;-)$ and $\Sigma(0;+,+)$ will be the reflectance and transmittance respectively of x_0 for upwelling irradiance (in the direction of k). The lower boundary

$x_1 = \{(x,y,z) : z = z_1'\}$ also has associated with it reflectance and transmittance factors for each stream of flux. These will be denoted by $\Sigma(n+1; -; +)$ and $\Sigma(n+1; -; -)$ for the downwelling stream, and $\Sigma(n+1; +; -)$ and $\Sigma(n+1; +; +)$ for the upwelling stream.

Next, choose and fix for the remainder of the discussion some integer $n \geq 1$. Divide z_1', into n (not necessarily equal) parts of magnitude Δ_j with $\sum_j \Delta_j = z_1'$. Let $z_j = \Delta_1 + \cdots + \Delta_{j-1} + \frac{\Delta_j}{2}$. Then we have the following partition X_{n+2} of X :

$$\{x_0, x_1, \ldots, x_n \; x_{n+1}\},$$

where x_0 and x_{n+1} are the upper and lower boundaries of X as defined above. Furthermore:

$$x_j = \left\{(x,y,z) : z_j - \frac{\Delta_j}{2} < z \leq z_j + \frac{\Delta_j}{2}\right\}, \quad j = 1, \ldots, n-1,$$

$$x_n = \left\{(x,y,z) : z_n - \frac{\Delta_n}{2} < z < z_n + \frac{\Delta_n}{2}\right\}.$$

This partition of the slab X, when the elements $x_1, \ldots, x_n$ are considered as "points," concentrated at z_j, becomes the associated linear lattice to the slab X. We now show how to find the values of $\Sigma(x_j; \cdot; \cdot)$, the local scattering function on X_{n+2}. We have already defined the values of Σ for the two extreme indices $j = 0$, $j = n+1$, so it remains only to define the values for $j = 1, \ldots, n$.

Construction of the Attenuation Functions

By means of the continuous theory[5] of radiative transfer on plane parallel media, we may associate with each X the attentuation functions $\alpha(\cdot,+)$ and $\alpha(\cdot,-)$ for the upwelling and downwelling irradiances in X. Thus, e.g., $\alpha(z,+)$ is the value of the attenuation function for upwelling irradiance at depth z. The attenuation functions may be decomposed into absorption, forward and backward scattering components, thus:

$$\alpha(z,\pm) = a(z,\pm) + f(z,\pm) + b(z,\pm) .$$

These absorption and scattering functions are now used to define the associated A and Σ functions for X_{n+2}. Let z_j as before, denote the mid-depth of the jth slab. Then for $j=1,\ldots,n$, set:

$$A(j,\pm) = a(z_j,\pm)\Delta_j$$

$$\Sigma(j;\pm;\mp) = b(z_j,\pm)\Delta_j \qquad (48)$$

$$\Sigma(j;\pm;\pm) = f(z_j,\pm)\Delta_j + e^{-\alpha(z_j,\pm)\Delta_j}$$

Demonstrating the Local Conservation Property

Now, for all z and Δ,

$$e^{-\alpha(z,\pm)\Delta} = 1 - \alpha(z,\pm)\Delta + o(\alpha_{\pm}\Delta), \tag{49}$$
$$(\alpha_{\pm}\Delta \equiv \alpha(z,\pm)\Delta)$$

where $o(\cdot)$ is a function which has the property: $o(\alpha\Delta)/\alpha\Delta \to 0$ as $\Delta \to 0$. Combining (48) and (49) we have:

$$A(j,+) + \Sigma(j;+;-) + \Sigma(j;+;+) = 1 + o(\alpha_{+}\Delta_j). \tag{50}$$

A similar statement holds for the downwelling stream:

$$A(j,-) + \Sigma(j;-;-) + \Sigma(j;-;+) = 1 + o(\alpha_{-}\Delta_j). \tag{51}$$

Thus to within an increment of order of magnitude smaller than $\alpha_{\pm}\Delta_j$, the local conservation property holds for X_{n+2} at each z_j, $j=0,\ldots,n+1$. The approximation improves with decreasing Δ, that is, with finer subdivisions of X. Since in general, $a(z,+) \neq a(z,-)$, $b(z,+) \neq b(z,-)$, $f(z,+) = f(z,-)$, we conclude that X_{n+2} is generally not isotropic, so that polarity of both the R and T factors is generally to be expected.

To summarize: Starting with a slab of depth z_1', with reflecting upper and lower boundaries, we divided the slab into $n+2$ parts: $x_0, x_1, \ldots, x_n, x_{n+1}$ where x_0 and x_{n+1} are the upper and lower boundaries, and $x_1, \ldots, x_n$ are the n internal partition elements of X. The quotient space $X_{n+2} = \{x_0, x_1, \ldots, x_n, x_{n+1}\}$ is the linear lattice associated with X. Each element x_j of X_{n+2} is assigned A and Σ functions for each stream in accordance with the given information about the boundaries and internal optical properties of X. It was shown that, to within a small calculable error which decreases as the fineness of the subdivision of X increases, the local conservation property holds at each $x_j \in X_{n+2}$.

It remains to show that the principle of local interaction holds on this linear lattice. But this task is readily accomplished by carefully retracing the steps (3) - (10) which were developed above for precisely this purpose.

Thus we have shown how the general two-flow transfer problem of continuous plane-parallel media may be transformed into the context of a linear lattice transfer problem, thereby permitting the application of the simple and powerful methods of the discrete theory to an otherwise intractable problem.

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DEFINITION OF A LINEAR LATTICE

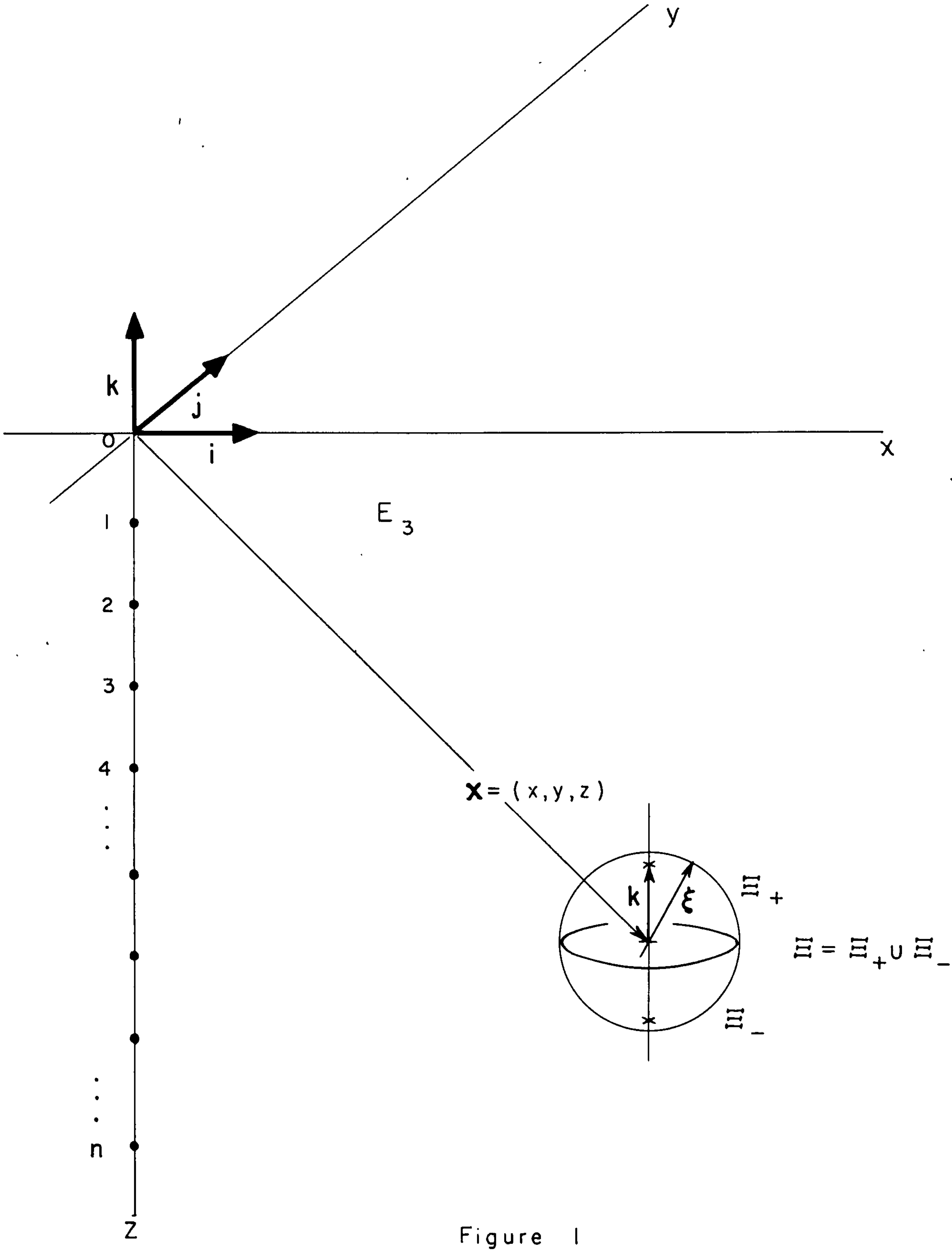

Figure 1

Rudolph W. Preisendorfer

GENERAL PARTITION OF A LINEAR LATTICE USED IN THE DERIVATION OF THE PRINCIPLES OF INVARIANCE

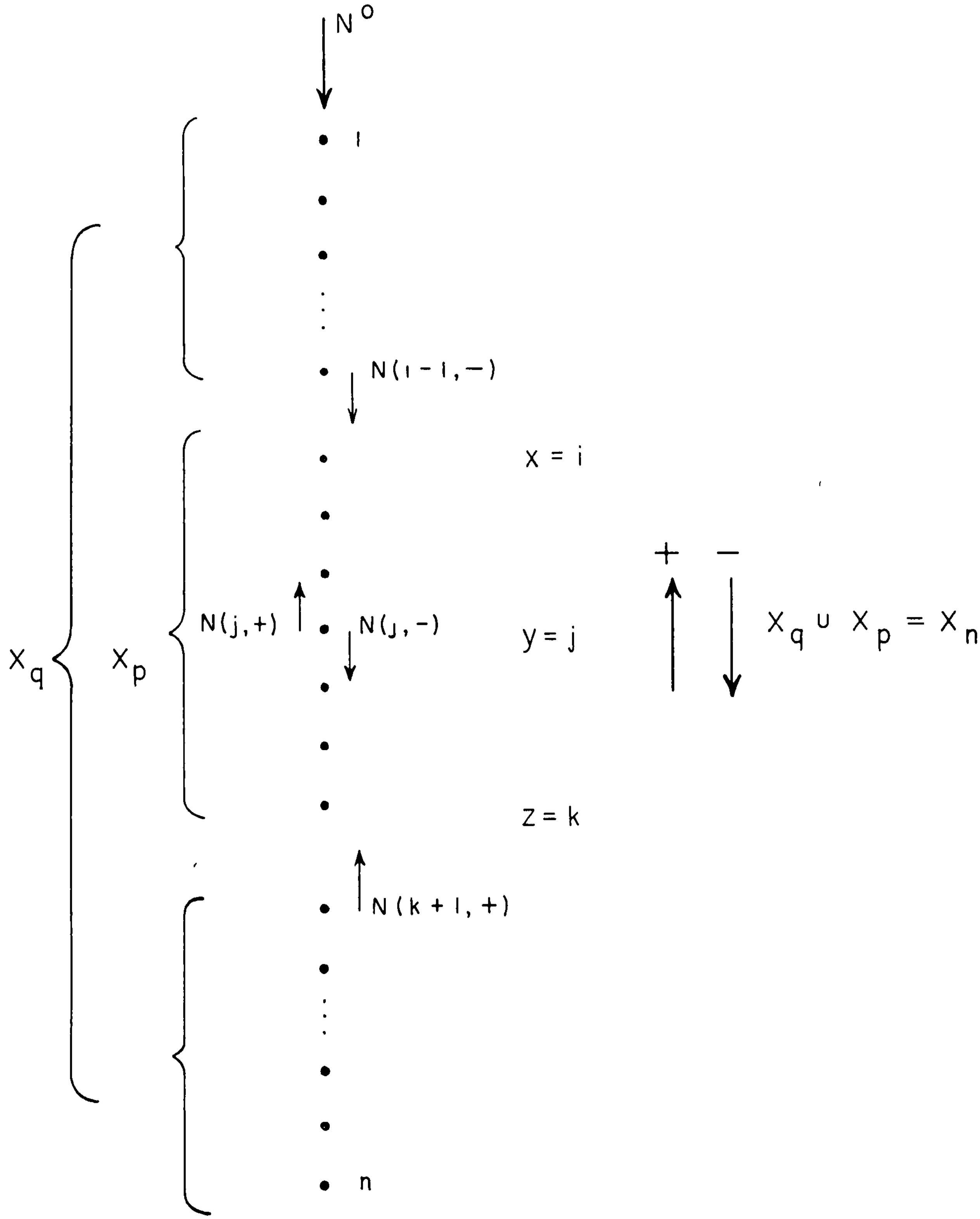

Figure 2

Rudolph W. Preisendorfer

R AND T FACTOR NOTATION FOR AN ARBITRARY CONNECTED SUBSET OF X_n

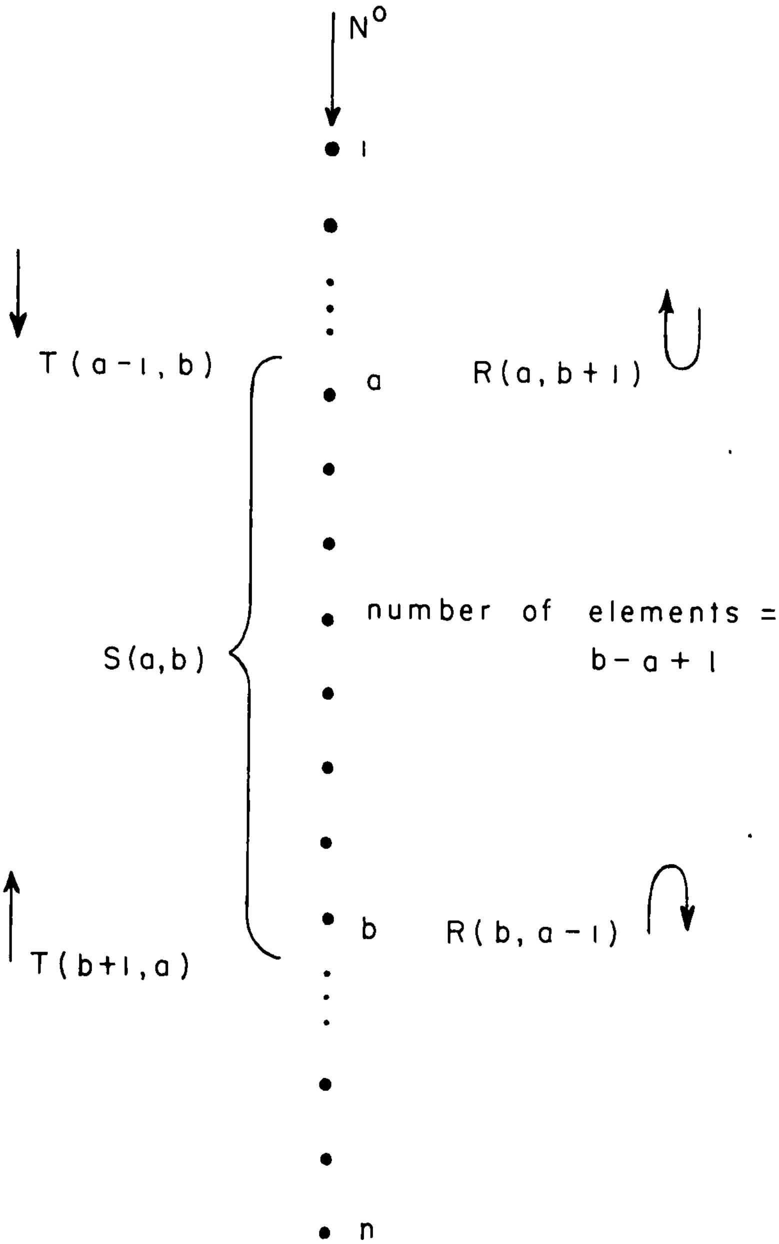

Figure 3

PARTITIONS USED IN RECURRENCE RELATIONS

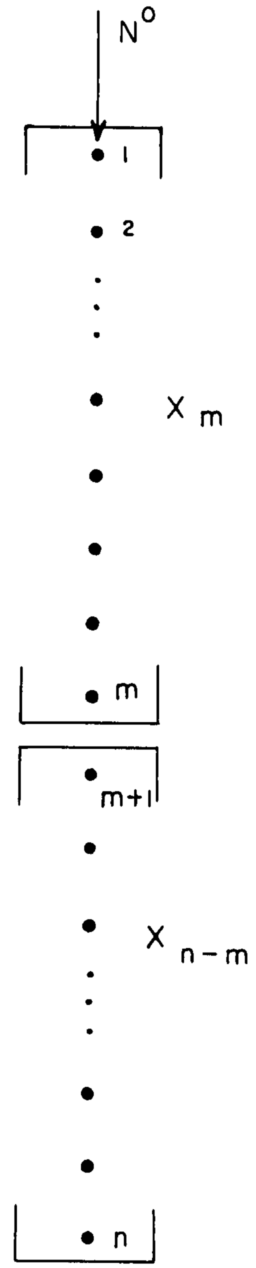

Figure 4

Rudolph W. Preisendorfer

PLANE-PARALLEL MEDIUM AND ITS ASSOCIATED LINEAR LATTICE

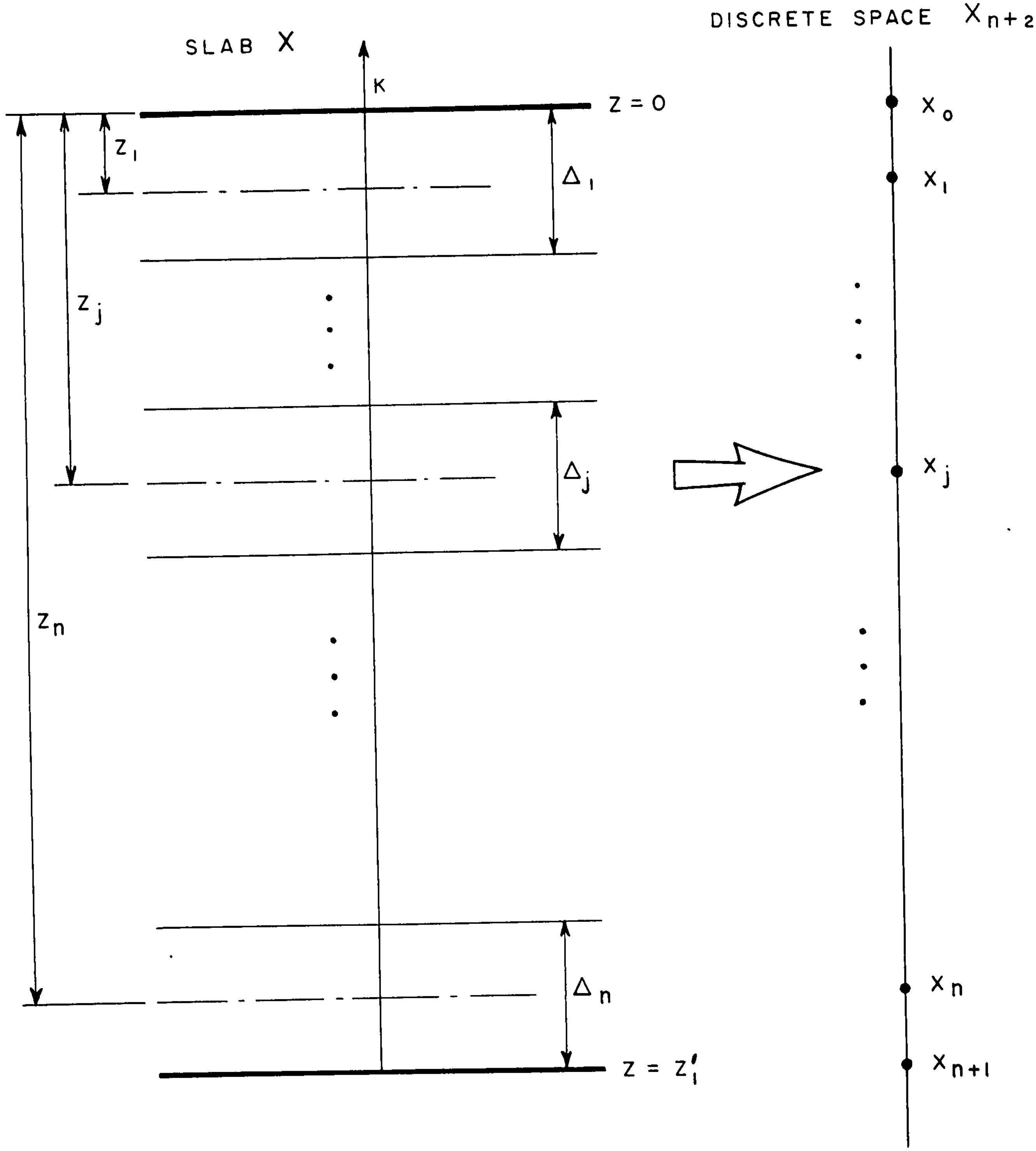

Figure 5

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