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RADIATIVE TRANSFER ON A CUBIC LATTICE

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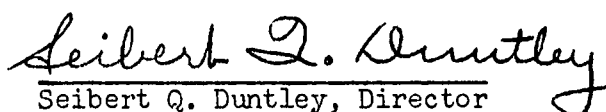
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# Radiative Transfer on a Cubic Lattice

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## INTRODUCTION

This work presents in complete detail an application of the general theory of radiative transfer on discrete spaces<sup>1-4</sup> to the problem of determining the radiance distributions at each depth in a plane-parallel medium with given external boundary conditions. Thus, in particular the present work gives an explicit solution of the complete two boundary Planetary Hydrosphere Problem<sup>5</sup> in the discrete-space context. All of the relatively intractable integral equations in reference 5 are thereby reduced to simple recurrence relations for relatively low-order matrices. Step by step details are given to show how the basic radiometric principle of discrete spaces, namely the local interaction principle,<sup>1</sup> ultimately leads to a workable computation procedure, suitable for use on automatic computers, which in principle can result in a complete numerical determination of the radiance distribution at each point of the discrete-space counterpart to a real optical medium, such as those associated with planetary atmospheres and hydrospheres.

The type of discrete space used here is known as the extended cubic lattice, which is a discrete-space model of an arbitrarily stratified plane-parallel optical medium. A simpler type of discrete space, known as the linear lattice, was used in an earlier work<sup>4</sup> to solve the general two-flow problem in arbitrarily stratified plane-parallel media. In essence, the extended cubic lattice replaces the continuous plane-parallel optical medium by a collection of points located at the centers of contiguous equal cubes contained within two parallel planes in Euclidean three-space. Each cube-center is a point of the lattice and each point may interact radiometrically with its twenty-six immediate neighbors (see Figure 1).

The problem solved in this work is the complete two boundary, twenty-six flow problem. Given: The incident plane radiance distributions on the upper and lower boundaries of a well-defined extended cubic lattice; Required: The radiance of the emergent flux in each of the twenty-six directions about each point of the lattice. The complete solution is given by a detailed seven stage computation procedure appended to the present work.

The overall plan of the present work is as follows: First the geometrical setting of the problem is defined, using the notion of an extended cubic lattice and its associated quotient space. Next, the principles of invariance are formulated for the

lattice. From these are deduced the requisite equations governing the  $R$  and  $T$  operators on the lattice. These operators are then used to express the required internal radiance distributions at each point of the lattice.

### THE EXTENDED CUBIC LATTICE

The definition of a general cubic lattice was given in reference 2. For the reader's convenience, we will repeat the definition here. Let  $E_3$  designate the usual Euclidean three-space (i.e., the set of all triples  $(x, y, z)$  of real numbers  $x$ ,  $y$ , and  $z$ ). Let  $a, b, c$  be three finite integers, such that  $0 \leq a \leq b$  and  $0 \leq c$ . Then the subset

$$X_n = \left\{ (x, y, z) : |x|, |y| \leq c, a \leq z \leq b, x, y, z \text{ integers} \right\}$$

of  $E_3$  is a cubic lattice of  $n = (2c+1)^2(b-a+1)$  points. The subset  $X_m$  of  $E_3$  is an extended cubic lattice of depth  $m = (b-a+1)$  if  $c \rightarrow \infty$ .

In a previous study,<sup>4</sup> the discrete space of central interest was the linear lattice, which is the special cubic lattice defined by setting  $c = 0$ . The linear lattice thus represents the simplest

extreme in the family of spaces encountered in radiative transfer theory, namely the one-dimensional space associated with the two-flow analysis of the light field. The extended cubic lattice is a discrete space of intermediate complexity whose continuous counterpart is the slab in  $E_3$  (i.e., the set of all points of  $E_3$  between and including the two planes defined by  $z=a$ ,  $z=b$ ) with plane boundary conditions. The extended cubic lattice will be the discrete space of central interest in this work.

#### The Plane Boundary Condition

The sources of radiant flux on the extended cubic lattice  $X_n$  will be limited to the upper boundary  $\alpha_a$  and lower boundary  $\alpha_b$  defined as all points  $(x, y, z)$  in  $X_n$  with  $z=a$  or  $z=b$ , respectively. Unless explicitly stated otherwise, the source radiance distribution function  $N^\circ(x_i, \cdot)$  will be zero on  $\Xi$  (the unit sphere in  $E_3$ , i.e., the set of all unit vectors  $\xi$  in  $E_3$ ) for all  $x_i \in X_n$ , except for those in  $\alpha_a$  and  $\alpha_b$  at each of which  $N^\circ(x_i, \cdot)$  will have prescribed values on  $\Xi_-$  and  $\Xi'_+$ , respectively. In particular, for the present work,  $N^\circ(x_i, \cdot)$  will be assumed independent of  $x_i$  on the upper and lower boundary. Thus, in the terminology of the continuous theory,  $X_n$  has a plane source of radiant flux at its upper

and lower boundaries. The directional structure of  $N^0(x, \cdot)$ , however, will be arbitrary and fixed during the remainder of this discussion.

### The Eclipse Convention

The first two steps in the simulation of the continuous slab geometry have been accomplished by the adoption of the notion of an extended cubic lattice  $X_n$  and the assumption of a plane source on  $X_n$ . The final step in the present attempt to simulate the continuous slab geometry will be accomplished by adopting the following eclipse convention for  $X_n$  (For some earlier remarks on the eclipse conventions, see reference 1.): Let  $(x, y, z)$  be a point in the extended cubic lattice  $X_n$ . Consider the subset  $C(x, y, z)$  of  $\mathbb{E}_3$  defined as:

$$C(x, y, z) = \left\{ (x', y', z') : |x' - x| \leq 1, |y' - y| \leq 1, |z' - z| \leq 1, (x, y, z) \in X_n \right\}$$

The subset  $C(x, y, z)$  clearly consists of at most twenty-seven points of  $X_n$  (Figure 1), and has the general configuration of a cube with  $(x, y, z)$  at its center. The subset  $C(x, y, z)$  is called the cell associated with  $(x, y, z)$ . The present eclipse convention now may be phrased: For each  $(x, y, z) \in X_n$ ,  $N(x, y, z, \cdot)$

is identically zero on  $\Xi - \Xi'$ , where  $\Xi'$  is the fixed local direction space defined by  $C(x, y, z)$ . In other words the present eclipse convention states: the radiometric activity of each point  $(x, y, z)$  of  $X_n$  will be explicitly limited to the points of  $C(x, y, z)$ .

#### THE ASSOCIATED QUOTIENT SPACE

The adoption of the present plane source boundary condition, and the above eclipse convention for the extended cubic lattice  $X_n$  allows a profound simplification in the essential structure of  $X_n$ . This simplification is achieved by constructing a special quotient space  $Y_n$  from  $X_n$ . The construction of the associated quotient space begins with the partitioning of  $X_n$  into columns  $Y_n(x, y)$  which are subsets\* of  $X_n$  of the form:

$$Y_n(x, y) = \{(x, y, z) : (x, y, z) \in X_n, x, y \text{ fixed}\}.$$

---

\* The logical choice of notation for these subsets would be  $X_n(x, y)$ . However, for the sake of simple quotient space notation later on in this work, the choice of a symbol other than  $X$  is imperative. Later on in the present series, the more logical notation is resumed.

There is a countably infinite number of columns  $Y_n(x, y) \subset X_n$ . The union of these columns is, of course,  $X_n$ . The sets  $Y_n(x, y)$  are the points of the quotient space  $Y_n$ . Because of the conventions adopted above the fact is clear that the description of the radiance function on all of  $X_n$  is already achieved by describing it at any point of  $Y_n$  i.e., along a column  $Y_n(x, y)$ . This follows from the fact that the radiance distribution about  $(x_1, y_1, z)$  in  $Y_n(x_1, y_1)$  is identical with that about  $(x_2, y_2, z)$  in  $Y_n(x_2, y_2)$  for each  $z$ ,  $a \leq z \leq b$ . Thus, from a radiometric point of view, the quotient space  $Y_n$  can be reduced to a single column. With this observation we may and shall henceforth restrict attention to the special cubic lattice formed of the cells associated with the points of the special single column  $Y_n(0, 0)$  in  $X_n$  which we will designate as  $Y_n$  for short.

Figure 1 may be used to depict  $Y_n$  if the reader imagines that  $x$  and  $y$  are set equal to zero. We now set  $a = 1$  and  $b = n$  until further notice. Observe that we may locate a point  $(0, 0, y)$  of  $Y_n$  by a single integer  $j = y$ ,  $1 \leq j \leq n$ . A point on the cell associated with  $(x, y, z)$  may be identified by means of a unit vector of  $\Xi'$ . (See preceding section.) For the present purposes these twenty-six directions are grouped into the following four subsets of the local direction space of a point in  $X_n$ : (See equation (26) below and Figure 3 for the detailed

identification of these vectors.)

$$\Xi_+ = \{ \xi : \xi \in \Xi', \xi \cdot k \geq 0 \}$$

$$\Xi'_+ = \{ \xi : \xi \in \Xi', \xi \cdot k > 0 \}$$

$$\Xi_0 = \{ \xi : \xi \in \Xi', \xi \cdot k = 0 \}$$

$$\Xi_- = \{ \xi : \xi \in \Xi', \xi \cdot k < 0 \}$$

where  $k = (0, 0, -1)$  is the unit outward normal to  $Y_n$ . Here  $\Xi_+$  is the set of all upward (outward) directions, and  $\Xi_-$  is the set of all downward (inward) directions. The subset  $\Xi_0$  is the set of all horizontal (singular) directions. The set  $\Xi'_+$  defines the proper upward directions. Clearly, the number of elements associated with  $\Xi_+$ ,  $\Xi_0$ , and  $\Xi_-$  is 17, 8, and 9, respectively. The number of elements in  $\Xi'_+$  is 9.

The radiance distribution at  $(0, 0, j)$  is then completely described by the specific radiance vector

$$N(j) = [ N(0, 0, j, \xi_1), \dots, N(0, 0, j, \xi_{26}) ], \quad (1)$$

where  $\xi_i \in \Xi'$ ,  $1 \leq i \leq 26$ . We will set, for brevity,

$$N(j, \xi_i) \equiv N(0, 0, j, \xi_i)$$

It will be convenient to partition vector (1) into three parts  $N'_+(j)$ ,  $N_0(j)$  and  $N_-(j)$ , each part being defined as:

$$N'_+(j) = [N(j, \xi_1), \dots, N(j, \xi_9)], \quad (2)$$

where  $\xi_i \in \Xi_+ - \Xi_0 = \Xi'_+$ ,  $1 \leq i \leq 9$ . Further:

$$N_0(j) = [N(j, \xi_{10}), \dots, N(j, \xi_{17})], \quad (3)$$

where  $\xi_i \in \Xi_0$ ,  $10 \leq i \leq 17$ . Finally

$$N_-(j) = [N(j, \xi_{18}), \dots, N(j, \xi_{26})], \quad (4)$$

where  $\xi_i \in \Xi_-$ ,  $18 \leq i \leq 26$ .

to set:

$$N_+(j) = [N'_+(j), N_0(j)] \quad (5)$$

so that  $N(j)$  may be written:

$$N(j) = [N_+(j), N_-(j)], \quad (6)$$

which completes the analog with the continuous radiance distributions in reference 6. We now go on to formulate the principles of invariance associated with the special cubic lattice  $\Upsilon_n$ .

## PRINCIPLES OF INVARIANCE

The principles of invariance have proved to be powerful and incisive tools in the formulation of radiative transfer problems. We now develop the requisite forms of the principles for the present problem.

The first step in the derivation of the four principles of invariance is the development of the appropriate form of the invariant imbedding relation. This step has been essentially accomplished in equation (47) of reference 2. It remains only to make explicit the values of  $\alpha, \gamma, \bar{z}$ , and the appropriate dimensions of the various vectors and matrices involved in that statement.

## Invariant Imbedding Relation

Following the general methodology of reference 2, we now partition  $Y_n$  into two subsets  $Y_p$  and  $Y_q$ , such that

$$Y_p = \{ (0, 0, j) : 1 \leq i \leq j \leq k \leq n ; i, j, k \text{ fixed integers}; \rho = (k-i+i) \}$$

Then define  $Y_q = Y_n - Y_p$ , so that  $n = p + q$ . The remainder of the derivation now proceeds in essentially the same way as

that used to establish the corresponding results for a linear lattice (reference 4). In particular, we designate by  $N_-(0)$  the given incident radiance distribution  $N^0(1, \cdot)$  on  $\Xi_-$ . Further we set  $N'_+(k+1) = N^0(n, \cdot)$  on  $\Xi'_+$ . The purpose of these new designations of  $N^0$  is to allow, by means of a uniform notation for radiance distributions, a relatively compact formulation of the invariant imbedding relation without having to explicitly display the incident radiance distributions in the form  $N^0(j, \cdot)$ . Finally, as in reference 4, we choose for consideration the particular partitioned subvector  $[N_+(j), N_-(j)]$  of  $N(\rho|n)$  which occurs in the general theory (see reference 2). This choice may be realized by extracting the appropriate components from the left and right hand sides of the general invariant imbedding relation ((47) of reference 2). Further, it is of some importance to observe at this point that the present exact counterparts to  $N_+(z)$  and  $N_-(x)$  (in (47) of reference 2) are, respectively,  $N'_+(k+1)$  and  $N_-(i-1)$  associated with the present partition  $\{Y_p, Y_q\}$  of  $Y_n$ . With these observations, (47) of reference 2 yields:

$$[N_+(j), N_-(j)] = [N'_+(k+1), N_-(i-1)] \begin{pmatrix} \mathcal{T}(k+1, j, i-1) & \mathcal{R}(k+1, j, i-1) \\ \mathcal{R}(i-1, j, k+1) & \mathcal{T}(i-1, j, k+1) \end{pmatrix}; \quad (7)$$

where  $1 \leq i \leq j \leq k \leq n$ . Recall that  $N'_+(j)$  and  $N_-(j)$  generally have 9 components and that  $N_+(j)$  has 17. It follows that  $\mathcal{T}(k+1, j, i-1)$  and  $\mathcal{Q}(i-1, j, k+1)$  are  $9 \times 17$  matrices, whereas the remaining two matrices in (1) are  $9 \times 9$ . (Compare these dimensions with those of the linear lattice case, reference 4).

#### Statement of the Principles

We now follow the general methodology of reference 6 to derive the two main statements of the principles of invariance from (7). First, we set  $i' = j'$ , so that (7) becomes:

$$[N_+(j), N_-(j)] \cong [N'_+(k+1), N_-(j-1)] \begin{pmatrix} \mathcal{T}(k+1, j, j-1) & \mathcal{Q}(k+1, j, j-1) \\ \mathcal{Q}(j-1, j, k+1) & \mathcal{T}(j-1, j, k+1) \end{pmatrix} \quad (8)$$

We now set

$$\begin{aligned} \mathcal{T}(k+1, j, j-1) &\cong T(k+1, j), \\ \mathcal{Q}(j-1, j, k+1) &\cong R(j, k+1). \end{aligned} \quad (9)$$

The  $9 \times 17$  matrix  $T(k+1, j)$  is associated with a slab whose initial level is at  $k$  and whose terminal level is at  $j$ , where  $j \leq k$ ; hence the slab has  $k - j + 1$  layers. The  $9 \times 17$  matrix  $R(j, k+1)$  is associated with the slab whose initial level is at  $j$  and terminal level at  $k$ . Reading off the first component in the vector equation (8), we have, with the definitions (9):

$$I. \quad N_+(j) = N'_+(k+1) T(k+1, j) + N_-(j-1) R(j, k+1) \quad (10)$$

This is the first main statement of the invariance principle.

The second main statement is obtained from (7) by setting  $k = j$ :

$$[N_+(j), N_-(j)] = [N'_+(j+1), N_-(j-1)] \begin{pmatrix} \mathcal{T}(j+1, j, j-1) & \mathcal{Q}(j+1, j, j-1) \\ \mathcal{Q}(j-1, j, j+1) & \mathcal{T}(j-1, j, j+1) \end{pmatrix} \quad (11)$$

Now set

$$\begin{aligned}\hat{T}(i-1, j, j+1) &\equiv T(i-1, j), \\ \hat{R}(j+1, j, i-1) &\equiv R(j, i-1).\end{aligned}\tag{12}$$

The  $9 \times 9$  matrix  $T(i-1, j)$  is associated with a slab whose initial level is at  $i$  and terminal level at  $j$ ,  $i \leq j$ . Hence the slab has  $j - i + 1$  layers. The  $9 \times 9$  matrix  $R(j, i-1)$  is associated with the slab whose initial level is at  $j$  and terminal level at  $i$ . Reading off the appropriate component relation in (11):

$$\text{II. } N_-(j) = N_-(i-1) T(i-1, j) + N'_+(j+1) R(j, i-1)\tag{13}$$

This is the second of the two main statements of the principles of invariance.

The remaining principles of invariance now follow automatically: In I, let  $j=1$ ,  $k=n$ ; and then let  $j=1$ , with  $k$  arbitrary:

$$\begin{aligned} \text{III. } N_+(1) &= N'_+(n+1) T(n+1, 1) + N_-(0) R(1, n+1) \\ &= N'_+(k+1) T(k+1, 1) + N_-(0) R(1, k+1) \end{aligned}$$

(14)

The vectors  $N'_+(n+1)$ ,  $N_-(0)$  are the arbitrary  $1 \times 9$  source radiance vectors at the lower and upper boundaries, respectively.

Finally, in II, let  $j \neq n$ ,  $i=1$ , and then  $j=n$  with  $i$  arbitrary:

$$\begin{aligned} \text{IV. } N_-(n) &= N_-(0) T(0, n) + N'_+(n+1) R(n, 0) \\ &= N_-(i-1) T(i-1, n) + N'_+(n+1) R(n, i-1), \end{aligned}$$

(15)

### The Standard Reflectance and Transmittance Operators

The quartet of  $R$  and  $T$  matrices introduced in (9) and (12) above are the standard reflectance and transmittance operators associated with  $Y_n$ . From the point of view of the principles of invariance, the global scattering properties of are known, and in fact the entire radiative transfer problem associated with  $Y_n$  is solved, once this quartet of operators is completely known on  $Y_n \times Y_n$ .

Some care must be exercised in the correct dimensioning of these standard operators. By recalling that we have set  $i' \leq j' \leq K$  throughout the present discussion, it is then easy to infer that  $T(K+1, j)$  and  $R(j, i-1)$  are associated with the upward (or outward) incident radiances on their respective slabs. Further  $T(i-1, j)$  and  $R(j, K+1)$  are associated with downward (or inward) incident radiances on their respective slabs. Finally, because of the unavoidable symbolic perversity of any notation associated with a plane-parallel slab (or an extended cubic lattice) which thus induces an unavoidable asymmetric partitioning of  $N(i)$  into the 17 component vector  $N_+(j)$  and the 9 component vector  $N_-(j)$ , it follows that the  $R$  and  $T$  operators are also unavoidably asymmetric with respect to the

up and downwelling streams.\* Thus  $T(k+1, j)$  and  $R(j, k+1)$  are, as noted earlier, necessarily  $9 \times 17$  matrices, whereas,  $T(i-1, j)$  and  $R(j, i-1)$  are necessarily  $9 \times 9$  matrices. This asymmetric state of affairs is also present in the continuous case, but is usually only a harmless nuisance there because the subset  $\Xi_0$  in the continuous case is of  $\Omega$ -measure zero. However, in the discrete case, the set  $\Xi_0$  of singular directions contains about 31% of all the directions of the local direction space  $\Xi'$ , and can no longer be conveniently ignored.

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\* If the  $R$  and  $T$  matrices are all redefined to be  $9 \times 9$  matrices so that a semblance of symmetry is attained, then it will be necessary to introduce (i): new operators which map  $N_+$  and  $N_-$  into  $N_0$  (ii): additional principles of invariance governing these new operators, and (iii): more detailed computation procedures. By retaining the present asymmetry, we avoid such needless complications here and, incidentally, preserve the classical methodology of the continuous radiative transfer theory, which employs only two general (the  $R$  and  $T$ ) operators. <sup>6</sup> In subsequent works (on the plane and point source problems) additional operators must be introduced. It turns out that the present point of view is adequate, and the present choice of notation, fortunate, when transferred to the context of these more general problems.

The standard  $R$  and  $T$  operators defined on the arbitrary discrete space  $Y_n \times Y_n$  have many of the formal properties of the  $R$  and  $T$  operators in the continuous case as can be seen by comparing the principles of invariance in the discrete case with those in the general continuous case (reference 6). In particular we observe that

$$R(j,j) = O \quad (\text{the } 9 \times 9 \text{ zero matrix}) \quad (16)$$

$$T(j,j) = I \quad (\text{the } 9 \times 9 \text{ identity matrix}) \quad (17)$$

which follows from principle II (Equation (13)) by formally setting  $i-1 = j$ . A similar statement follows from principle I (Equation (10)) by setting  $k+1 = j$ , but now the statement is appropriately tailored to the  $9 \times 17$  matrices occurring there: The "tailoring" is accomplished by operating on each side of I with the contracting matrix  $C'$  (defined in (24) below) which reduces principle I to the  $9 \times 9$  case considered in (16) and (17); the result is:

$$R'(j,j) = O \quad (\text{the } 9 \times 9 \text{ zero matrix contracted from a } 9 \times 17 \text{ zero matrix}). \quad (18)$$

$T'(j,j) = I$  (the  $9 \times 9$  identity matrix contracted from a particular  $9 \times 17$  matrix). (19)

Furthermore, the particular  $9 \times 17$  matrix  $T(j,j)$  of which  $T'(j,j)$  is a contraction, is defined by the property:

$$T'(j+1,j) T(j,j) = T(j+1,j), \quad 1 \leq j \leq n. \quad (20)$$

Property (20) is the identity transmittance convention for upwelling  $9 \times 17$  transmittance matrices, of the form  $T(j,j)$ , which we adopt henceforth. Examples of the use of (20) occur in (48) when  $n = 1$ , and in (101) when  $\rho = 1$ .

## EQUATIONS GOVERNING THE R AND T OPERATORS FOR MULTILAYERS

If the reader has been following the present development of the principles of invariance by comparing each of the preceding steps with its linear lattice counterpart in reference 4, he should by this stage have discerned the remarkable formal similarity between the simple two-flow formulation of the linear lattice and the relatively more complex twenty-six flow case. The principle of invariance approach in the discrete space setting is of such fundamental strength that the dimensionality and geometric complexity of the basic space  $X_n$  has no discernable affect on the apparent complexity of the formulations of the principles of invariance on  $X_n$ . Thus the preceding statements I - IV (Equations (10), (13), (14), (15)) of the principles of invariance on the cubic lattice are apparently no more complex than their simple two-flow counterparts on the linear lattice of reference 4. However, the most remarkable feature of the discrete formulations is yet to come: In this section we show that the formal similarity between the linear and cubic lattice contexts persists up to the stage of

developing the recurrence relations for the  $R$  and  $T$  operators.\* In fact, one merely verifies that he may repeat here the steps in the corresponding section of reference 4. The only difference in the resulting functional relations is that they are matrix statements, and not scalar statements. Thus a scalar statement of the form:

$$\frac{A_1}{1 - A_2 A_3}$$

found in reference 4 is now to be interpreted as:  $A_1 [I - A_2 A_3]^{-1}$  where  $I$ ,  $A_1$ ,  $A_2$ , and  $A_3$  are any four mutually commensurate matrices such that  $[I - A_2 A_3]$  is a square matrix whose inverse  $[I - A_2 A_3]^{-1}$  exists. The question of the existence of this inverse in the discrete space setting was settled in references 1 and 2.

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\* This suggests that one may expect the general partition relations and general recurrence relations in a cubic lattice to be independent of the structure of the local direction space associated with the points of the lattice. It turns out that this is true. The proof is accomplished by an appeal to the general partition relations (21)-(24) of reference 2. Thus, to within the limitations of the capability of a computer, the radiometric interconnections between points of a cubic lattice may be arbitrarily increased beyond the presently considered set of twenty-six. The present method, with only minor changes, remains applicable.

As in the scalar or two-flow case (reference 4), to establish the  $R$  and  $T$  equations we first establish the desired equations for a general partition of  $Y_n$ .

#### General Partition Relations for Downward Flux

Suppose  $Y_n$  is partitioned into two subsets  $Y_m$  and  $Y_{n-m}$  such that  $Y_m = \{(0,0,1), \dots, (0,0,m)\}$  and  $Y_{n-m} = \{(0,0,m+1), \dots, (0,0,n)\}$ . To obtain the requisite relation for  $R(1,n)$ , start with principle III (Equation (14)) by setting  $K = m$ ,  $N'_+(n+1) = 0$ , and writing:

$$\begin{aligned} N_+(1) &= N_-(0) R(1,n) \\ &= N'_+(m+1) T(m+1, 1) + N_-(0) R(1, m+1). \end{aligned} \tag{21}$$

Next, in I (Equation (10)) set  $j = n+1$ ,  $K = n$  so that

$$N_+(m+1) = N_-(m) R(m+1, n+1). \tag{22}$$

Finally, in II (Equation (13)) set  $j = m$ ,  $i = 1$  :

$$N_-(m) = N_-(0) T(0, m) + N'_+(m+1) R(m, 0) . \quad (23)$$

Now if this were the linear lattice derivation we would next eliminate  $N_-(m)$  from (22) and (23) in order to obtain an expression for  $N_+(m+1)$  . Observe, however, that unlike the linear lattice case, we cannot make immediate use of the intermediate relation (23), derived from II, to effect the desired elimination because (23) uses  $N'_+(m+1)$  and not  $N_+(m+1)$  . Recall that  $N_+(m+1)$  is a 17-component vector, whereas  $N'_+(m+1)$  is a 9-component vector. The way out of this impasse is clear: Define a 17 x 9 matrix:

$$C' = \begin{pmatrix} I \\ O \end{pmatrix} , \quad (24)$$

where  $I$  is a  $9 \times 9$  identity matrix, and the zero denotes a zero-matrix of dimensions to complement those of  $C'$  and  $I$ , namely  $8 \times 9$ . The matrix  $C'$  is called the contracting matrix.<sup>\*</sup> It has the property:

$$N_{+(j)} C' = N'_{+(j)}, \quad (25)$$

for all  $j$ ,  $1 \leq j \leq n$ .

The contracting matrix can generally operate on matrices of dimension  $y \times 17$ ,  $y$  arbitrary, if  $C'$  is used as a post-multiplier. Thus if  $A$  is a  $y \times 17$  matrix we will henceforth denote by  $A'$  the  $y \times 9$  matrix:

$$A' = AC'. \quad (26)$$

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\* This process of contracting a 17-component vector to a 9-component vector is a special case of the general process of extracting an arbitrary submatrix from a general matrix, which generally requires two contracting matrices: one as a pre-multiplier, the other as a postmultiplier of the given general matrix.

Returning now to the problem of determining the partition relation governing  $R(l, n+1)$ , we postmultiply relation (22) by the contracting matrix  $C'$ ; the result is:

$$N'_+(m+1) = N_-(m) R'(m+1, n+1), \quad (27)$$

Inserting this expression for  $N'_+(m+1)$  into (23):

$$N_-(m) = N_-(0) T(0, m) + N_-(m) R'(m+1, n+1) R(m, 0), \quad (28)$$

and then solving this for  $N_-(m)$ , the result is:

$$N_-(m) = N_-(0) T(0, m) [I - R'(m+1, n+1) R(m, 0)]^{-1}. \quad (29)$$

Equations (27) and (29) are now used to obtain the equation governing  $N'_+(m+1)$ :

$$\begin{aligned} N'_+(m+1) &= \\ &= N_-(0) T(0, m) [I - R'(m+1, n+1) R(m, 0)]^{-1} R'(m+1, n+1), \quad (30) \end{aligned}$$

This, in turn, is used in (21) to obtain the desired relation governing  $R(1, n+1)$  :

$$N_-(0)R(1, n+1) = N_-(0)R(1, m+1) + \\ + N_-(0)T(0, m) [I - R'(m+1, n+1)R(m, 0)]^{-1} R'(m+1, n+1)T(m+1, 1) .$$

Since  $N_-(0)$  is an arbitrary vector, we have, finally:

$$R(1, n+1) = \\ = R(1, m+1) + T(0, m) [I - R'(m+1, n+1)R(m, 0)]^{-1} R'(m+1, n+1)T(m+1, 1) \quad (31)$$

Relation (31) is the desired partition relation governing the standard reflectance operator for downward flux in a cubic lattice  $X_n$  . The partition  $\{Y_m, Y_{n-m}\}$  of  $Y_n$  is arbitrary. An important recurrence relation for  $R(1, n+1)$  may be obtained by setting  $m=1$  . This will be discussed in detail later on. Now we go on to obtain the partition relation for  $T(0, n)$  .

Starting with principle IV (Equation (15)) by setting  $i-1 = m$  and  $N'_+(n+1) = 0$ , we have:

$$\begin{aligned} N_-(n) &= N_-(0) T(0, n) \\ &= N_-(m) T(m, n). \end{aligned} \tag{32}$$

From (29) and (32) it follows that

$$\begin{aligned} N_-(0) T(0, n) &= \\ &= N_-(0) T(0, m) \left[ I - R'(m+1, n+1) R(m, 0) \right]^{-1} T(m, n). \end{aligned}$$

Since  $N_-(0)$  is arbitrary,

$$\boxed{T(0, n) = T(0, m) \left[ I - R'(m+1, n+1) R(m, 0) \right]^{-1} T(m, n)} \tag{33}$$

Relation (33) is the desired partition relation governing the standard transmittance operator  $T(0, n)$  for downward flux in a cubic lattice  $\Upsilon_n$ .

## General Partition Relations for Upward Flux

The functional relations governing the  $\mathcal{R}$  and  $\mathcal{T}$  operators for upward flux in an arbitrary  $Y_n$ , namely  $R(n, 0)$  and  $T(n+1, 1)$  may be obtained in precisely the manner used to obtain  $R(1, n+1)$  and  $T(0, n)$  associated with the downward flux, after making the appropriate changes in the boundary lighting conditions. Since it is important that the partition relations for  $R(n, 0)$  and  $T(n+1, 1)$  enjoy the same confidence as their downward flux counterparts, they will also be derived step by step from the principles of invariance. It may be of interest to observe, however, that the desired expressions for  $R(n, 0)$  and  $T(n+1, 1)$  can be read off directly from (31) and (33) when the reader takes full cognizance of the order of occurrence of the individual matrices in their expressions along with their physical significance of each. This, incidentally, was the manner in which  $R(n, 0)$  and  $T(n+1, 1)$  for the linear lattice (of reference 4) were obtained.

We begin with the derivation of the partition relation for  $R(n, 0)$  with respect to the general partition  $\{Y_m, Y_{n-m}\}$  of  $Y_n$ ,  $1 \leq m \leq n$ . In principle IV (Equation (15)) set

$N_-(0) = 0$ ,  $i = m+1$  ; then

$$\begin{aligned} N_-(n) &= N'_+(n+1) R(n,0) \\ &= N_-(m) T(m,n) + N'_+(n+1) R(n,m). \end{aligned} \tag{34}$$

In principle II (Equation (13)), set  $j' = m$ ,  $k' = 1$ , then

$$N_-(m) = N'_+(m+1) R(m,0). \tag{35}$$

In principle I. (Equation (10)), set  $j = m+1$ ,  $k = n$ , then

$$N_+(m+1) = N'_+(n+1) T(n+1, m+1) + N_-(m) R(m+1, n+1). \tag{36}$$

Next, contract each side of (36) by postmultiplying with the contracting matrix  $C'$  (see Equation (24)). The result is:

$$N'_+(m+1) = N'_+(n+1) T'(n+1, m+1) + N_-(m) R'(m+1, n+1). \tag{37}$$

From (35) and (37):

$$N_-(m) = \\ = N'_+(n+1) T'(n+1, m+1) R(m, 0) + N_-(m) R'(m+1, n+1) R(m, 0), \quad (38)$$

which can be solved for  $N_-(m)$  :

$$N_-(m) = \\ = N'_+(n+1) T'(n+1, m+1) R(m, 0) [I - R'(m+1, n+1) R(m, 0)]^{-1} \quad (39)$$

Using this in (34), it follows that

$$N'_+(n+1) R(n, 0) = \\ = N'_+(n+1) R(n, m) + \\ + N'_+(n+1) T'(n+1, m+1) R(m, 0) [I - R'(m+1, n+1) R(m, 0)]^{-1} T(m, n),$$

which in view of the fact that  $N'_+(n+1)$  is arbitrary, yields the operator equation:

$$R(n, 0) = \\ = R(n, m) + T'(n+1, m+1) R(m, 0) [I - R'(m+1, n+1) R(m, 0)]^{-1} T(m, n) \quad (40)$$

This is one form of the partition relation for  $R(n,0)$  in an arbitrary cubic lattice  $Y_n$ . The partition  $\{Y_m, Y_{n-m}\}$  of  $Y_n$  is arbitrary.

An alternate form of (40) may be obtained by making use of a general theorem in matrix theory which states that: For any two equal-order square matrices  $A_1, A_2$ , for which  $[I - A_2 A_1]^{-1}$  exists, then

$$A_2 [I - A_1 A_2]^{-1} = [I - A_2 A_1]^{-1} A_2 .$$

If now we make the following identifications:  $A_1 = R'(m+1, n+1)$ ,  $A_2 = R(m, 0)$ , then using the above matrix theorem, (40) may be written:

$$R(n,0) = R(n,m) + T'(n+1, m+1) [I - R(m,0)R'(m+1, n+1)]^{-1} R(m,0) T(m,n) \quad (41)$$

This is the desired form of the partition relation for in an arbitrary cubic lattice  $Y_n$ . This is also the form obtained if a careful study of (31) is made and then (31) is translated, matrix by matrix, into the upward flux case.

It now remains to derive the expression for  $T(n+1, 1)$ .

Starting with principle III (Equation (14)) in which we set

$N_-(0) = 0$ ,  $\kappa = m$ , it follows that

$$N'_+(n+1) T(n+1, 1) = N'_+(m+1) T(m+1, 1) \quad (42)$$

Using (35) and (37), we find that

$$N'_+(m+1) = N'_+(n+1) T'(n+1, m+1) + N'_+(m+1) R(m, 0) R'(m+1, n+1),$$

which may be solved for  $N'_+(m+1)$ :

$$N'_+(m+1) = N'_+(n+1) T'(n+1, m+1) [I - R(m, 0) R'(m+1, n+1)]^{-1} \quad (43)$$

When this is used in (42), the result is:

$$\begin{aligned} N'_+(n+1) T(n+1, 1) &= \\ &= N'_+(n+1) T'(n+1, m+1) [I - R(m, 0) R'(m+1, n+1)]^{-1} T(m+1, 1), \end{aligned}$$

which in view of the fact that  $N'_+(n+1)$  is arbitrary, yields the operator equation:

$$T(n+1,1) = T'(n+1,m+1) [I - R(m,0)R'(m+1,n+1)]^{-1} T(m+1,1)$$

(44)

This is the desired partition relation for  $T(n+1,1)$  in an arbitrary cubic lattice  $Y_n$ . The partition  $\{Y_m, Y_{n-m}\}$  of  $Y_n$  is arbitrary. Equation (44) is also the form one would obtain by a careful direct translation of (33) into the upward flux geometry.

#### General Recurrence Relations

The act of setting  $m=1$  in the general partition  $\{Y_m, Y_{n-m}\}$  of the cubic lattice  $Y_n$  may be interpreted as the addition of a point to the sub-lattice  $Y_{n-1} = \{(0,0,2), \dots, (0,0,n)\}$ . Hence if the  $R$  and  $T$  operators are known for the sub-lattice  $Y_{n-1}$ , it becomes a simple inductive step to arrive at the  $R$  and  $T$  operators for  $Y_n = (0,0,1) \cup Y_{n-1}$ . Thus setting  $m=1$  in (31) yields the desired recurrence relation for  $R(1,n+1)$ :

$$R(1,n+1) = R(1,2) + T(0,1) [I - R'(2,n+1)R(1,0)]^{-1} R'(2,n+1) T(2,1)$$

(45)

From this it may be observed that the  $9 \times 17$  matrix  $R(1, n+1)$  associated with the lattice  $Y_n = \{(0, 0, 1), \dots, (0, 0, n)\}$  is determinable once we know the  $9 \times 17$  matrix  $R(2, n+1)$  for  $Y_{n-1} = \{(0, 0, 2), \dots, (0, 0, n)\}$ , along with the operators  $R(1, 2)$ ,  $T(0, 1)$ ,  $R(1, 0)$ , and  $T(2, 1)$ .

The latter matrices are the reflectance and transmittance operators for the cell associated with the point  $(0, 0, 1)$  of  $Y_n$ . Equivalently, from the point of view of the extended cubic lattice, these operators are the reflectance and transmittances associated with the single layer at level 1 in the extended cubic lattice from which  $Y_n$  was constructed. The single layer at level  $j$  in the extended cubic lattice will be called the monolayer at level  $j$ . In the simple linear lattice case (reference 4) the reflectance and transmittance of a monolayer are obtainable directly from the  $\Sigma$ -function of the linear lattice. The case of a monolayer in cubic lattice, however, is more complex. This follows from the fact that the cell at level  $j$  in  $Y_n$  can interact with the eight neighboring cells at level  $j$  in the adjacent  $Y_n'$  of the extended cubic lattice, a situation which could not occur in the linear lattice. Hence it becomes a separate task to evaluate the four operators  $R(j, j+1)$ ,  $T(j-1, j)$ ,  $R(j, j-1)$ ,  $T(j+1, j)$ , associated with the monolayer at level  $j$  in a cubic lattice.

A special section below is devoted to this task.

We now continue with the derivations of the general recurrence relations. By setting  $m=1$  in (33), the result is

$$T(0,n) = T(0,1) [I - R'(2,n+1)R(1,0)]^{-1} T(1,n)$$

(46)

Hence if  $T(1,n)$  and  $R'(2,n+1)$  are known for the sublattice  $Y_{n-1} = \{(0,0,2), \dots, (0,0,n)\}$  of  $Y_n$ , along with  $T(0,1)$  and  $R(1,0)$  for the monolayer at level 1, then  $T(0,n)$  for the lattice  $Y_n = \{(0,0,1), \dots, (0,0,n)\}$  is determinable. Because of the presence of  $R'(2,n+1)$  in (46), it follows that any systematic numerical computation procedure must first determine the  $R(j,n+1)$  operators,  $1 \leq j \leq n$  by means of (45). This fact will be taken into account in the computation procedure below.

Equations (45) and (46) are the general recurrence relations for downward flux in an arbitrary cubic lattice  $Y_n$ . Equations (47) and (48) below are the general recurrence relations for upward flux in the arbitrary cubic lattice  $Y_n$ . They are obtained from (41) and (44), respectively, by setting  $m=n-1$ :

$$\begin{aligned}
 R(n,0) &= \\
 &= R(n,n-1) + T'(n+1,n) [I - R(n-1,0)R'(n,n+1)]^{-1} R(n-1,0) T(n-1,n)
 \end{aligned}
 \tag{47}$$

$$T(n+1,1) = T'(n+1,n) [I - R(n-1,0)R'(n,n+1)]^{-1} T(n,1)
 \tag{48}$$

The recurrence relations (47) and (48) may be given a physical interpretation as follows: Suppose the  $R$  and  $T$  operators for upwelling flux are known for the sub-lattice  $\{(0,0,1), \dots, (0,0,n-1)\}$ , i.e., we know  $R(n-1,0)$ ,  $T(n,1)$ . Furthermore, the  $R$  and  $T$  operators for the monolayer at level  $n$  are known, i.e., we know  $R(n,n-1)$ ,  $T(n+1,n)$ ,  $R(n,n+1)$ , and  $T(n-1,n)$ . Then the recurrence formulae (47) and (48) show how to obtain  $R(n,0)$  and  $T(n+1,1)$  for the lattice  $\{(0,0,1), \dots, (0,0,n)\}$  where  $n$  is an arbitrary integer,  $n > 1$ .

An examination of the recurrence relations (45), (46), (47), (48), shows that before they can be used the various  $R$  and  $T$  operators for an arbitrary monolayer at level  $j$  in  $Y_n$  must be known.

Observe that the general partition relations cannot be used to obtain the monolayer operators; for by setting  $\eta$  (and therefore  $r$ ,  $m$ ) equal to unity in (31), (33), (41) and (44), one obtains mere identities by virtue of Equations (16) - (19). However, the way out of this difficulty is easily effected by appealing once more to the principle of local interaction. This will now be done.

## THE R AND T OPERATORS FOR A MONOLAYER

### Introductory Remarks

In the course of deriving the four recurrence relations governing the  $R$  and  $T$  operators for a general multilayer (preceding section) it was found that in order to initiate an actual computation procedure, each of the four recurrence relations required explicit knowledge of the four  $R$  and  $T$  operators for a general monolayer of the extended cubic lattice. It was further noted that the general partition relations for a lattice are unable to supply the required monolayer operators, since the partition relations reduce--as they should--to identities when an extended cubic lattice of unit depth is considered.

The purpose of this section is to derive the required general formulae governing the four  $R$  and  $T$  operators associated with an arbitrary monolayer in the lattice. Toward this end, the principle of local interaction may once again be called into service. By means of it we may quickly and completely solve the present complex multiple scattering problem.

To prepare the ground for the application of the principle of local interaction, we now outline the essential physical and geometric aspects of the present problem. Figure 2 shows the eight immediate neighbors of the point  $(0,0,j)$  (which is the  $j$ 'th member of the column  $Y_n(0,0)$ , see Figure 1). To fix ideas, suppose the entire monolayer is irradiated by a steady collimated specific radiance in the direction of the unit vector  $\hat{k}$ . The point  $(0,0,j)$ , as each of the countably infinite other points in the monolayer, redirects some of this radiance into the eight directions about it toward its eight immediate neighbors in the cell associated with  $(0,0,j)$ . An inter-reflection (a multiple scattering) process is thereby initiated within the monolayer. At steady state a certain fraction of the incident flux escapes from the monolayer, another fixed fraction is continuously cycled within the monolayer, and, finally, a third fixed fraction is lost by absorption. On one hand the local conservation property allows us to conclude that

the sum of these fractions is unity. On the other hand, the principle of local interaction allows us to deduce directly the individual steady state magnitudes of the radiance distribution in each of the twenty six directions about  $(\vartheta, \vartheta, j)$ . In short, the principle allows a means of explicitly calculating the components of the specific radiance vector  $N(j)$  at  $(\vartheta, \vartheta, j)$ . (See Equation (1).)

The requisite vector form of the principle of local interaction is given in Equation (18) of reference 1. This equation, written in its most general form, is:

$$N_+(x_j) = N_-(x_j) \Sigma(x_j) + N^o(x_j) \Sigma^o(x_j). \quad (49)$$

The mathematical definitions of the terms are given in complete detail in reference 1; therefore, they need not be repeated here. However, it will be helpful to restate the physical interpretations of the terms.  $N_+(x_j)$  is the specific radiance output vector of the point  $x_j$  in the general discrete space  $X_n$ ;  $N_-(x_j)$  is the field radiance input vector to the point  $x_j$ . Both  $N_+(x_j)$  and  $N_-(x_j)$  generally have  $n^2$  components.

In the present case, because of the adopted eclipse conventions, they each have 26 components. The vector  $N^o(x_j)$  is the source vector at  $x_j$ , and represents input sources of incident radiant flux on  $x_j$  originating from outside  $X_n$ . The matrices  $\Sigma(x_j)$  and  $\Sigma^o(x_j)$  are the local scattering matrices which transform the incident radiances on  $x_j$  into the output radiance from  $x_j$ .

To adapt the general formula (49) to our present needs, we imagine the nine points of the horizontal section of the cell associated with  $(\theta, \theta, j)$  to form a closed system imbedded within the given extended cubic lattice. This is rigorously permissible in view of the properties of the associated quotient space  $Y_n$  of the extended cubic lattice. (See the discussion above in the section entitled, "The Associated Quotient Space".) Thus the term  $N^o(x_j) \Sigma^o(x_j)$  of (49) may be used to describe the effect of the radiances  $N_-(j-1)$ ,  $N'_+(j+1)$  incident on the monolayer from its upper and lower neighbor layers on levels  $j-1$  and  $j+1$ , respectively. The output of the  $j$ th monolayer in the direction of the  $j-1$  and  $j+1$  layers will then be described respectively by the components  $N'_+(j)$  and  $N_-(j)$ , of the vector  $N(j)$  which are defined in (2) and (4), respectively. The cycling radiance within the  $j$ th monolayer will then be described by the component  $N_o(j)$  of  $N(j)$ , which is defined in (3). Recall that, by virtue of equation (1) above,

the mutual relations between these vectors are:

$$N(j) = [N_+(j), N_-(j)], \quad (50)$$

and

$$N_+(j) = [N'_+(j), N_o(j)] \quad (51)$$

#### The Local Interaction Principle for Horizontal Radiance

With these preliminaries in mind, it should then be clear that the requisite special form of (49) for the present closed system is:

$$N_o(j) = N_o(j) \Sigma_o(j) + N^o(j) \Sigma^o(j). \quad (52)$$

Observe that the vector  $N_o(j)$  plays a dual role in (52). As it stands on the left side of (52) it is an output vector. On the right side of (52) it plays the role of an input vector. This dual role is a consequence of the mathematical fact that

$$N_o(j) = N_o(j) M,$$

where  $M$  is the permutation matrix for the present closed system. (For the detailed description of  $M$ , see Equation (21), reference 1.) The dual role of  $N_o(j)$  may also be understood from physical considerations of the radiant flux pattern within the  $j$ th monolayer. (See Figure 2.) Here, any given output component of  $N_o(j)$ , say the one which directs flux from  $(0,0,j)$  in the  $\underline{j}$ -direction, is at the same time an input radiance on  $(0,0,j)$  coming from the neighbor of  $(0,0,j)$  which lies in the  $(-\underline{j})$ -direction from  $(0,0,j)$ .

#### The General Solution for Horizontal Radiance

With these observations, it follows that  $\Sigma_o(j)$  is an 8 x 8 matrix of the form:

$$\Sigma_o(j) = \begin{pmatrix} \Sigma(j; \xi_{10}; \xi_{10}) & \cdots & \Sigma(j; \xi_{10}; \xi_{17}) \\ \vdots & \ddots & \vdots \\ \Sigma(j; \xi_{17}; \xi_{10}) & \cdots & \Sigma(j; \xi_{17}; \xi_{17}) \end{pmatrix}, \quad (53)$$

For notational reasons which will become clear in the discussion that follows we relabel the matrix (53) as:  $S(j; 0; 0)$ . Formally, then, we set:

$$\Sigma_o(j) \equiv S(j; 0; 0) \quad (8 \times 8) \text{ matrix} \quad (54)$$

The notation  $\Sigma_o(j)$  has now completed its purpose of allowing a conceptually smooth transition from the general context of (49) to the present special monolayer context.

The solution of (52) for  $N_o(j)$  is straight forward:

$$N_o(j) = N^o(j) \Sigma^o(j) [I - S(j; 0; 0)]^{-1} \quad (55)$$

The existence of the inverse of  $I - S(j; 0; 0)$  is guaranteed on the grounds that  $S(j; 0; 0)$  is a norm-contracting operator (Equation (28), reference 1.) Equation (55) then represents the horizontal flux within the  $j$ 'th monolayer induced by radiant flux incident on the monolayer either from the directions within  $\Xi_o$  or  $\Xi - \Xi_o$ , or a combination of these possibilities. The matrix  $\Sigma^o(j)$  in the present general case directs the flux from the directions of the source vector into the direction space  $\Xi_o$  within the monolayer.

## The Auxiliary Scattering Matrices

In order to describe the response of the  $j$ th monolayer to irradiation from the  $j-1$  and  $j+1$  monolayers, we must formulate several special forms of the local interaction principle (49). We now define the special scattering matrices which arise in such a formulation, and collect them for convenient reference below. The symbol  $(\alpha \times \beta)$  to the left of each matrix gives the dimensions of the matrix. For completeness, we include  $S(j; 0; 0)$  once again in (60) below.

$$(9 \times 17) \quad S(j; +; +) = \left( \sum (j; \xi_\alpha; \xi_\beta) \right) \quad (56)$$

$$1 \leq \alpha \leq 9, \\ 1 \leq \beta \leq 17$$

$$(9 \times 8) \quad S(j; +; 0) = \parallel \quad \begin{array}{l} 1 \leq \alpha \leq 9 \\ 10 \leq \beta \leq 17 \end{array} \quad (57)$$

$$(9 \times 9) \quad S(j; +; -) = \parallel \quad \begin{array}{l} 1 \leq \alpha \leq 9 \\ 18 \leq \beta \leq 26 \end{array} \quad (58)$$

$$(8 \times 17) \quad S(j; 0; +) = \parallel \quad \begin{array}{l} 10 \leq \alpha \leq 17 \\ 1 \leq \beta \leq 17 \end{array} \quad (59)$$

$$(8 \times 8) \quad S(j, 0; 0) = \parallel \quad \begin{array}{l} 10 \leq \alpha \leq 17 \\ 10 \leq \beta \leq 17 \end{array} \quad (60)$$

$$(8 \times 9) \quad S(j; 0; -) = \left( \sum (j; \xi_\alpha; \xi_\beta) \right) \quad (61)$$

$$\begin{array}{l} 10 \leq \alpha \leq 17 \\ 18 \leq \beta \leq 26 \end{array}$$

$$(9 \times 17) \quad S(j; -; +) = \quad " \quad \begin{array}{l} 18 \leq \alpha \leq 26 \\ 1 \leq \beta \leq 17 \end{array} \quad (62)$$

$$(9 \times 8) \quad S(j; -; 0) = \quad " \quad \begin{array}{l} 18 \leq \alpha \leq 26 \\ 10 \leq \beta \leq 17 \end{array} \quad (63)$$

$$(9 \times 9) \quad S(j; -; -) = \quad " \quad \begin{array}{l} 18 \leq \alpha \leq 26 \\ 18 \leq \beta \leq 26 \end{array} \quad (64)$$

It may be helpful to point out that the four matrices  $S(j; +; +)$ ,  $S(j; +; -)$ ,  $S(j; -; +)$ , and  $S(j; -; -)$  defined above have the same dimensions respectively as  $T(j+1, j)$ ,  $R(j, j-1)$ ,  $R(j, j+1)$ ,  $T(j-1, j)$ , the four required  $R$  and  $T$  operators for the monolayer. In fact, if there were no cycling horizontal radiant flux within the level  $j$  (which is essentially the case in the linear lattice in which there is essentially only one point comprising the monolayer) then we would have  $R(j, j+1) = S(j, -; +)$ , etc. However, an interreflection process at a given point within the  $j$ th monolayer exists by virtue of the presence of neighboring points in the layer. Thus an interreflection term must be added to  $S(j; -; +)$  before it can be equated to  $R(j, j+1)$ .

## The Required Forms of the R and T Monolayer Operators

1. Downward Flux.

Let the  $j^{\text{th}}$  monolayer be irradiated by an arbitrary  $(j-1)$  monolayer radiance distribution,  $N_{-(j-1)}$ . This vector then acts as the source vector  $N^o(j)$  in (52). There are no other sources on the monolayer at this time. Let  $N_o(j)$  now denote the cycling horizontal flux induced by this source. The general local interaction principle (49) now applied once again to the monolayer as a discrete space imbedded in the original extended cubic lattice, states that the reflected radiance vector  $N_+(j)$  is governed by the equation:

$$N_+(j) = N_o(j) S(j; 0; +) + N_{-(j-1)} S(j; -; +). \quad (65)$$

For this same source, the transmitted radiance vector  $N_-(j)$  is governed by:

$$N_-(j) = N_o(j) S(j; 0; -) + N_{-(j-1)} S(j; -; -). \quad (66)$$

Now using the reflectance operator  $R(j, j+1)$  and the transmittance operator  $T(j-1, j)$  for downward flux incident on the  $j$ 'th monolayer, we may also write

$$N_+(j) = N_-(j-1) R(j, j+1), \quad (67)$$

$$N_-(j) = N_-(j-1) T(j-1, j), \quad (68)$$

which follow from the two main Principles of Invariance I, II (Equations (10), (13)) by setting  $\lambda = j = K$ , and using the present source hypothesis that  $N_+'(j+1) = 0$ .

From (55), the vector  $N_0(j)$  is now given by

$$N_0(j) = N_-(j-1) S(j; 0; -) [I - S(j; 0; 0)]^{-1} \quad (69)$$

Combining (65), (67), and (69):

$$\begin{aligned} N_-(j-1) R(j, j+1) &= N_+(j) = \\ &= N_-(j-1) S(j; -; 0) [I - S(j; 0; 0)]^{-1} S(j; 0; +) \\ &+ N_-(j-1) S(j; -; +). \end{aligned}$$

Since  $N_{-(j-1)}$  is arbitrary, it follows that

$$R(j, j+1) = S(j; -; +) + S(j; -; 0) [I - S(j; 0; 0)]^{-1} S(j; 0; +) \quad (70)$$

Repeating this process for the combination of (66), (68), (69):

$$\begin{aligned} N_{-(j-1)} \bar{F}(j-1, j) &= N_{-(j)} = \\ &= N_{-(j-1)} S(j; -; 0) [I - S(j; 0; 0)]^{-1} S(j; 0; -) + N_{-(j-1)} S(j; -; -) \end{aligned}$$

and since  $N_{-(j-1)}$  is arbitrary,

$$T(j-1, j) = S(j; -; -) + S(j; -; 0) [I - S(j; 0; 0)]^{-1} S(j; 0; -) \quad (71)$$

## 2. Upward Flux.

The pattern of derivation is now clear. To determine the  $R$  and  $T$  operators  $R(j, j-1)$  and  $T(j+1, j)$  for upward flux incident on the  $j$ th monolayer, we first write down the requisite special form of (49) with  $N^0(j)$  now replaced by an

arbitrary upward radiance distribution  $N'_+(j+1)$  from the  $j+1$  monolayer: The cycling horizontal flux  $N_o(j)$  within the  $j$ th monolayer is then given by

$$N_o(j) = N'_+(j+1) S(j, +; 0) [I - S(j; 0; 0)]^{-1} \quad (72)$$

The reflected downward radiance distribution  $N_-(j)$  and transmitted upward radiance distribution  $N_+(j)$  are then described by:

$$N_-(j) = N_o(j) S(j; 0; -) + N'_+(j+1) S(j; +; -) \quad (73)$$

$$N_+(j) = N_o(j) S(j; 0; +) + N'_+(j+1) S(j; +; +) \quad (74)$$

The main principles of invariance I and II (Equations (10), (13)) now yield:

$$N_-(j) = N'_+(j+1) R(j, j-1) \quad (75)$$

$$N_+(j) = N'_+(j+1) T(j+1, j) \quad (76)$$

by setting  $i = j = k$  and using the present source hypothesis,  
 $N_-(j^-) = 0$  .

Combining (72), (73) and (75), we have

$$\begin{aligned} N_+^i(j+1) R(j, j-1) &= N_-(j) = \\ &= N_+^i(j+1) S(j; +, 0) [I - S(j; 0; 0)]^{-1} S(j; 0; -) \\ &+ N_+^i(j+1) S(j; +, -), \end{aligned}$$

whence

$$R(j, j-1) = S(j; +, -) + S(j; +; 0) [I - S(j; 0; 0)]^{-1} S(j; 0; -) \quad (77)$$

In a similar way, we find:

$$T(j+1, j) = S(j; +, +) + S(j; +; 0) [I - S(j; 0; 0)]^{-1} S(j; 0; +) \quad (78)$$

Equations (70), (71), (77), (78) give the required reflectance operators for the  $j$ th monolayer in an extended cubic lattice under a plane boundary lighting condition.

## REMARKS ON THE POLARITY OF THE R AND T OPERATORS

## The Polarity Theorem

Knowing under what conditions polarity of an optical medium is to be expected, allows the theorist or computer to occasionally simplify his work by choosing the appropriate form of the  $R$  and  $T$  functions on the medium. The concept of polarity in radiative transfer theory was defined and discussed in reference 4. The discussion of the polarity of the  $R$  and  $T$  operators for a cubic lattice proceeds in a manner similar to that for the  $R$  and  $T$  factors associated with a linear lattice. We shall therefore not enter into any further analytical details here.

Because of the more complex local direction space  $\Xi'$  in a cubic lattice, there is more of a chance, so to speak, for polarity to arise. One would then expect the Polarity Theorem for the linear lattice to be modified in the cubic lattice setting. The form which the theorem now takes is:

Polarity Theorem for Cubic Lattices: (i) If  $\Upsilon_n$  is isotropic at every point and is homogeneous, then the  $R$  and  $T$  operators possess no polarity. That is:

$$R'(1, n+1) = R(n, 0) , \quad (79)$$

$$T(0, n) = T'(n+1, 1) \quad (80)$$

(ii) If  $\Upsilon_n$  is isotropic at every point but is not homogeneous, then the  $R$  and  $T$  operators generally possess polarity, That is:

$$R'(1, n+1) \neq R(n, 0) , \quad (81)$$

$$T(0, n) \neq T'(n+1, 1) . \quad (82)$$

One implication of the theorem is that if  $\Upsilon_n$  is isotropic and homogeneous for all points of  $\Upsilon_n$  except one point then one may generally expect polarity of the  $R$  and  $T$  operators associated with  $\Upsilon_n$  . Another implication is that isotropy of all the points of  $\Upsilon_n$  does not generally guarantee isotropy of  $\Upsilon_n$  : Statement (ii) asserts that the presence of inhomogeneities in an otherwise isotropic medium gives rise to anisotropy on the quotient space level.

The proof of the present polarity theorem may be based on the general recurrence relations (45), (46), (47), and (48). The proof, which is quite simple, begins by setting  $n=2$  in all of these equations. This simulates the case of a two-layer cubic lattice. Because of the possibility of a quotient space interpretation, the two layer lattice may well represent the partition into two layers of a continuous slab of arbitrary finite depth. An examination of the resulting two-layer expressions under various assumptions on the homogeneity and isotropy properties on the point level yields the two conclusions of the theorem.

#### A Specific Example of Polarity

Instead of the formal proof, we will give an example of sufficient simplicity to allow the reader to perceive intuitively the polarity of the  $R$  and  $T$  operators in a specific case.

Consider a continuous slab  $X$  which consists of two homogeneous isotropic contiguous layers A and B whose boundaries are parallel to the  $xy$  plane. The upper layer A is purely absorbing. Hence in A the volume scattering function  $\sigma$  is identically zero on  $X \times \Xi$  and the volume absorption function has some constant

fixed value  $\alpha$ . The layer B is purely scattering so that  $\alpha \equiv 0$  on B, and  $\mathcal{T}$  has some fixed directional structure independent of depth in B. The slab considered as a whole is therefore inhomogeneous, and isotropic at each point.

Suppose a single pencil of radiant flux of unit radiance is normally incident on the upper boundary of A from above. The pencil is transmitted, unscattered, to the boundary between A and B. Thus as the attenuated flux reaches this internal boundary, it is still in collimated form and normally incident on the boundary. The layer B transmits the incident flux which emerges in some fanned out directional pattern, say  $P_-$ , at the lower boundary of B. The pattern  $P_-$  is characteristic of the  $\mathcal{T}$  operator for B.

Now suppose the lower boundary of B is irradiated by a normally incident pencil of radiant flux of unit radiance in the upward direction. Layer B then transmits this flux and the flux emerges in some fanned out pattern  $P_+$  at the boundary between A and B. Of course the absolute magnitude of the transmitted radiance through B differs from the previous case, but it should be intuitively clear that the directional structures of  $P_-$  and  $P_+$ , and the relative amounts transmitted by B are identical. Symbolically,  $P_- = P_+$ .

Now the pattern  $P_+$  is transmitted to the upper boundary of A. Because A is purely absorbing, the emergent directional pattern  $P'_+$  of the radiance at the upper boundary of A is generally different from the common pattern of  $P_-$  and  $P_+$ . The pattern  $P'_+$  can, in fact, be easily calculated, given  $P_+$ . Thus the downward and upward radiance transmittance operators of the slab  $X$  are different. A similar argument can be made for the downward and upward radiance reflectance operators associated with  $X$ .

Of course, in real life no such slabs exist; but one may imagine a continuous, gradual departure within A and B from these ideal extremes. Since polarity is a yes-or-no phenomenon by definition (i.e., either there is polarity or there is not polarity) it is clear that polarity will persist as the departure from this extreme case is made as long as the medium  $X = A + B$  is inhomogeneous. This completes the example.

## GENERAL SOLUTION OF THE TWENTY-SIX-FLOW PROBLEM

## Solution of the Problem

All of the pieces of the solution of the twenty-six-flow problem in an extended cubic lattice have now been manufactured. It remains only to assemble them in the proper order. Suppose then that  $N_-(0)$  represents a 9-component radiance vector at each point of the upper boundary of an extended cubic lattice  $X_n$  of depth  $n$ . Then the downward radiance vector  $N_-(j)$  at level  $j$ , is given in terms of  $N_-(0)$  by (29), and  $N_+(j)$  by (22) once  $N_-(j)$  is known. The  $R$  and  $T$  operators appearing in (22) and (29) are defined by (31) and (33) for the downward flux, and by (41) and (44) for the upward flux. Using equations (45), (46), (47), (48) these recurrence formulae may be used to obtain the general  $R$  and  $T$  operators once the  $R$  and  $T$  operators for a monolayer are known. The formulae for the monolayer operators for the present problem are given in terms of the basic  $\Sigma$ -function of  $X_n$  in equations (70), (71), (77), (78). This entire procedure may be applied mutatis mutandis to the problem of an incident radiance distribution at level  $n$ , i.e., when  $N_-(0) = 0$  and  $N'_+(n+1) \neq 0$ .

### Boundary Effects

The preceding general solution may be used to take into specific account the effects of reflecting boundaries of a medium. This can be done in two ways: (i) By assigning to  $\Sigma$ -functions of layers 1 and  $n$  the reflectance and transmittance properties of the boundaries of the medium. (ii) By adjoining to  $\chi_n$  the additional layers labeled 0 and  $n+1$  which are then assigned  $\Sigma$ -functions which represent the boundary reflectance and transmittance data. This latter method will be used in the computation procedure given below.

### Superposition of Solutions

Since all of the presently used principles and operators of discrete space radiative transfer theory are linear, the basic solution above may be used to find  $N_+(j)$  and  $N_-(j)$  when an arbitrary set of incident source conditions are given at the upper and lower boundaries simultaneously. This is accomplished by adding together the contributions to  $N_+(j)$  and to  $N_-(j)$  from each of the boundary sources.

### Homogeneous Isotropic Media

In the event that a homogeneous isotropic extended cubic lattice  $X_n$  is considered, the basic solution procedure for the medium is considerably simplified in view of the Polarity Theorem. (See Equations (79) - (82).) It would in this case be sufficient to compute only  $R(j, n+1)$ ,  $1 \leq j \leq n$  and  $T(j, n)$ ,  $0 \leq j \leq n-1$  for downward flux. Under these conditions we have in addition:  $R(j, K) = R(j', K')$  and  $T(j, K) = T(j', K')$  whenever  $K-j = K'-j' \geq 0$ .

### The Internal-Source Problem

The present work has considered the so called complete two boundary problem (using the classification scheme of plane-parallel transfer problems, reference 5). In such a problem, the radiance distribution is required at each point of a medium which is arbitrarily stratified, has generally two reflecting-transmitting boundaries, and the only sources of radiance on the medium are those which irradiate the boundaries from without. The internal source problem on the other hand hypothesizes that the same medium generally has its internal layers directly irradiated by sources, and requires the determination of the resultant radiance

distribution at each point of the medium. This problem may be solved using the techniques of discrete space radiative transfer theory. To solve this problem it is found that the principle of local interaction must be pressed into use in all its guises: in the form of the invariant imbedding relation, the principles of invariance, and the local interaction principle itself. To formulate and solve the internal-source problem here would constitute too great a digression from the present goals; this task will be reserved for a subsequent report.

#### THE PLANE-PARALLEL MEDIUM AND ITS ASSOCIATED CUBIC LATTICE

The purpose of this section is to outline the sequence of steps to be followed whenever a cubic lattice representative of a plane-parallel arbitrarily stratified optical medium is to be constructed.\*

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\* With only minor notational changes ( $\mathbf{x}$ -notation to  $(x, y, z)$ -notation) the following construction procedure is immediately applicable to an arbitrary space  $X_n$  which is to be imbedded in a general cubic lattice. The only material change in the formulations is the appropriate generalization of the distribution factors  $D_i$  (see (84) below) which are here specially designed for the plane-parallel setting.

In this way the complex multiple scattering problem in a continuous medium may be converted into its more tractable discrete counterpart. There are two main steps in such a construction: (a) The partitioning of the body of the given plane-parallel medium into a finite number  $n$  of a contiguous slabs. These  $n$  slabs along with the (generally) two boundaries of  $X$  are assembled to form an extended cubic lattice  $X_{n+2}$  of depth  $n+2$ . (b) Each point of a column of the associated quotient space  $Y_{n+2}$  is assigned a local scattering function  $\Sigma$  and local absorption function  $A$  in such a way that these functions are related as naturally and as closely as possible to their continuous counterparts in  $X$ , and such that they satisfy the local conservation property (Equation (11), reference 1).

## Construction of the Lattice

We begin with an outline of the first step in the construction of the associated lattice.  $X$  is the given slab, i.e.,  $X = \{(x, y, z) : 0 \leq z \leq z_1\}$  with unit outward normal  $k$  (See Figure 1). Let  $X$  be arbitrarily stratified, i.e., let its optical properties be constant on planes parallel to the plane  $x_0 = \{(x, y, z) : z = 0\}$  which is the upper boundary of  $X$ . As in reference 4, divide  $X$  into  $n$  slabs  $x_j$ ,  $1 \leq j \leq n$  whose boundaries are parallel to  $x_0$ , and such that the  $j$ th slab  $x_j$  is of vertical extent  $\Delta_j$ . Let  $x_{n+1} = \{(x, y, z) : z = z_1\}$  be the lower boundary of  $X$ . The ordered collection  $\{x_0, x_1, \dots, x_n, x_{n+1}\}$  of subsets of  $X$  assumes the role of the column  $Y_{n+2}(0, 0)$  in the associated quotient space  $Y_{n+2}$  of the cubic lattice. The subset  $x_j$ ,  $0 \leq j \leq n+1$  of  $E_3$  will assume the role of the  $j$ th point of the column  $Y_{n+2}(0, 0)$ . Hence  $\Delta_0 = \Delta_{n+1} = 0$ . For computation purposes, the subsets  $x_j$ ,  $1 \leq j \leq n$  will be considered concentrated at the point  $(0, 0, z_j)$  in  $E_3$  where  $z_j$  is the mid-depth of the slab  $x_j$ :

$$z_j = \Delta_0 + \dots + \Delta_{j-1} + \frac{\Delta_j}{2}, \quad 1 \leq j \leq n.$$

We will once again abbreviate the column notation  $Y_{n+2}(0, 0)$  to  $Y_{n+2}$ .

## Construction of the Local Direction Space

The second main step in the construction of the cubic lattice is the assignation of the  $\Sigma$  and  $A$  functions to the points of the column  $Y_{n+2}$ . For this part it is necessary to know first of all  $\sigma$  and  $a$ , the volume scattering and volume absorption functions on  $X$ , and secondly, to know the reflectances and transmittances of the upper and lower boundaries  $\alpha_0$  and  $\alpha_{n+1}$  of  $X$ . The latter information is coded in the form:  $\Sigma(0; \xi_i; \xi_k)$  and  $\Sigma(n+1; \xi_i; \xi_k)$ , respectively, where  $\xi_i, \xi_k$  are elements of the set of twenty-six directions of the common local direction space  $\Xi'$  of each point of  $Y_{n+2}$ .

Before this assignation can be performed, however, it is necessary to explicitly give the formulas for the elements of  $\Xi'$ . The most natural display of  $\Xi'$  is as follows (next page):

$$\equiv + :$$

$$\left\{ \begin{array}{l} \xi_1 = (0, 0, -1) \\ \xi_2 = \frac{(1, 0, -1)}{\sqrt{2}} \quad \xi_3 = \frac{(1, 1, -1)}{\sqrt{3}} \quad \xi_4 = \frac{(0, 1, -1)}{\sqrt{2}} \quad \xi_5 = \frac{(-1, 1, -1)}{\sqrt{3}} \\ \xi_6 = \frac{(-1, 0, -1)}{\sqrt{2}} \quad \xi_7 = \frac{(-1, -1, -1)}{\sqrt{3}} \quad \xi_8 = \frac{(0, -1, -1)}{\sqrt{2}} \quad \xi_9 = \frac{(1, -1, -1)}{\sqrt{3}} \end{array} \right. \quad (83)$$

$$\equiv 0 :$$

$$\left\{ \begin{array}{l} \xi_{10} = (1, 0, 0) \quad \xi_{11} = \frac{(1, 1, 0)}{\sqrt{2}} \quad \xi_{12} = (0, 1, 0) \quad \xi_{13} = \frac{(-1, 1, 0)}{\sqrt{2}} \\ \xi_{14} = (-1, 0, 0) \quad \xi_{15} = \frac{(-1, -1, 0)}{\sqrt{2}} \quad \xi_{16} = (0, -1, 0) \quad \xi_{17} = \frac{(1, -1, 0)}{\sqrt{2}} \end{array} \right.$$

$$\equiv - :$$

$$\left\{ \begin{array}{l} \xi_{18} = \frac{(1, 0, 1)}{\sqrt{2}} \quad \xi_{19} = \frac{(1, 1, 1)}{\sqrt{3}} \quad \xi_{20} = \frac{(0, 1, 1)}{\sqrt{2}} \quad \xi_{21} = \frac{(-1, 1, 1)}{\sqrt{3}} \\ \xi_{22} = \frac{(-1, 0, 1)}{\sqrt{2}} \quad \xi_{23} = \frac{(-1, -1, 1)}{\sqrt{3}} \quad \xi_{24} = \frac{(0, -1, 1)}{\sqrt{2}} \quad \xi_{25} = \frac{(1, -1, 1)}{\sqrt{3}} \\ \xi_{26} = (0, 0, 1) \end{array} \right.$$

This display is patterned after the usual  $(\theta, \phi)$  representation of directions using polar coordinates. The vector  $\xi_1$  is the zenith or basic outward direction, the vector  $\xi_{26}$  is the nadir or basic inward direction. All the others are arranged so that they cover each  $\theta$ -level in an increasing  $\phi$ -fashion, starting with  $\theta = 0$ , and ending with  $\theta = \pi$ . (See Figure 3, which depicts the cell associated with the origin of  $E_3$ .)

#### Construction of the Attenuation Functions

The task that remains is the assigning of the  $\Sigma$  and  $A$  functions to the remaining  $n$  points of  $Y_{n+2}$ . Toward this end, partition the unit sphere  $\Xi$  in  $E_3$  about  $x_j$  into twenty-six regions  $\Xi_j$  of equal solid angle measure  $\Omega(\Xi_j) = 4\pi/26$  such that  $\xi_j \in \Xi_j$ . The precise geometric description of these subsets of  $\Xi$  is best left to the individual programmer; even equal solid angle requirements may be relaxed for individual cases. We will require, in addition, the twenty-six distribution factors  $D_i$ :

$$D_i = \frac{\int_{\Xi_i} d\Omega(\xi)}{\int_{\Xi_i} |S \cdot \xi| d\Omega(\xi)}, \quad 1 \leq i \leq 26 \quad (84)$$

Now by means of the basic relation <sup>\*</sup>

$$\alpha(z) = a(z) + \Delta(z), \quad (85)$$

where

$$\Delta(z) = \int_{\Xi} \sigma(z; \xi'; \xi) d\Omega(\xi), \quad (86)$$

is the value of the volume total scattering function  $\Delta$  at depth  $z$ , and  $\alpha$  is the volume attenuation function, we may write

$$\alpha(z, \xi_2) = a(z, \xi_2) + \Delta(z, \xi_2), \quad (87)$$

---

\* If  $X$  is anisotropic, relations (85) and (86) still hold. In this event the direction of the incident flux should be explicitly included. This is done in (88). Hence from (88) onward, the possibility of anisotropy is explicitly carried along in the notation.

where

$$\begin{aligned}\alpha(z, \xi_\lambda) &= D_\lambda \alpha(z) \\ a(z, \xi_\lambda) &= D_\lambda a(z) \\ \Delta(z, \xi_\lambda) &= D_\lambda \Delta(z)\end{aligned}\tag{88}$$

Furthermore, for  $\xi_j \in \Xi'$ , define

$$\Sigma(j; \xi_i; \xi_\lambda) = e^{-\alpha(z_j, \xi_i) \Delta_j} + \Delta_j D_i \int_{\Xi_i} \sigma(z_j; \xi_i; \xi) d\Omega(\xi), \tag{89}$$

and for  $\xi_\lambda \neq \xi_\kappa$ ,  $\xi_i, \xi_\kappa \in \Xi'$ , define

$$\Sigma(j; \xi_i; \xi_\kappa) = \Delta_j D_i \int_{\Xi_\kappa} \sigma(z_j; \xi_i; \xi) d\Omega(\xi), \tag{90}$$

and finally, define

$$A(j, \xi_\lambda) = a(z_j, \xi_\lambda) \Delta_j, \tag{91}$$

where  $j$  in (89) - (91) takes on the values  $j = 1, \dots, n$ .

## Demonstration of the Local Conservation Property

The local conservation property for the functions  $\Sigma(\sigma; \cdot; \cdot)$  and  $A(\sigma; \cdot)$  associated with  $\mathcal{X}_0$  (and also those for  $\mathcal{X}_{n+1}$ ) is an immediate consequence of the properties of the reflectance and transmittances of surfaces. For the case of the points  $\mathcal{X}_j$ ,  $1 \leq j \leq n$ , however, the local conservation property is not supplied automatically and must be verified. Now we have defined  $\Sigma$  and  $A$  above in such a way that the local conservation property is arbitrarily closely approximated, the finer the subdivision of  $X$ . To see this we merely recall that  $e^{-\alpha(z_j, \xi_i) \Delta_j}$  may be represented exactly as:

$$e^{-\alpha(z_j, \xi_i) \Delta_j} = 1 - \alpha(z_j, \xi_i) \Delta_j + o(\alpha(z_j, \xi_i) \Delta_j), \quad (92)$$

where the "o" symbol stands for a function with the property:

$$\frac{o(\epsilon)}{\epsilon} \rightarrow 0$$

as  $\epsilon \rightarrow 0$ . Hence for sufficiently small  $\Delta_j$ ,  $o(\alpha(z_j, \xi_i) \Delta_j)$  can, for example, be an order of magnitude smaller than  $\alpha(z_j, \xi_i) \Delta_j$ . Combining (87) - (92), it follows

that

$$A(j; \xi_i) + \sum_{k=1}^{26} \Sigma(j; \xi_i; \xi_k) = 1 + o(\alpha(z_j, \xi_i) \Delta_j), \quad (93)$$

for every  $j$ ,  $1 \leq j \leq n$  and each  $\xi_i \in \Xi'$ . Hence for a given slab  $X$  over  $[0, \bar{x}_1]$  and a sufficiently large  $n$  the local conservation property becomes arbitrarily closely satisfied on  $Y_{n+2}$ . (See Equation (11), reference 1 for general statement of local conservation property.)

An alternative to (89) and (90) would be:

$$\Sigma(j; \xi_i; \xi_i) = e^{-\alpha(z_j, \xi_i) \Delta_j} + \Delta_j D_i \Omega_i \sigma(z_j; \xi_i; \xi_i) \quad (94)$$

$$\Sigma(j; \xi_i; \xi_k) = \Delta_j D_i \Omega_i \sigma(z; \xi_i; \xi_k) \quad (95)$$

This alternative way of defining  $\Sigma$  would perhaps be more welcome in the tedious preparation period prior to a numerical calculation program. Now, however, unless  $\sigma$  is isotropic, (93) can only be approximated, even as  $\max \{ \Delta_j \} \rightarrow 0$ .

## COMPUTATION PROCEDURE

This section contains a finite sequence of rules which constitute a computation procedure leading to the numerical tabulation of the radiance distributions  $N(j) = [N_+(j), N_-(j)]$ ,  $j = 0, 1, \dots, n, n+1$ , on a cubic lattice  $Y_{n+2}$  with reflecting boundaries and source conditions at the upper and lower boundaries. Before proceeding with the actual details, some preliminary remarks are made which can insure efficient use of the computation procedure.

## Preliminary Definitions and Observations

Definition:\* Let  $X$  represent an arbitrarily stratified isotropic plane-parallel medium between the planes at geometric depths  $z_0$ , and  $z_1 \geq z_0$ . Then if  $\alpha$  is the volume attenuation function on  $X$ , the optical depth  $\tau(z', z'')$  of a general subslab of  $X$  defined by the interval  $[z', z''] \subset [z_0, z_1]$  is :

$$\tau(z', z'') \equiv \int_{z'}^{z''} \alpha(z) dz. \quad (96)$$

\* If  $X_n$  is anisotropic, then instead of  $\alpha$  use  $\bar{\alpha}(\cdot) = \frac{1}{2} [\alpha(\cdot, \mathfrak{E}_1) + \alpha(\cdot, \mathfrak{E}_2)]$  over  $[z_0, z_1]$ .

Hence, to any geometrical depth  $z - z_0$  in  $X$  below the  $z_0$ -boundary,  $z_0 \leq z \leq z_1$ , there is assigned an optical depth :

$$\tau \equiv \int_{z_0}^z \alpha(z) dz. \quad (96')$$

Definition: Let  $X$  represent a plane-parallel medium. Let  $\alpha$  and  $\sigma$  represent the volume attenuation and volume scattering functions on  $X$  and  $X \times \Xi \times \Xi$  respectively. Then the medium  $X$  is said to be separable if and only if the phase function  $\rho$  :

$$\rho = 4\pi\sigma/\alpha \quad (97)$$

defined on  $X \times \Xi \times \Xi$ , is independent of optical depth  $\tau$  in  $Z$ .

Observation: If a plane-parallel medium  $X$  is separable, then  $X$  becomes a homogeneous medium under the replacement of geometric depth  $z - z_0$  by optical depth  $\tau$ , i.e., under the adoption of the optical-depth coordinate system.

Observation: It follows from the preceding observation and the Polarity Theorem ( (79) - (82) ) that the  $R$  and  $T$  operators of an isotropic separable medium are polarity-free in the optical-depth coordinate system.

Observation: If  $X$  is isotropic at every point, then there are only three basically distinct directions of  $\Xi$  - along which pencils of unit radiance may impinge on a point of  $\mathcal{X}_0$  (the upper boundary). These directions are distinct in the sense that the resultant light field from a unit radiance impinging on the point along any of the other 6 directions of  $\Xi$  - is, after a suitable rotation about the  $Z$  axis, identical to that associated with one of these three directions. In the notation of (83), we will choose  $\xi_{24}$ ,  $\xi_{25}$ , and  $\xi_{26}$  as these basic directions.

## The Seven Stages of the Computation Procedure

STAGE ONE: DEFINITION OF  $Y_{n+2}$ 

- (a) Basic Data Assumed Known: Reflectance and transmittance of upper and lower boundaries of the given plane-parallel medium  $X$ ; the volume attenuation function  $\alpha$  of  $X$ ; the volume scattering function  $\sigma$  of  $X$ ; source condition giving  $N^0$  at upper and lower boundaries of  $X$ .
- (b) Choose an optical depth  $\tau(0, z_1)$ , by means of (96), over which  $X$  is to be considered.
- (c) Choose an integer  $n$  which defines the number of slabs into which  $X$  is to be partitioned. Let  $\Delta\tau \equiv \tau(0, z_1)/n$ . The geometric depth  $\Delta_j$  of the  $j$ th slab is then defined recursively by the equation:

$$j \Delta\tau \equiv \int_0^{\Delta_1 + \dots + \Delta_j} \alpha(z) dz.$$

Net result: A partition of  $X$  into  $n$  slabs of fixed optical depth  $\Delta\tau$ . This partition, together with the upper and lower boundaries  $x_0$  and  $x_{n+1}$  of  $X$  defines  $Y_{n+2} = \{x_0, x_1, \dots, x_n, x_{n+1}\}$ , the associated quotient space of  $X_n$ .

- (d) Define the local direction space  $\Xi'$ .  
(Equation (83).)
- (e) Determine the local scattering functions  $\Sigma(j; \cdot; \cdot)$  and local absorption functions  $A(j; \cdot)$  for each point of  $Y_{n+2}$  so that  $\Sigma$  and  $A$  satisfy the local conservation property. (See Equations (84) - (93).)

STAGE TWO: MONOLAYER MATRICES.

- (a) Determine the  $R$  and  $T$  operators for the  $j$ 'th monolayer;  $0 \leq j \leq n+1$  in  $Y_{n+2}$ . Use Equations (70), (71), (77), and (78).
- (b) Notation:  $R(j, j+1)$ ,  $T(j-1, j)$  are  $R$  and  $T$  matrices for downward flux at level  $j$ ,  $0 \leq j \leq n+1$ .  $R(j, j-1)$ ,  $T(j-1, j)$  are  $R$  and  $T$  matrices for upward flux at level  $j$ ,  $0 \leq j \leq n+1$ .

STAGE THREE: DOWNWARD REFLECTANCE MATRICES

- (a) The reflectance of the  $\rho$ -layer subset of  $Y_{n+2}$ ,  $1 \leq \rho \leq n+2$ , whose initial layer is at level  $n+2-\rho$  and whose terminal layer is at  $n+1$  (Figure 4) is given by:

$$\begin{aligned}
 R(n+2-\rho, n+2) = & R(n+2-\rho, n+3-\rho) + \\
 & + T(n+1-\rho, n+2-\rho) \left[ I - R'(n+3-\rho, n+2) R(n+2-\rho, n+1-\rho) \right]^{-1} \times \\
 & \times R'(n+3-\rho, n+2) T(n+3-\rho, n+2-\rho)
 \end{aligned}$$

(98)

This follows by induction on  $p$ , starting with recurrence relation (45) applied to  $Y_{n+2}$ . This formula may be checked by setting  $p=1$ ,  $p=n+2$ . The former case yields an identity, the latter yields (45) for  $Y_{n+2}$ .

- (b) Compute  $R(n+2-p, n+2)$  for  $p=1, \dots, n+2$ . Observe that for each  $p > 1$ , all the matrices on the right of (98) are known, either from STAGE TWO, on the preceding step for a  $(p-1)$  layer.

STAGE FOUR: UPWARD REFLECTANCE MATRICES

- (a) The reflectance of a  $p$ -layer subset of  $Y_{n+2}$ ,  $1 \leq p \leq n+2$  whose initial layer is at level  $p-1$  and whose terminal layer is at level 0 (Figure 5) is given by:

$$R(p-1, -1) = R(p-1, p-2) + T'(p, p-1) \times \\ \times \left[ I - R(p-2, -1) R'(p-1, p) \right]^{-1} R(p-2, -1) T(p-2, p-1)$$

(99)

This follows by induction on  $p$  starting with the recurrence relation (47) applied to  $Y_{n+2}$ .

This formula may be checked by setting  $p=1, p=n+2$   
 The former case yields an identity, the latter  
 yields (47) for  $Y_{n+2}$ .

- (b) Compute  $R(p-1, -1)$  for  $p=1, \dots, n+2$   
 Observe that for each  $p > 1$ , all matrices on  
 the right of (99) are known, either from STAGE  
 TWO, or the preceding step for a  $(p-1)$ -layer.

STAGE FIVE: DOWNWARD TRANSMITTANCE MATRICES

- (a) The transmittance of a  $p$ -layer subset of  $Y_{n+2}$   
 $1 \leq p \leq n+2$ , whose initial  
 layer is at level 0 and whose terminal layer  
 is at  $p-1$  (Figure 5) is given by:

$$T(-1, p-1) = T(-1, p-2) \left[ I - R'(p-1, p) R(p-2, -1) \right]^{-1} \times$$

$$\times T(p-2, p-1)$$

(100)

This follows from (46) by making the following  
 substitutions: layer symbol 0  $\rightarrow$  - | layer symbol  
 $m \rightarrow p-2$ , layer symbol  $n \rightarrow p-1$ .

This formula may be checked by setting  $p=1$ ,  
 $p=n+2$ . The former case yields an  
 identity, the latter yields (46) for  $Y_{n+2}$ .

(b) Compute  $T(-1, p-1)$  for  $p=1, \dots, n+2$

Observe that for each  $p > 1$ , all matrices  
 on the right of (100) are known, either from  
 STAGE FOUR, STAGE TWO, or the results of the  
 preceding step for a  $(p-1)$  - layer.

STAGE SIX: UPWARD TRANSMITTANCE MATRICES

(a) The transmittance of a  $p$  layer subset of  $Y_{n+2}$ ,  
 $1 \leq p \leq n+2$ , whose initial layer is at  
 level  $n+1$  and whose terminal layer is a  
 level  $n+2-p$  (Figure 4) is given by:

$$T(n+2, n+2-p) = T'(n+2, n+3-p) \left[ I - R(n+2-p, n+1-p) \times \right. \\ \left. \times R'(n+3-p, n+2) \right]^{-1} T(n+3-p, n+2-p)$$

This follows from (44) by making the following substitutions: layer symbol  $n \rightarrow n+1$ , layer symbol  $m \rightarrow n+2-p$ , layer symbol  $l \rightarrow n+2-p$ . This formula may be checked by setting  $p=1$ ,  $p=n+2$ . The former case yields an identity, the latter case yields (44) for  $\Upsilon_{n+2}$ .

- (b) Compute  $T(n+2, n+2-p)$  for  $p=1, \dots, n+2$ . Observe that for each  $p > 1$ , all matrices on the right side of (101) are known, either from STAGE THREE, STAGE TWO, or the results of the preceding step for a  $(p-1)$  - layer.

STAGE SEVEN: RADIANCE VECTORS

Part (A). Source Condition: Irradiate  $\Upsilon_{n+2}$  at the upper boundary  $\chi_0$ , in turn, by a unit radiance along each of the nine directions of  $\Xi$ . The vector  $N'_+(n+2)$  is set equal to the zero vector throughout Part (A) of STAGE SEVEN.

- (a) For each of the nine incidence directions of  $\Xi_-$ , compute  $N_-(j)$  at each level,  $0 \leq j \leq n+1$ , using (29) now adapted to  $Y_{n+2}$ :

$$N_-(j) = N_-(-1) T(-1, j) [I - R'(j+1, n+2) R(j, -1)]^{-1} \quad (102)$$

where  $N_-(-1) = [N^0(0, \xi_{1\theta}), \dots, N^0(0, \xi_{2\phi})]$  and in which each component, in turn, is unity with all others zero. The matrices required in this step are given by the results of STAGES THREE, FOUR, and FIVE.

- (b) For each of the nine incidence directions of  $\Xi_-$ , compute  $N_+(j)$  at each level,  $0 \leq j \leq n+1$  using (22) now adapted to  $Y_{n+2}$ :

$$N_+(j) = N_-(j-1) R(j, n+2) \quad (103)$$

The matrices required in this step are given by the results of STAGE THREE.

Part (B). Source Condition: Irradiate  $Y_{n+2}$  at the lower boundary  $\mathcal{X}_1$ , in turn, by a unit radiance along each of the nine directions of  $\Xi'_+$ . The vector  $N_-(-1)$  is set equal to the zero vector throughout part (B) of STAGE SEVEN.

- (a) For each of the nine incident directions of  $\Xi'_+$ , compute  $N_+(j)$  at each level,  $0 \leq j \leq n+1$  using:

$$\begin{aligned}
 N_+(j) &= \\
 &= N'_+(n+2) \left\{ T(n+2, j) + T'(n+2, j) [I - R(j-1, -)R'(j, n+2)]^{-1} R(j-1, -)R(j, n+2) \right\}
 \end{aligned}
 \tag{104}$$

which is a combination of the formulae (35), (36), and (43) now applied to  $Y_{n+2}$ . In (104),  $N'_+(n+2)$  is of the form:

$$N'_+(n+2) = [N^0(n+1, \xi_1), \dots, N^0(n+1, \xi_9)] ,$$

in which each component in turn is unity with all others zero. The matrices required in this step are given by the results of STAGES THREE, FIVE, and SIX.

- (b) For each of the nine incidence directions of  $\Xi_+$  compute  $N_-(j)$  at each level,  $0 \leq j \leq n+1$  using (35) now applied to  $Y_{n+2}$  :

$$N_-(j) = N'_+(j+1) R(j, -1) \quad (105)$$

The matrices required in this step are given by the results of STAGE FIVE.

Part (C): Source Condition:  $Y_{n+2}$  irradiated at  $x_0$  and  $x_{n+1}$  by arbitrary source vectors  $N_-(-1)$ ,  $N'_+(n+1)$  .

- (a) Each of the resultants:  $N_+(j)$  and  $N_-(j)$ , at each level  $0 \leq j \leq n+1$ , is the sum of the contributions from each of the boundary sources, computed in accordance with Parts (A) and (B) above.

THIS ENDS THE COMPUTATION PROCEDURE.

## REFERENCES

1. Preisendorfer, R. V., Radiative Transfer on Discrete Spaces, SIO Ref. 59-53, Scripps Institution of Oceanography, University of California, La Jolla, California, (1959).
2. \_\_\_\_\_ Principles of Invariance on Discrete Spaces, SIO Ref. 59-55, Scripps Institution of Oceanography, University of California, La Jolla, California, (1959).
3. \_\_\_\_\_ The Theory of Polarized Light Fields in Discrete Spaces, SIO Ref. 59-66, Scripps Institution of Oceanography, University of California, La Jolla, California, (1959).
4. \_\_\_\_\_ Radiative Transfer on a Linear Lattice, SIO Ref. 59-67, Scripps Institution of Oceanography, University of California, La Jolla, California, (1959).
5. \_\_\_\_\_ The Planetary Hydrosphere Problem, SIO Ref. 58-40, Scripps Institution of Oceanography, University of California, La Jolla, California (1957).
6. \_\_\_\_\_ Functional Relations for the R and T Operators on Plane-Parallel Media, Proc. Nat. Acad. Sci. 44, 323-327 (1958).

## TABLE OF NOTATION

<u>Notation</u>	<u>Definition</u>	<u>Page</u>
A	Local absorption function. $A(j, \xi_i) = a(z_j, \xi_i) \Delta_j$	67
$a(z, \xi_i)$	Volume absorption function.	63
$a(z)$	$a(z, \xi_i) = D_i a(z)$	
a, b, c	Finite integers $\exists 0 \leq a \leq b; 0 \leq c$	3
$C'$	Contracting matrix (17x9). $C' = \begin{pmatrix} I \\ 0 \end{pmatrix}$	24
$C(x, y, z)$	Cell associated with $(x, y, z)$ , its center. $C(x, y, z) = \{(x', y', z') :  x' - x  \leq 1,  y' - y  \leq 1,  z' - z  \leq 1; (x, y, z) \in E_3\}$	5
$D_i$	Distribution factor for direction $\xi_i$ . $D_i = \frac{\int_{\Xi_i} d\Omega(\xi)}{\int_{\Xi_i}  \xi \cdot k  d\Omega(\xi)} \quad 1 \leq i \leq 26$	65

$E_3$	Euclidean 3-space	3
$I$	Identity matrix	24
$j$	$j = y, 1 \leq j \leq n, \text{ in } (0, 0, y) \in \Psi_n,$ corresponds to depth.	7
$k$	unit outward normal to $\Psi_n$ $k = (0, 0, -1)$	8
$M$	Permutation matrix	42
$m$	Arbitrary partition index of $\Psi_n$	23
$N(j)$	Radiance distribution at $(0, 0, j)$ $N(j) = [N(0, 0, j, \xi_1), \dots, N(0, 0, j, \xi_{26})]$ $= [N'_+(j), N_0(j), N_-(j)]$ $= [N_+(j), N_-(j)]$	8 10
$N_+(j)$	$N_+(j) = [N'_+(j), N_0(j)]$	10
$N'_+(j)$	$N'_+(j) = [N(j, \xi_1), \dots, N(j, \xi_9)]$ where $\xi_i \in \Xi_+ - \Xi_0 = \Xi'_+$ ; $1 \leq i \leq 9$	9
$N_0(j)$	$N_0(j) = [N(j, \xi_{10}), \dots, N(j, \xi_{17})]$ where $\xi_i \in \Xi_0$ , $10 \leq i \leq 17$ .	9
$N_-(j)$	$N_-(j) = [N(j, \xi_{18}), \dots, N(j, \xi_{26})]$ where $\xi_i \in \Xi_-$ ; $18 \leq i \leq 26$	9
$N'_+(n+1)$	Incident upward radiance distribution on $(0, 0, n)$ in $\Psi_n$ $N'_+(n+1) = N^0(n, \cdot)$ on $\Xi'_+$	12

$N_-(0)$	Incident downward radiance distribution on $(0,0,1)$ in $\Psi_n$ $N_-(0) = N^0(1, \cdot)$ on $\Xi_-$	12
$N(j, \xi_i)$	$N(j, \xi_i) \equiv N(0,0,j, \xi_i)$	9
$N^0(x_i, \cdot)$	Source radiance distribution function Generally zero on $\Xi$ for all $x_i \in X_n$ except $x_a$ and $x_b$	4
$n$	number of elements in $\Psi_n$	7
$O$	Zero matrix	24
$o$	Order of magnitude function $\frac{o(\epsilon)}{\epsilon} \rightarrow 0$ as $\epsilon \rightarrow 0$	68
$p$	Phase function, $p = 4\pi \frac{\sigma}{\alpha}$	71
$R(j, i-1)$	$R(j, i-1) = R(j+1, j, i-1)$ , $i \leq j$ Standard reflectance operator (upward flux) $9 \times 9$ matrix	15
$R(j, k+1)$	$R(j, k+1) = R(j-1, j, k+1)$ , $j \leq k$ Standard reflectance operator (downward flux) $9 \times 17$ matrix	13
$R(j, j)$	$R(j, j) = O$ , $9 \times 9$ zero matrix	19

$\mathcal{Q}$	See page 12	12
$S(z)$	Volume Total scattering function $S(z) = \int_{\Xi} \sigma(z, \xi^i, \xi) d\Omega(\xi)$	66
$S(z, \xi_i)$	$S(z, \xi_i) = D_i S(z)$	
$S(j, \cdot, \cdot)$	$S(j, \cdot, \cdot) = \sum (j, \xi_\alpha, \xi_\beta)$ $S(j, +, +): 1 \leq \alpha \leq 9, 1 \leq \beta \leq 17, (9 \times 17)$ $S(j, +, 0): 1 \leq \alpha \leq 9, 10 \leq \beta \leq 17, (9 \times 8)$ $S(j, +, -): 1 \leq \alpha \leq 9, 18 \leq \beta \leq 26, (9 \times 9)$ $S(j, 0, +): 10 \leq \alpha \leq 17, 1 \leq \beta \leq 17, (8 \times 17)$ $S(j, 0, 0): 10 \leq \alpha \leq 17, 10 \leq \beta \leq 17, (8 \times 8)$ $S(j, 0, -): 10 \leq \alpha \leq 17, 18 \leq \beta \leq 26, (8 \times 9)$ $S(j, -, +): 18 \leq \alpha \leq 26, 1 \leq \beta \leq 17, (9 \times 17)$ $S(j, -, 0): 18 \leq \alpha \leq 26, 10 \leq \beta \leq 17, (9 \times 8)$ $S(j, -, -): 18 \leq \alpha \leq 26, 18 \leq \beta \leq 26, (9 \times 9)$	45
$T(i-1, j)$	$T(i-1, j) = \mathcal{T}(i-1, j, j+1), \quad i \leq j$ Standard transmittance operator (downward flux) $(9 \times 9)$ matrix	15
$T(k+1, j)$	$T(k+1, j) = \mathcal{T}(k+1, j, j-1), \quad j \leq k$ Standard transmittance operator (upward flux) $(9 \times 17)$ matrix	13
$T(j, j)$	$T(j, j) = I, (9 \times 9)$ Identity matrix	19
$\mathcal{T}$	See page 12	12
$X_n$	Cubic lattice of $n = (2c+1)^2(b-a+1)$ points $X_n = \{(x, y, z) :  x ,  y  \leq c; a \leq z \leq b; \\ x, y, z \text{ integers}\}$	3

$x, y, z$	Real numbers. $x, y, z$ integers when $(x, y, z) \in X_n$	3
$x_a$	Upper boundary. $(x, y, z) \in X_n: z = a$	4
$x_b$	Lower boundary. $(x, y, z) \in X_n: z = b$	4
$Y_n$	Quotient space of $X_n$ . Column partitioning of $X_n$ $Y_n(x, y) = \{(x, y, z) : (x, y, z) \in X_n; x, y \text{ fixed}\}$	6
$Y_n(0, 0)$	The basic column of cubic lattice considered in this treatment.	7
$\alpha(z)$	$\alpha(z) = a(z) + s(z)$ , volume attenuation function.	66
$\alpha(z, \xi_i)$	$\alpha(z, \xi_i) = a(z, \xi_i) + s(z, \xi_i)$ $= D_i \alpha(z)$	66
$\Delta_j$	Vertical extent of $j$ th slab $x_j$	62
$\theta$	Vertical angle from zenith	65
$\Xi$	Unit sphere in $E_3$ ; the set of all $\xi \in E_3$	4
$\Xi'$	Fixed local direction space defined by $C(x, y, z)$	6 64
$\Xi_+$	Set of all upward (outward) directions $\Xi_+ = \{\xi : \xi \in \Xi'; \xi \cdot k \geq 0\}$	8

- $\Xi'_+$  Set of all proper upward (outward) directions 8  
 $\Xi'_+ = \{ \xi : \xi \in \Xi' ; \xi \cdot R > 0 \}$   
 $= \Xi_+ - \Xi_0$
- $\Xi_0$  Set of all horizontal (singular) directions 8  
 $\Xi_0 = \{ \xi : \xi \in \Xi' ; \xi \cdot R = 0 \}$   
 $\Sigma_0 \equiv S(\cdot, 0, 0)$
- $\Xi_-$  Set of all downward (inward) directions 8  
 $\Xi_- = \{ \xi : \xi \in \Xi' ; \xi \cdot R < 0 \}$
- $\xi_i$  Unit vectors on  $\Xi'$ .  $1 \leq i \leq 26$  63
- $\Sigma_0(j)$ 

$$\Sigma_0(j) = \begin{pmatrix} \Sigma(j; \xi_{10}; \xi_{10}) & \cdots & \Sigma(j; \xi_{10}; \xi_{17}) \\ \cdots & \cdots & \cdots \\ \Sigma(j; \xi_{17}; \xi_{10}) & \cdots & \Sigma(j; \xi_{17}; \xi_{17}) \end{pmatrix}$$
 43
- $\Sigma_0(j) \equiv S(j, 0, 0)$  44
- $\Sigma(j; \xi_i; \xi_k)$  Local scattering function on a discrete space 67  

$$\Sigma(j; \xi_i; \xi_i) = e^{-\alpha(z_j; \xi_i) \Delta_j} + \Delta_j D_i \int_{\Xi_i} \sigma(z_j; \xi_i; \xi) d\Omega(\xi)$$
  

$$\Sigma(j; \xi_i; \xi_k) = \Delta_j D_i \int_{\Xi_k} \sigma(z_j; \xi_i; \xi) d\Omega(\xi)$$
- Or:
- $\Sigma(j; \xi_i; \xi_i) = e^{-\alpha(z_j; \xi_i) \Delta_j} + \Delta_j D_i \Omega_i \sigma(z_j; \xi_i; \xi_i)$  69  
 $\Sigma(j; \xi_i; \xi_k) = \Delta_j D_i \Omega_i \sigma(z_j; \xi_i; \xi_k)$

$\sigma$	Volume scattering function	63
$T(z', z'')$	Optical depth of the interval defined by $[z', z''] \in [z_0, z_1]$ in an isotropic medium. $T(z', z'') = \int_{z'}^{z''} \alpha(z) dz, z' \leq z''$ $T = \int_{z_0}^z \alpha(z) dz$	70
$\phi$	Horizontal angle	65
$\Omega(\Xi_j)$	Solid angle division of the unit sphere about $\xi_j, \xi_j \in \Xi_j$ Specifically: $\Omega(\Xi_j) = \frac{4\pi}{26}$	65

THE CELL  $C(x,y,z)$  ASSOCIATED WITH THE POINT  $(x,y,z)$  IN  $X_n$

Note: For simplicity, the unit cubes, which partition  $E_3$ , and at whose centers the points of  $X_n$  lie, have been omitted from the figure. The light lines connecting the points of  $C(x,y,z)$  are drawn in to achieve a 3-dimensional effect

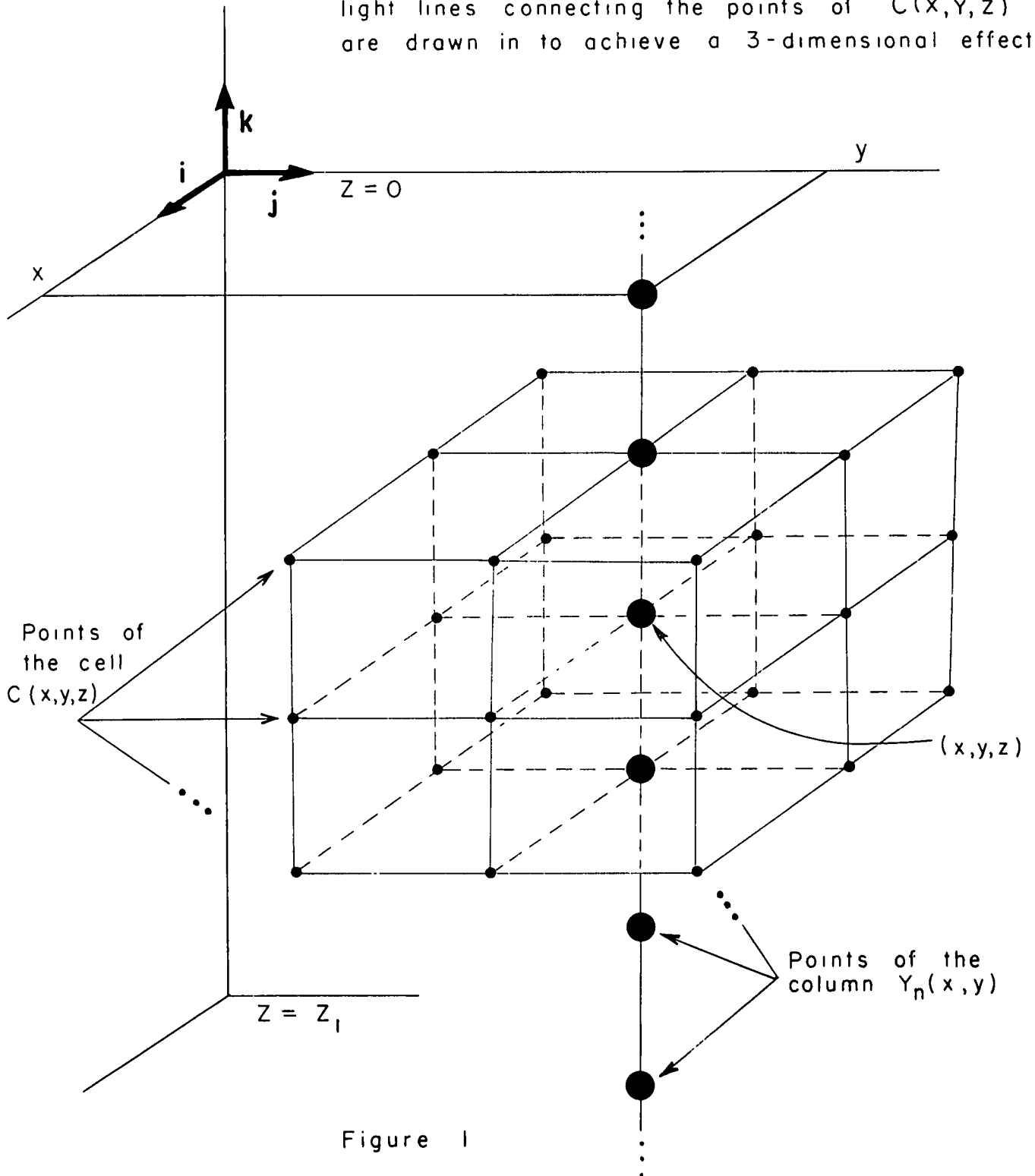
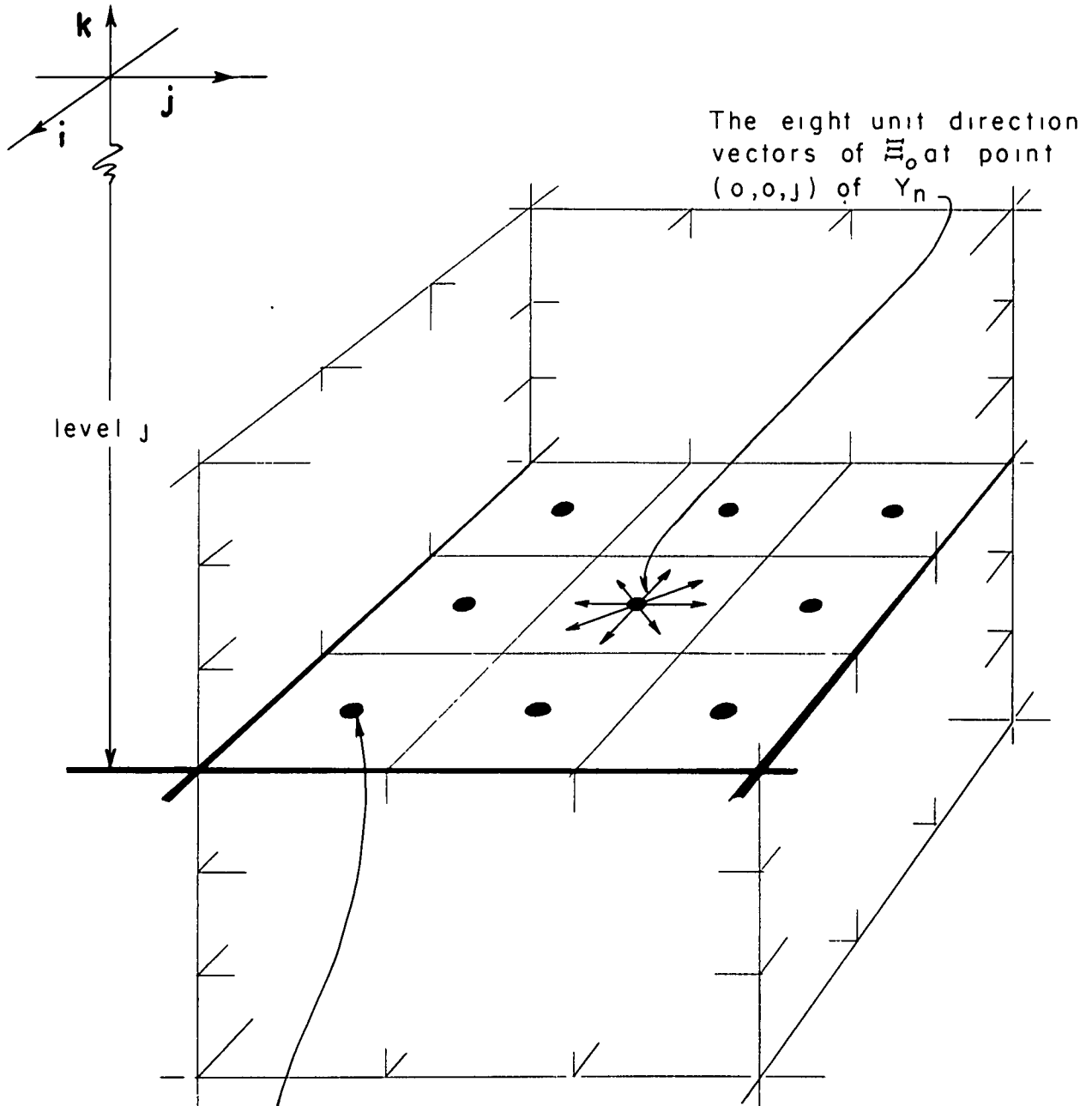


Figure 1

RADIOMETRIC ACTIVITY WITHIN A MONOLAYER  
OF THE EXTENDED CUBIC LATTICE



The heavy dots denote the eight points of the cell associated with  $(0,0,j)$  which lie in the monolayer at level j.

Figure 2

THE LOCAL DIRECTION SPACE  $\Xi'$

The numbers in the circles correspond to the indices of the elements of  $\Xi'$ . The circles correspond to the elements of the cell associated with the origin  $(0,0,0)$  of  $E_3$ .

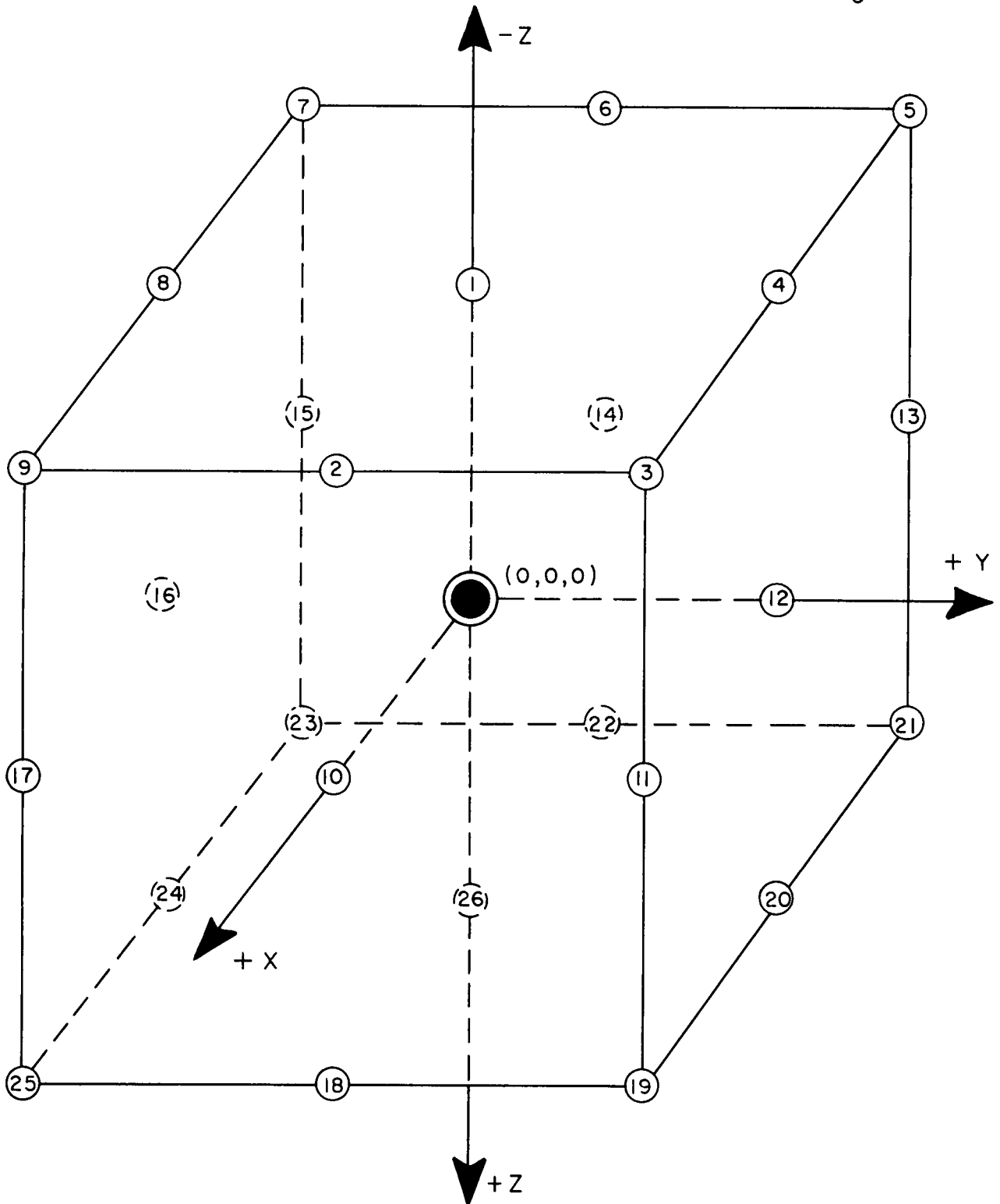


Figure 3

THE FOUR R AND T OPERATORS FOR THE P-LAYER SUBSETS OF  $Y_{n+2}$  USED IN THE COMPUTATION PROCEDURE

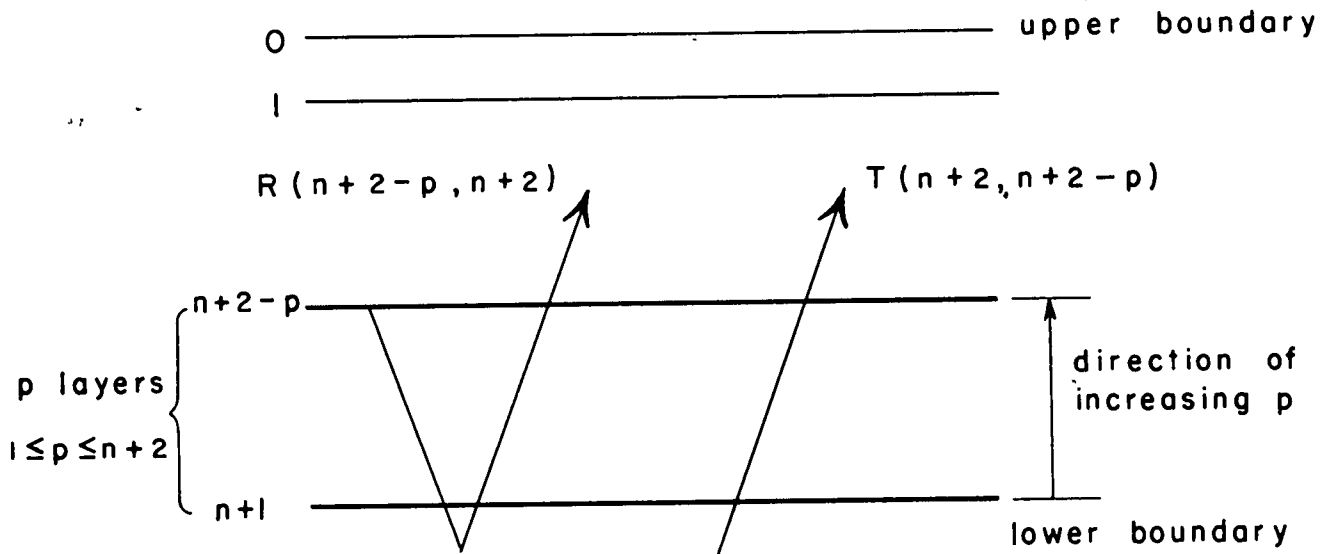


Figure 4

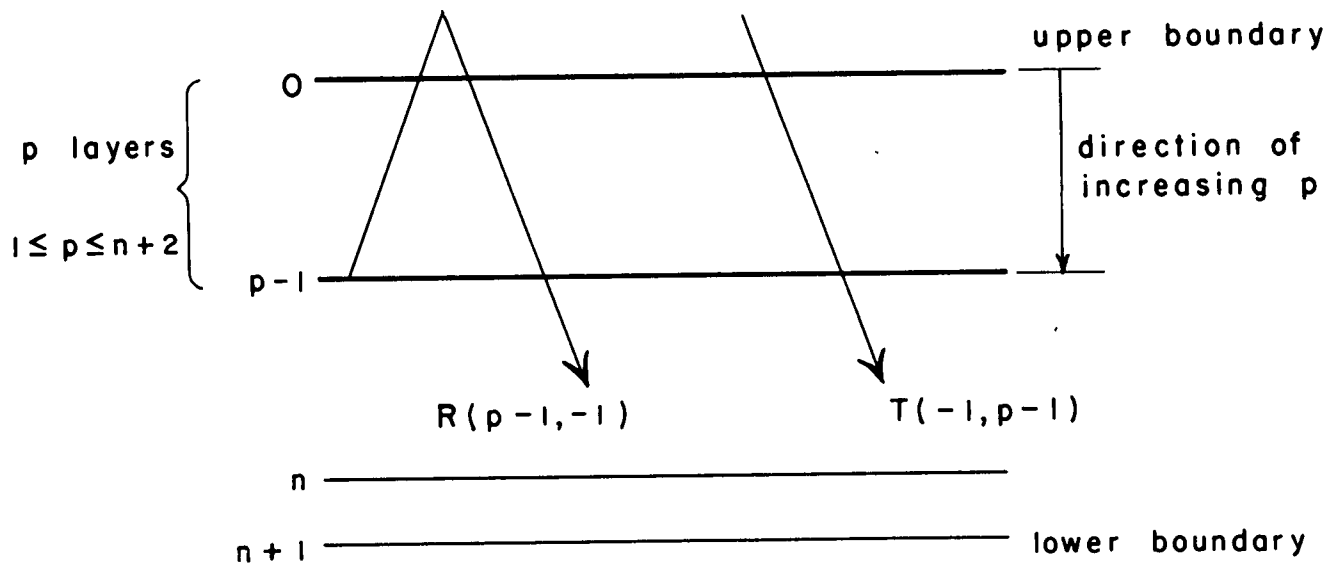


Figure 5