Visibility Laboratory
University of California
Scripps Institution of Oceanography
San Diego 52, California

TEMPORAL METRIC SPACES IN RADIATIVE TRANSFER THEORY

II. Epoch Times and Characteristic Functions

Rudolph W. Preisendorfer

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Approved:

Seibert Q. Duntley

Seibert Q. Duntley, Director
Visibility Laboratory

Approved for Distribution:

Roger Revelle

Roger Revelle, Director
Scripps Institution of Oceanography

Temporal Metric Spaces in Radiative Transfer Theory

II. Epoch Times and Characteristic Functions

Rudolph W. Preisendorfer

Scripps Institution of Oceanography, University of California

La Jolla, California

INTRODUCTION

This paper continues the study of the time dependent multiple scattering problem begun in paper I. We introduce the notions of least epoch time and local epoch time by means of the radiative process on an arbitrary carrier space and show under what conditions they allow various temporal metrics to be introduced into the carrier space. It turns out that the least local epoch time operator has the properties of a pseudo-metric (defined below) whenever the carrier space has a temporally homogeneous radiative process. Thus a pseudo-metric topology may be introduced on all such carrier spaces.

The notion of the characteristic function is also introduced, and several of its key properties are illustrated by observations and theorems. The characteristic function is not only a convenient conceptual tool which describes the interchange of flux between two points of space but also forms the basis for a "temporal metric calculus" consisting of a collection of symbols which may be manipulated according to fixed rules, and which yield physically meaningful statements about the time-dependent radiation field. In this way the formalism based on the characteristic function (and later the characteristic ellipsoids and spheroids) can be used to solve practical problems in much the same way that Boolean algebra and the rules of elementary logic can solve the practical problems of computer encoding and switch-circuit analyses.

As in paper I a necessary prerequisite for the present discussions is the material developed in reference 1. Most of the theorems in this and succeeding papers of the series are generalizations of some earlier work contained in reference 2 which has been brought up to date and considerably extended, using the foundation developed in reference 1.

EPOCH TIMES

Let $(X, \underset{\sim}{S}, v)$ be an arbitrary carrier space. Let X contain a single point source (the fundamental source) $p_0 \in X$ which initiates emission of radiant energy at time t_0. The temporal variation of the output of p_0 is arbitrary but well defined. If $t \geq t_0$, we define $T = t - t_0$ as the epoch time with respect to the source p_0.

In the interests of simplicity we limit the set of fundamental sources to a single point. All of the present results can be generalized to the case of an arbitrary set of sources; but such a generalization does not add materially to the conclusions obtained for a single source. In the (phenomenological) theory of radiative transfer the effect of a set of sources is postulated to be representable by a (suitable) linear superposition of the effects of the individual sources.

Least Epoch Time

The Radiative Process Axiom[1] on $(X, \underset{\sim}{S}, v)$ gives rise to a non-negative valued function γ on $T \times X \times T \times X$, the γ-density, where T stands for the time domain. This function describes in abstract form the propagation of radiation from a point in $T \times X$ to another point in $T \times X$ in the presence of reflecting, refracting, scattering, absorbing, and emission mechanisms in X.

For example, the value

$$\gamma(t_o, p_o;\ t_o + T, p) \quad (\geq 0)$$

of γ gives the radiance at $p \in X$ at time $t_o + T$ as induced by the radiance at p_o at time t_o. It may happen that for a fixed pair of points (p_o, p) and a given t_o, that at epoch time T the value of the γ-density is zero. This may be interpreted by saying that p has not, at epoch time T, received any flux from p_o. On the other hand a certain value of T may yield a positive value of γ, indicating that p is receiving at this epoch time radiant flux from p_o. It is clear that the least of these times is a measure of the temporal separation between p_o and p.

We may now formulate the notion of least epoch time $T_{p_o p}$ between the fundamental source p_o and an arbitrary point p:

$$T_{p_o p} = \inf \{ T : \gamma(t_o, p_o;\ t_o + T, p) > 0 \}. \tag{1}$$

This is somewhat different than the method of determining the temporal separation $t(p_o, p)$ of p_o and p by means of the Transfer Process, as was done in paper I. However, under suitable conditions on the space X and its Radiative Process (namely certain special classical carrier space conditions) these two methods yield the same measure of temporal separation of points of X (Theorem 3 below).

Least epoch time $T_{p_0 p}$ is a more general concept than $t(p_0, p)$ in the sense that the latter time is defined only if p_0 and p are on a natural path in X ; $T_{p_0 p}$ is, however, defined for all pairs of points in X . The most commonly occurring minimum radiative transfer route in real media is defined by the direct traversal of the natural path $P(p_0, p)$ between p_0 and p whenever this path exists. However, in media with opaque obstacles or transparent media with relatively bizarre variations of index of refraction, the extremal time $t(p_0, p)$ associated with $P(p_0, p)$ may be larger than $T_{p_0 p}$. $T_{p_0 p}$ may be described as the epoch time at which the "first burst" of energy from p_0 arrives at p --regardless of the path taken to reach p .

Local Epoch Time

The <u>local epoch time</u> $T_{p_0}(p)$ at p relative to the fundamental source p_0 is defined as:

$$T_{p_0}(p) = T - T_{p_0 p} . \qquad (2)$$

$T_{p_0}(p)$ is clearly the time elapsed since p has first received radiant flux from p_0 .

In each of the preceding definitions, the fundamental source was referred to as the starting point for the supply of radiant energy in X. By means of the mechanisms of scattering, etc., referred to above, each point p' of X may be considered as a potential secondary source of radiant energy to which may be assigned a local epoch time $T_{p_0}(p')$. This time plays the same role for p' as T plays for p_0. Therefore if p' is a point in X which acts as a secondary local source for still another point p in X, let $T_{p'p}$ be the least local epoch time $T_{p_0}(p')$ at which p receives flux from p'. Formally,

$$T_{p'p} = \inf \left\{ T_{p_0}(p') : \gamma\left(T_{p_0 p'}, p'; T_{p_0 p'} + T_{p_0}(p'), p \right) > 0 \right\} . \quad (3)$$

Furthermore, at the instant p receives flux from p' we may begin to reckon a local epoch time $T_{p'}(p)$ at p with respect to the secondary source p':

$$T_{p'}(p) = T_{p_0}(p') - T_{p'p} . \quad (4)$$

Observations on the Properties of Epoch Times

1. If $T_{\rho_0}(\rho) > 0$, the element of "volume" about $\rho \neq \rho_0$ is definitely receiving (and possibly "re-emitting") flux from ρ_0; if $T_{\rho_0}(\rho) = 0$, it is not. In the special case where $\rho = \rho_0$ then $T_{\rho_0}(\rho_0) = T - T_{\rho_0\rho_0} = T$. Observe that no assumptions have been made about the temporal behavior of the radiant flux output of ρ_0 (or for any local source). The fundamental source may radiate for just an instant, or for some finite length of time in any manner during that time and then be permanently shut off. It may start, on the other hand, at t_0 and radiate uniformly, or in any prescribed manner for all $T > 0$, etc.

2. It follows trivially from the definitions and the properties of the Radiative Process that for all $\rho \in X$,
$T_\rho(\rho) = T_{\rho_0}(\rho)$.

3. If ρ' and ρ are in X and $P(\rho', \rho)$ exists, then $t(\rho', \rho) \geq T_{\rho'\rho}$.
For the most general Radiative Process both $t(\rho', \rho)$ and $T_{\rho'\rho}$ are dependent on t_0. Recall that the temporal semimetric t was formulated in I by means of the Transfer Process in such a way that $t(\rho_1, \rho_2) = |t_1 - t_2|$.

Here p_1, p_2 are any two points on a natural path defined by a time t_0, a point $p_0 \in X$, and two times t_1, t_2 such that

$$p_1 = T_{t_0, t_1}(p_0),$$
$$p_2 = T_{t_0, t_2}(p_0).$$

The Transfer Process Axiom is so general that the transformation T_{t_1, t_2} for any two times may depend jointly on t_1 and t_2. Whenever T_{t_1, t_2} depends only on $t_2 - t_1$, the Transfer Process is said to be <u>temporally homogeneous</u>. Such a case arises, for example, when the index of refraction n does not explicitly depend on time. In the present series both the Radiative and Transfer Process $\mathcal{R}_\phi$, $\mathcal{T}_\psi$ will be assumed temporally homogeneous whenever the symmetry property of a particular metric-function is required.

A simple example of how temporal inhomogeneity can destroy the symmetry property of the temporal semimetric t can be drawn from Example 3 of paper I. We observed that for the natural path $A A_1 A_2 C$,

$$t(A, C) = \frac{1}{c}\left[d(A, A_1) + n d(A_1, A_2) + d(A_2, C)\right].$$

If at some time t_1, a ray traverses AC (in that sense) when $n = n_1$, in X_0, we obtain a corresponding value for $t(A, C)$, call it $t_1(A, C)$. If at some time t_2, $t_2 > t_1$, $n = n_2 \neq n_1$ in X_0, then a ray traverses CA (in that sense) in time $t_2(C, A)$. Clearly, in this case,

$$t_1(A,C) \neq t_2(C,A),$$

and, in fact,

$$t_2(C,A) - t_1(A,C) = \frac{n_2 - n_1}{c} d(A_1, A_2).$$

Even more severe break-downs of the symmetry property occur when spatial variations of n are piled on top of its temporal variations. We observe, however, that the symmetry property of t holds for quite arbitrary <u>spatial</u> variations of n as long as they are independent of time.

4. Definition (4) may be given a wider context. Consider any three points p^*, p' and p in X such that p^* is a local source for p',

and p' is a local source for p ;
schematically:

$$p^* \longrightarrow p' \longrightarrow p \, .$$

A local epoch time $T_{p'}(p)$ can now be defined at p as

$$T_{p'}(p) = T_{p^*}(p') - T_{p'p} \, .$$

The point p^* now takes the role of p_0 in (4), but is not necessarily the point p_0 . An immediate corollary of this definition holds for a finite sequence $\{p_0, p_1, p_2, \ldots, p_n\}$ in X such that p_i is a local source for p_{i+1}. The local time $T_{p_{k-1}}(p_k)$ at p_k with p_{k-1} as an <u>immediate local source</u> is given by

$$T_{p_{k-1}}(p_k) = T_{p_{k-2}}(p_{k-1}) - T_{p_{k-1}p_k}$$

$$= T - \sum_{j=0}^{k-1} T_{p_j p_{j+1}} \, .$$

Summary of Epoch Times

The various epoch times described in this section may now be summarized:

T	epoch time	time elapsed since fundamental source ρ_0 in X began to emit.
$T_{\rho^*}(\rho)$	local epoch time	time elapsed since ρ first received flux from source ρ^*.
$T_{\rho'\rho}$	least local epoch time	temporal separation of two points ρ' and ρ in X, i.e., local epoch time $T_{\rho^*}(\rho')$ required for flux leaving ρ' to first reach ρ, where ρ^* is a source for ρ.

TEMPORAL CONNECTIVITY

By defining the least epoch time $T_{p'p}$ between any two points p' and p by means of the γ-density associated with the Radiative Process, we were able to bypass explicit consideration of the actual path or paths taken by the radiant energy to go from p' to p. In our subsequent studies it will be necessary to know something about these paths. We prepare the groundwork for this knowledge with a review of some basic properties of classical carrier spaces.

Classical Carrier Spaces

If $(\Phi, \underset{\sim}{\Phi}, \nu)$ is a classical carrier space then, by definition, $\Phi = X_0 \times \Xi$, where X_0 is some well-defined subset of E_3, and Ξ is the unit sphere in E_3. X_0 is the location space component and Ξ is the direction space component of the phase space Φ. A point p of Φ is then a pair $p = (x, \xi)$, $x \in X_0$, $\xi \in \Xi$.

Now in the usual run of classical carrier spaces it is possible to imbed <u>any two points of X_0</u> in a natural path; the associated ξ's are determined by the extremals of the Euler equations (see paper I). The emphasis here is on <u>points of X_0</u>. Because of the necessity of satisfying the boundary conditions on the extremals, we see that even in the most simple of classical carrier spaces, <u>any two points of Φ</u> are generally not on a natural path. Fortunately it is the connectivity of points of X_o by means of natural paths that is of central interest in the multiple scattering problem.

We now turn to a specific example of this connectivity in classical carrier spaces. Besides being of illustrative value, the following discussion supplies the motivation for the adoption of the postulate of temporal connectivity in classical and general carrier spaces.

Temporal Connectivity in Classical Carrier Spaces

Let $P_{\rho'\rho}$ designate a path associated with $T_{\rho'\rho}$. We will show that $P_{\rho'\rho}$ is not necessarily unique, and secondly that $P_{\rho'\rho}$ is not necessarily a natural path $P(\rho',\rho)$ when the latter exists. These facts may be illustrated by examining Figure 1 of Example 3, paper I. The natural path between A and C is defined in the Figure by the straight line segment AA_1A_2C. However, it was shown that the extremal time associated with this path can be larger than that associated with a combination of two natural paths AB and BC. Therefore,

$$T_{AC} < t(A,C),$$

where T_{AC} is the least epoch time associated with the points A and C (observe we are designating for simplicity the points of the carrier space only by their location space coordinates). It is also clear that

$$T_{AC} < t(A,B) + t(B,C),$$

for by considering a new path $A23C$, we see that

$$t(A,2) + t(2,3) + t(3,C) < t(A,B) + t(B,C).$$

It turns out that in the present space, for suitable choices of the length a, and the index of refraction n, the three natural paths $P(A,2)$, $P(2,3)$, $P(3,C)$ may represent an actual route taken by a set of photons which form part of the "first burst" of scattered energy received at C from the source at A. Therefore, in the space X of Example 3, a path P_{AC} associated with T_{AC} can be represented by the formal sum:

$$P_{AC} = P(A,2) + P(2,3) + P(3,C)$$

of the indicated natural paths; the corresponding representation of T_{AC} is:

$$T_{AC} = t(A,2) + t(2,3) + t(3,C).$$

The symmetrical arrangement of X_o of Figure 1 (paper I) about line AC shows that the above representation of P_{AC} may just as well have been of the form:

$$P_{AC} = P(A,1) + P(1,4) + P(4,C),$$

and

$$T_{AC} = t(A,1) + t(1,4) + t(4,C).$$

This illustrates the assertion that P_{AC} may not necessarily be a unique path in X. However, by definition of T_{AC}, the two representations of T_{AC} are numerically equal (in this case term by term).

This example also illustrates an important property possessed by many (but not all) classical carrier spaces, namely the property which we shall call temporal connectivity: any two points p, p' of Φ may be connected by a finite number of contiguous natural $P(p_i, p_j)$ paths in such a way that $T_{p'p}$ may thus be represented as a finite sum of the associated extremal times $t(p_i, p_j)$ of the natural-path segments $P(p_i, p_j)$. We will now establish the appropriate generalizations of this notion needed in the context of arbitrary carrier spaces.

ϵ-Temporal Connectivity in Classical Carrier Spaces

Suppose that the space X_0 of Example 3 were not a parallelepiped, but rather some region bounded by curved surfaces, such as that shown in Figure 1 of the present paper. The basic dimensions of the figure are identical with Figure 1 of paper I, and $n(p) = n > 1$ on X_0 as before, but now the straight lines 23 and 14 are replaced by the gently curving arcs $(2,3)$ and $(1,4)$ as shown. It follows once more that

$t(A,B) + t(B,C) < t(A,C)$ is quite possible by suitable choice of the values of a and n. Without going into all the details it should be clear that P_{AC} can actually be the formal sum of the paths $P(A,2)$, $(2,3)$ (the curved arc), and $P(3,C)$. This would be true if, for example, scattering mechanisms were in force throughout X. But the curved arc $(2,3)$ is certainly not a natural path in this space, so that no simple extremal time can be associated with it. Now extremal times, and, more generally, temporal semimetrics are the only temporal yard sticks at our disposal in <u>general</u> carrier spaces. How, then, can the temporal metrics be applied to such paths as $(2,3)$? One way of solving this problem is suggested by the following considerations. If we represent P_{AC} <u>formally</u> by the sum

$$P_{AC} = P(A,2) + (2,3) + P(3,C) ,$$

and the corresponding least epoch time T_{AC} tentatively by

$$T_{AC} = t(A,2) + t_{23} + t(3,C) ,$$

where t_{23} is the "time" required for radiant flux to traverse $(2,3)$, it is clear that at any rate

$$t(A,B) + t(B,C) > t(A,2) + t_{23} + t(3,C) = T_{AC} .$$

The left hand side of this inequality may be considered as a crude estimate of T_{AC} which uses extremal times associates with natural paths. A better estimate of T_{AC} would be (see Figure 1)

$$t(A, B_1) + t(B_1, B_2) + t(B_2, C) > T_{AC} .$$

A still better estimate would be (Figure 1):

$$t(A, B_1) + t(B_1, B_3) + t(B_3, B_4) + t(B_4, C) > T_{AC} .$$

The left hand sides of the preceding three inequalities are progressively smaller numbers (times). It is clear that by choosing a sufficiently large finite number of contiguous natural paths we can come as close as we wish to approximating T_{AC} by the sum of their associated extremal times; however, the estimate of T_{AC} obtained in this manner will generally be too large.

We summarize these observations as follows: Let $(\Phi, \underline{\underline{\Phi}}, v)$ be a classical carrier space, and let $p', p \in \Phi$. Then given any $\epsilon > 0$, there always exists a finite set of $(k-1)$ points, $p_1, \ldots, p_{k-1}$, such that $P(p', p_1)$, $P(p_1, p_2)$, $\ldots$, $P(p_{k-1}, p)$, are natural paths and

$$T_{p'p} = t(p', p_1) + t(p_1, p_2) + \cdots + t(p_{k-1}, p) - \epsilon_k$$

where $0 \le \epsilon_k \le \epsilon$.

ϵ – Temporal Connectivity in General Carrier Spaces

It turns out that most of the important results deduced for abstract temporal metric spaces below, and in subsequent papers of this series, do not require the representation of $P_{p'p}$ by a single natural path. All that is required is the ϵ-type of representation enunciated above, or at most the possibility of a formal exact decomposition of $P_{p'p}$ into a finite set of contiguous natural paths.

Now, the general carrier spaces will be considered as product carrier spaces without any loss of generality (Reference I, §§ 16, 17). Hence, as in the case of the classical spaces, the abstract product carrier spaces can be formally two dimensional, with location and direction space components. The temporal connectivity statements will then apply as before to the points of the location space component.

A carrier space (X, S, v) is said to be <u>temporally connected</u> if for every two points $p_0, p_K \in X$ there is a finite (possibly empty) set $\{p_1, \dots, p_{K-1}\}$ of points in X such that $P(p_i, p_{i+1})$, $i = 0, \dots, K-1$, are natural paths with the property:

$$T_{p_0 p_K} = \sum_{i=0}^{K-1} t(p_i, p_{i+1}),$$

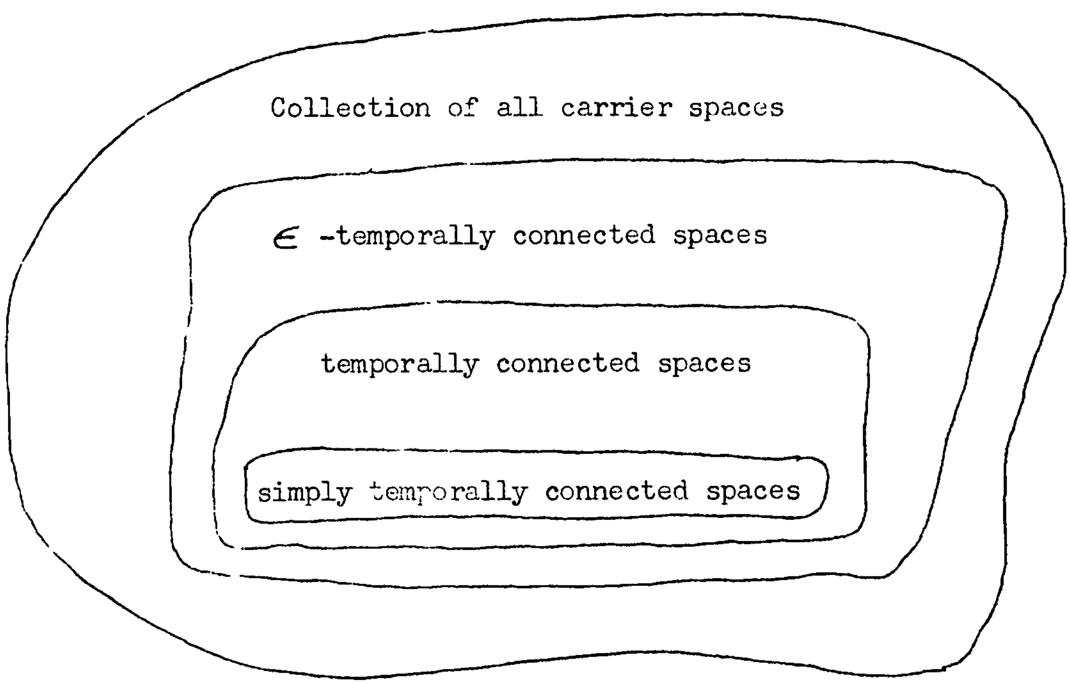
A carrier space (X, S, v) is said to be <u>ϵ-temporally connected</u> if for every two points $p_0, p_K \in X$, and arbitrary $\epsilon > 0$, there is a finite (possibly empty) set of points $\{p_1, \dots, p_{K-1}\}$ (depending possibly on ϵ) such that $P(p_\lambda, p_{\lambda+1})$, $\lambda = 0, \dots, k-1$, are natural paths with the property:

$$T_{p_0 p_k} = \sum_{\lambda=0}^{k-1} t(p_\lambda, p_{\lambda+1}) - \epsilon_k$$

where $0 \leq \epsilon_k \leq \epsilon$.

The notion of temporal connectivity is by definition more restrictive than that of ϵ-temporal connectivity in the sense that the former implies the latter. The converse, however, is not generally true, as was demonstrated in the preceding discussion. Furthermore, the demand for temporal connectivity is clearly less restrictive than the demand for the existence of a single natural path between every two points in the carrier space (<u>simple temporal connectivity</u>).

The hierarchy of connectivity notions introduced so far may be summarized diagrammatically:



Whenever possible, we will prove a theorem under the least restrictive assumptions. Thus all carrier spaces will henceforth be considered to be ϵ-temporally connected, unless explicitly stated otherwise.

TEMPORAL METRICS

Triangle Inequality for $T_{p'p}$

We now establish three basic theorems about the properties of least local epoch times $T_{p'p}$. The first theorem asserts that the triangle inequality holds for $T_{p'p}$ in arbitrary carrier spaces. The second supplies proof that $T_{p'p}$ is a temporal (pseudo) metric in carrier spaces with temporally homogeneous radiative processes. Finally, it is proved that the notions of temporal metric $t(p',p)$ and least local epoch time $T_{p'p}$ coincide in temporal metric spaces.

<u>Theorem 1. Let $(X, \mathcal{S}, v)$ be an arbitrary (not necessarily ϵ-temporally connected) carrier space with arbitrary radiative transfer process $(Q_\phi, \mathcal{T}_\phi)$. Then the triangle inequality holds for least local epoch times.</u>

<u>Proof</u>. Let p'', p', and p be three points of X. Let $T_{p'p}$ $T_{p''p'}$ be the least local epoch times associated with the pairs (p',p) (p'',p') respectively. Therefore, the sum $T_{p''p'} + T_{p'p}$ can be interpreted as <u>a</u> local epoch time (with respect to p'' as a local source for p) for the point pair (p'',p). But $T_{p''p}$ is the <u>least</u>

local epoch time associated with the pair (ρ'', ρ) . Therefore

$$T_{\rho''\rho} \leq T_{\rho''\rho'} + T_{\rho'\rho},$$

which completes the proof.

Identity Property for $T_{\rho'\rho}$

In the discussion of the epoch times — in particular, paragraph 3 above — we observed and illustrated the fact that the symmetry property of the temporal semimetric was a necessary consequence of the temporal homogeneity of the transfer process. Thus, such a simple and completely accepted fact in classical carrier spaces as the reciprocity of extremal times is denied validity in the more general spaces which do not possess temporally homogeneous radiative or transfer processes. Now while the identity property ((ii) in paper I) of t generally holds, a careful examination shows that it must be established by proof or fiat in the case of the least epoch time function. In any event, its validity cannot be taken for granted for $T_{\rho'\rho}$ in an arbitrary carrier space. We are referring to the property:

$T_{\rho'\rho} = 0$ if and only if $\rho' = \rho$ (identity property).

In ordinary euclidean space, if we are told that $d(p',p) = 0$, i.e., that the spatial distance between two points is zero, then we may conclude that $p' = p$. On the basis of simple intuition, it is inconceivable that $d(p',p) = 0$ for two distinct points where d is the usual euclidean spatial metric. Even more so, if $p' = p$, then we expect on the basis of experience in the real world that $d(p',p) = 0$. However, the condition that "$p' = p$ if $d(p',p) = 0$" is not logically necessary in certain contexts. Consider for example the space-time metric d' of special relativity applied to two spatially distinct points on the four dimensional light cone $\mathcal{C}$ of a point source of light. Any two points p', p on $\mathcal{C}$ have the property that

$$d'(p',p) = 0$$

Such a situation has been built into the axiomatic foundation of radiative transfer theory. From definition (1) and the statements of the radiative measure axiom R_1 (Reference 1) we have the following property:

$$\text{if } p' = p, \text{ then } T_{p'p} = 0.$$

However, the converse of the statement cannot be deduced from Axiom R_1:

$$\text{if } T_{p'p} = 0, \text{ then } p' = p. \qquad (*)$$

It is only when we come to the consideration of the abstract counterparts to the classical carrier spaces and study them via product carrier spaces that we have an analytical need for such a property. The property is present for example if one assumes that the radiative process is restricted on a carrier space (for definitions of terms, see Reference 1).

$T_{\rho'\rho}$ as a Pseudo-metric

The condition (*) above is inessential for many purposes. A function T on $X \times X$ which satisfies all the usual properties of a metric except (*) is customarily called a pseudo-metric.[3] As will be proved below, the temporal metric $T_{\rho'\rho}$ provided by least local epoch times is such a metric in carrier spaces with unrestricted temporally homogeneous radiative processes.

Theorem 2. Let $(X, \mathcal{S}, v)$ be an arbitrary carrier space on which is defined a temporally homogeneous radiative transfer process. Then the least local epoch time $T_{\rho'\rho}$ satisfies the usual postulates for a pseudo-metric, and $(X, \mathcal{S}, v)$ becomes a temporal pseudo-metric space under $T_{\rho'\rho}$.

Proof: $T_{\rho'\rho}$ is a temporal pseudo-metric on $X \times X$ if $T_{\rho'\rho}$ is non-negative valued and

(a) $T_{\rho'\rho} = 0$ if $\rho' = \rho$,

(b) $T_{\rho'\rho} = T_{\rho\rho'}$ for every pair (ρ', ρ) composed of points in X,

(c) $T_{\rho''\rho} < T_{\rho''\rho'} + T_{\rho'\rho}$ for every triple (ρ'', ρ', ρ) composed of points in X .

First of all, by definition (1), $T_{\rho'\rho}$ is clearly non-negative for all pairs (ρ', ρ) composed of points in X . Property (a) holds by virtue of the general Radiative Process properties. To prove (b), let $\epsilon > 0$ and let $(\rho_1, \ldots, \rho_{k-1})$ be a set of points in such that X

$$T_{\rho'\rho} = t(\rho', \rho_1) + \sum_{\lambda=1}^{k-2} t(\rho_\lambda, \rho_{\lambda+1}) + t(\rho_{k-1}, \rho) - \epsilon_k$$

where $0 \leq \epsilon \leq \epsilon$, which follows from the ϵ-connectivity of X . Since the function t is a temporal semimetric, we have

$$t(x_1, x_2) = t(x_2, x_1)$$

for all $x_1, x_2 \in X$. therefore:

$$T_{\rho'\rho} + \epsilon_k = t(\rho_1, \rho') + \sum_{\lambda=1}^{k-2} t(\rho_{\lambda+1}, \rho_\lambda) + t(\rho, \rho_{k-1}) \geq T_{\rho\rho'},$$

by definition of $T_{\rho\rho'}$.

Since $\epsilon > 0$ was arbitrary, we have

$$T_{\rho'\rho} \geq T_{\rho\rho'}.$$

An argument of this kind now applied to $T_{\rho\rho'}$ yields a similar result:

$$T_{\rho\rho'} \geq T_{\rho'\rho} ,$$

so that (b) holds. Finally, (c) is known to hold in all carrier spaces (Theorem 1). This completes the proof.

Compatibility of t and T

To understand the full implications of the final theorem of this section, we must recall the distinction between the temporal semimetric t on $X \times X$ and the least local epoch time function T on $X \times X$: the former was defined by means of the Transfer Process $\{T_{t_1, t_2}\}$ on $(X, \mathcal{S}, v)$, the latter by means of the γ-density which is supplied by the Radiative Process $\{R_{t_1, t_2}\}$ on $(X, \mathcal{S}, v)$. Now suppose that t is a temporal metric on $X \times X$. This situation is encountered, for example, on all classical carrier spaces with temporally and spatially invariant inherent optical properties. The following question now arises: given that t is a temporal metric on $X \times X$, and (ρ', ρ) is an arbitrary point pair on a natural path in X, what is the relation between the magnitudes $t(\rho', \rho)$ and $T_{\rho'\rho}$? It would be unfortunate if these two "shortest" temporal distances were found to be unequal. The impossibility of such a pathological state of affairs is guaranteed by the following:

<u>Theorem 3. If t is a temporal metric on a carrier space $(X, \leq, v)$ and (p', p) is any pair of points on a natural path in X, then $t(p', p) = T_{p'p}$.</u>

<u>Proof</u>. Consider the two paths $P(p', p)$ and $P_{p'p}$ associated with $t(p', p)$ and $T_{p'p}$ respectively. Since X is ϵ-temporally connected, for every $\epsilon > 0$ there exists the following representation of $T_{p'p}$:

$$T_{p'p} + \epsilon_K = t(p', p_1) + \sum_{i=1}^{K-2} t(p_i, p_{i+1}) + t(p_{K-1}, p),$$

where $0 \leq \epsilon_K \leq \epsilon$. By the triangle inequality property of t, we have

$$T_{p'p} + \epsilon_K \geq t(p', p),$$

and since ϵ is arbitrary,

$$T_{p'p} \geq t(p', p).$$

However, by definition of $T_{p'p}$, we require

$$T_{p'p} \leq t(p', p),$$

whence

$$T_{p'p} = t(p', p),$$

which completes the proof.

Observe that no corresponding assertion can be generally made about the identity of the paths $\rho_{\rho'\rho}$ and $\rho(\rho';\rho)$. Examples can be found under the hypothesis of Theorem 3 in which $\rho_{\rho'\rho}$ and $\rho(\rho';\rho)$ are distinct.

SUMMARY OF TEMPORAL METRIC CONCEPTS

Before going on to the applications of the preceding concepts, we pause to summarize the distinctions between the various metric concepts introduced so far. We begin by reviewing the several different types of metric spaces. To this end, let Y be an arbitrary set, and ρ a real valued function on $Y \times Y$. We are interested in the following five possible properties of ρ :

(a) $\rho(y_1, y_2) \geq 0$ for all $y_1, y_2 \in Y$ (non-negative property)

(b) $\rho(y_1, y_2) = \rho(y_2, y_1)$ for all $y_1, y_2 \in Y$ (symmetry property)

(c) $\rho(y_1, y_2) = 0$ if $y_1 = y_2$ (trivial identity property)

(d) $y_1 = y_2$ if $\rho(y_1, y_2) = 0$ (non-trivial identity property)

(e) $\rho(y_1, y_3) \leq \rho(y_1, y_2) + \rho(y_2, y_3)$ for every y_1, y_2, y_3 in Y.
(triangle inequality)

If ρ satisfies (a), (b), (c), and (d), ρ is a <u>semimetric</u> and Y is a <u>semimetric space</u>.

If ρ satisfies (a), (b), (c) and (e), ρ is a pseudo-metric and Y is a pseudo-metric space.

If ρ satisfies (a) - (e) inclusive, ρ is a metric and Y is a metric space.

If $Y = X_o$ is the spatial component of a two-dimensional carrier space $(X, \underset{\sim}{S}, \nu)$ and $\rho = t$ is defined by means of the Transfer Process such that t is a semimetric when applied to point pairs on natural paths (t is not defined on arbitrary point pairs) then t is a temporal semimetric space. The necessary condition that t be a semimetric is that the Transfer Process on $(X, \underset{\sim}{S}, \nu)$ is temporally homogeneous. In the most general situation presently encountered, t can only have properties (a), (c) and (d). If temporal homogeneity is introduced into the Transfer Process, then (b) becomes a property of t . If, finally (as an example) η is spatially homogeneous, then (e) also holds for t .

Let $Y = X_o$ be the spatial component of a carrier space $(X, \underset{\sim}{S}, \nu)$ and let $\rho = T$ be the least local epoch time function defined by means of the Radiative Process. In the most general situation presently encountered, T can only have the properties (a), (c) and (e) and T is defined on all point pairs. If temporal homogeneity is introduced into the Radiative Process, then (b) becomes a property of T and X becomes a pseudo-metric (Theorem 1) and X becomes a temporal pseudo-metric space. If, finally, (as an example) $(X, \underset{\sim}{S}, \nu)$ is a two-dimensional product carrier space on which the radiative process is 1-factored,[1] then $\mathcal{T}$ satisfies (d), and $\mathcal{T}$ becomes a <u>metric</u>, and X a <u>temporal metric space</u>.

These observations may be summarized schematically:

$$\text{I} \quad \begin{array}{l}\text{arbitrary}\\ \text{carrier}\\ \text{space}\end{array} \left\{ \begin{array}{ll} t: & \text{(a), (c), (d) (induced by Transfer Process)} \\ \mathcal{T}: & \text{(a), (c), (e) (induced by Radiative Process)} \end{array} \right.$$

$$\text{II} \quad \begin{array}{l}\text{arbitrary carrier space}\\ \text{with temporal homogeneity}\\ \text{of corresponding process}\end{array} \left\{ \begin{array}{ll} t: & \text{(a), (b), (c), (d) (semimetric)} \\ \mathcal{T}: & \text{(a), (b), (c), (e) (pseudo-metric)} \end{array} \right.$$

$$\text{III} \quad \begin{array}{l}\text{arbitrary carrier space}\\ \text{with temporal homogeneity}\\ \text{of corresponding process}\end{array} \left\{ \begin{array}{l} \text{+ spatial homogeneity, } t: \text{(a) - (e)} \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{(metric)} \\ \text{+ factored Radiative Process, } \mathcal{T}: \text{(a) - (e)} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(metric)} \end{array} \right.$$

The assumption of ϵ-connectivity is by definition associated with $\mathcal{T}$, but is not needed in set I above; it was however, needed to establish (b) of set II for $\mathcal{T}$. Furthermore, it was not needed to establish (d) of set III for $\mathcal{T}$.

The set of properties II will be the one mostly used throughout the remainder of this study; the carrier space and Radiative Transfer Process will be such that t is a semimetric and that $\mathcal{T}$ is a pseudo-metric.

CHARACTERISTIC FUNCTION

Let ρ^* be a local source in X, an arbitrary carrier space. Let ρ', ρ be an arbitrary pair of points in X. Then the characteristic function χ on $X \times X \times T$ into the non-negative reals is defined as

$$\chi(\rho', \rho ; T_{\rho^*}(\rho')) = \begin{cases} 0 & \text{if } T_{\rho^*}(\rho') < T_{\rho'\rho}, \\ 1 & \text{if } T_{\rho^*}(\rho') \geq T_{\rho'\rho}. \end{cases} \quad (5)$$

Observations on χ

1. By definition (5), a value of 1 for χ means that ρ is receiving flux from ρ' which acts as a local source for ρ. The local epoch time at ρ' is referred to some fixed local (not necessarily the fundamental) point source ρ^*. Now since the least local epoch time $T_{\rho'\rho}$ is basically a fixed magnitude (for the space X and its radiative process) there is a period of local epoch time at ρ' during which $T_{\rho^*}(\rho') < T_{\rho'\rho}$. During this time ρ receives no flux from ρ', (which has been passed on from ρ^*) and $\chi = 0$. However, $T_{\rho^*}(\rho')$ steadily grows until it becomes equal to $T_{\rho'\rho}$. At this instant the value of χ becomes 1 and remains 1 for all $T_{\rho^*}(\rho')$. afterward.

2. As a special case of definition (5) let $\rho^* = \rho_0$, the fundamental source. In addition, let ρ' coincide with ρ_0. Then according to the observation 1 in the section on epoch times, $T_{\rho^*}(\rho') = T$; and (5) reduces to:

$$\chi(\rho_0, \rho; T) = \begin{cases} 0 & \text{if } T < T_{\rho_0\rho} , \\ 1 & \text{if } T \geq T_{\rho_0\rho} , \end{cases}$$

which reveals the basically simple idea behind χ.

3. In the case of simply temporally connected spaces in which is a metric, the preceding form for χ may be simplified even further. For by Theorem 3, $T_{\rho_0\rho}$ may be replaced by $t(\rho_0, \rho)$:

$$\chi(\rho_0, \rho; T) = \begin{cases} 0 & \text{if } T < t(\rho_0, \rho) , \\ 1 & \text{if } T \geq t(\rho_0, \rho) . \end{cases}$$

Examples of Some Theorems Using χ

As an example of the use of χ and the various notions of epoch times, consider

<u>Theorem 4. Let ρ^* be a fixed local source in an arbitrary carrier space (X, ξ, ν). If (ρ_1, ρ) and (ρ_2, ρ) are two</u>

point pairs of χ such that

$$\text{(i)} \quad \chi(\rho_1, \rho\, ;\, T_{\rho^*}(\rho_1)) = 0\, ,$$

$$\text{(ii)} \quad T_{\rho^* \rho_2} + T_{\rho_2 \rho} \geq T_{\rho^* \rho_1} + T_{\rho_1 \rho}\, ,$$

then

$$\chi(\rho_2, \rho\, ;\, T_{\rho^*}(\rho_2)) = 0\, .$$

<u>Proof</u>. By (i), $T_{\rho_1 \rho} > T_{\rho^*}(\rho_1) = T_{\rho_0}(\rho^*) - T_{\rho^* \rho_1}$; so that $T_{\rho_1 \rho} + T_{\rho^* \rho_1} > T_{\rho_0}(\rho^*)$. But from (ii) we conclude that $T_{\rho^* \rho_2} + T_{\rho_2 \rho} > T_{\rho_0}(\rho^*)$; so then $T_{\rho_2 \rho} > T_{\rho_0}(\rho^*) - T_{\rho^* \rho_2} =$ $= T_{\rho^*}(\rho_2)$, which implies $\chi(\rho_2, \rho\, ;\, T_{\rho^*}(\rho_2)) = 0$. This completes the proof.

The simple geometric content of this theorem is depicted in Figure 2. This theorem was chosen for its extreme simplicity so that its conclusion (once the meaning of the various symbols in (i) and (ii) are digested) is intuitively evident. The point being made, however, is that the problem of deciding whether or not ρ is receiving flux from

p_i may be reduced to a formal methodical manipulation of symbols — a temporal calculus as it were — representing the time dependent multiple scattering problem. We arrive at a formalism analogous to that used in coding automatic computers, or that used in elementary logic, namely a formalism which on one level may be manipulated without knowing (or caring about) the physical meaning of the symbols, and on the other level (usually after the manipulation is over) may be interpreted for physical meaning.

As a final example of this calculus let us consider the following simple problem: If (p', p) is a pair of points in a space X with temporally homogeneous radiative process, and p is at a certain instant receiving radiant flux from p', what condition will allow the conclusion: p' is also, at this instant, receiving radiant flux from p (Figure 3). We suppose that the radiant flux field is generated by the fundamental source p_o. On intuitive grounds the required criterion is evidently that $T_{p_o p} \leq T_{p_o p'}$. This is borne out by

<u>Theorem 5. Let (p', p) be a point pair in X with p_o as a fundamental source. Then the conditions</u>

(i) $\chi(p', p, T_{p'}(p)) = 1$,

(ii) $T_{p_o p} \leq T_{p_o p'}$,

imply

$$\chi(\rho, \rho'; T_\rho(\rho)) = 1 .$$

Proof. By (i) $T_{\rho'}(\rho') > T_{\rho'\rho}$. By definition (4) and the trivial identity property for $T_{\rho'\rho}$:

$$T_{\rho'}(\rho') = T_{\rho_0}(\rho') - T_{\rho'\rho'} = T_{\rho_0}(\rho') ,$$

and by definition (2) :

$$T_{\rho_0}(\rho') = T - T_{\rho_0\rho'} .$$

Thus

$$T - T_{\rho_0\rho'} > T_{\rho'\rho} .$$

By (ii) and the symmetry property of $T_{\rho'\rho}$,

$$T > T_{\rho\rho'} + T_{\rho_0\rho} .$$

This implies

$$T_\rho(\rho) = T_{\rho_0}(\rho) = T - T_{\rho_0\rho} > T_{\rho\rho'} ,$$

which, by definition of χ , implies

$$\chi(\rho, \rho'; T_\rho(\rho)) = 1 ,$$

and the theorem is proved.

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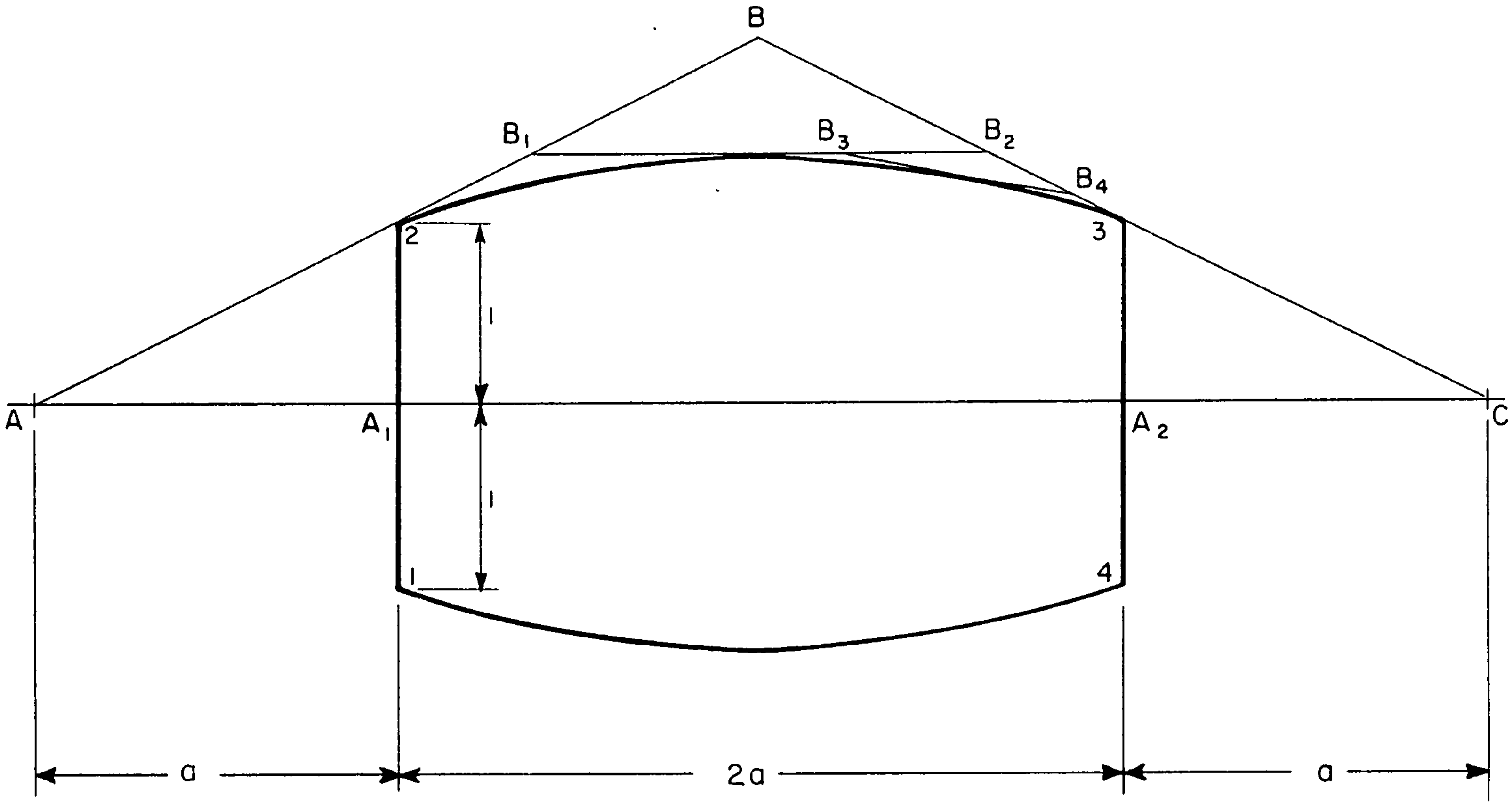

Figure 1

Rudolph W. Preisendorfer

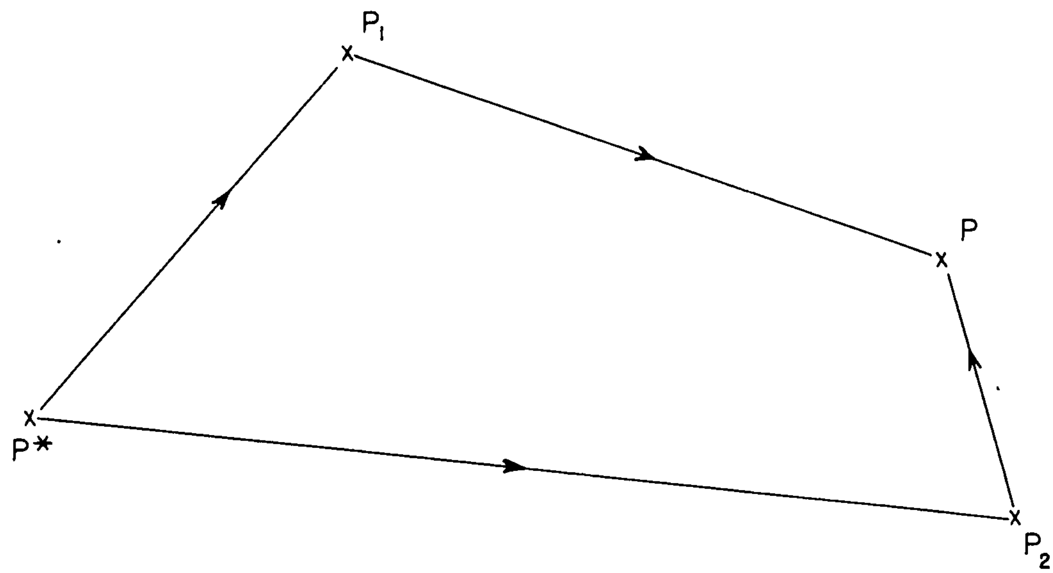

Figure 2

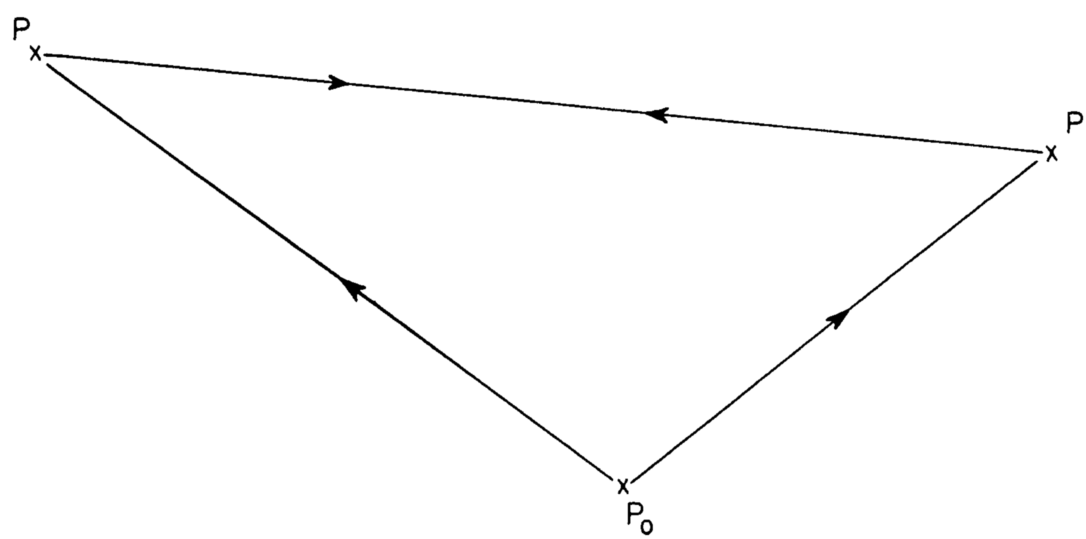

Figure 3

Rudolph W. Preisendorfer