Visibility Laboratory
University of California
Scripps Institution of Oceanography
San Diego 52, California

PLANE-SOURCE GENERATED LIGHT FIELDS IN DISCRETE SPACES

Rudolph W. Preisendorfer

December 1959
Index Number NS 714-100

Bureau of Ships
Contract NObs-72092

SIO Reference 59-70

Approved:

Seibert Q. Duntley

Seibert Q. Duntley, Director
Visibility Laboratory

Approved for Distribution:

Roger Revelle

Roger Revelle, Director
Scripps Institution of Oceanography

TABLE OF CONTENTS

SUMMARY

Plane-Source Generated Light Fields in Discrete Spaces

Rudolph W. Preisendorfer

Scripps Institution of Oceanography, University of California
La Jolla, California

INTRODUCTION

In this, the sixth of a series of papers on discrete-space radiative transfer theory, [1-5] we direct attention to the problem of determining the light field generated by internal plane-sources in an arbitrarily stratified plane-parallel medium. Using the methods of discrete-space theory established in the earlier studies of this series, a complete solution of the internal-source problem is developed. This is given in compact form in the summary section below.

The origin of the plane-source problem lies in the fact that within a real optical medium such as an ocean or an atmosphere (stellar or planetary) there may be radiative emission processes of either physical or biological origin distributed within a layer. In the ocean, for example, bioluminescence in layers at various depths may take place, thereby giving rise to a local source of radiant flux in the natural light field. This locally emitted flux initiates a multiple scattering process of its own, the net result of which is to augment at each point in the medium the natural light field induced by the flux incident from the sun and sky on the upper boundary of the medium.

Similar remarks apply to the internal-source problems that arise in stellar and planetary atmospheres.

The geometric setting of the present problem is an extended cubic lattice, which is the discrete-space counterpart to the continuous plane-parallel optical medium. The plane-sources irradiate arbitrary monolayers within the lattice.

The theory of radiative transfer on an extended cubic lattice was developed in detail in reference 5. In that reference, the complete two-boundary problem was solved, thereby giving explicit values of the twenty-six radiance flows at each point of a lattice which is irradiated by arbitrary radiance distributions at its upper and lower boundaries. The various standard $\mathcal{R}$ and $\mathcal{T}$ operators associated with an extended cubic lattice played an essential role in the details of the solution of that problem. The theory of these operators was therefore worked out in complete detail and computation procedures leading to their numerical evaluation were developed. It will be shown that the present problem can be reduced to components which once again depend only on the $\mathcal{R}$ and $\mathcal{T}$ operators; hence the discussion will draw heavily on the results of reference 5. Any undefined concepts and terminology within the present report will be found fully defined in reference 5 or earlier papers of the discrete-space series.

The plane-source problem in the <u>continuous</u> theory is a relatively intractable problem. Only by making stringent assumptions on both the inherent optical structure of the medium and the directional structure of the source radiance distributions can any theoretical progress be made in the continuous setting. An examination of some early work on the internal-source problem in the continuous setting illustrates this point.[6] By adopting a discrete-space setting, however, it is possible to explore the internal-source problem under very general conditions on the radiative processes: The plane-sources may have arbitrary directional structure, and may be located arbitrarily within the space; the medium may have an arbitrary volume scattering and volume attenuation function; the medium may be arbitrarily inhomogeneous, and even anisotropic, so that the $\mathcal{R}$ and $\mathcal{T}$ operators may possess polarity. Thus the discrete space approach permits a comparatively wide range of exploration of the effects of arbitrary internal sources within optical media.

In this note we explore in complete detail the general plane-source problem associated with arbitrary plane sources distributed throughout an extended cubic lattice. In a subsequent note, the internal source problem associated with arbitrary sets of isolated point sources will be considered. By means of these two studies we supply two concrete examples of how the methods of discrete-space theory may be applied to this difficult problem of radiative transfer theory.

FORMULATION OF THE PROBLEM

The general internal source problem on an arbitrary discrete space was formulated as a matter of course in the local interaction principle.[1] However, in accordance with our present purposes, we restate the problem in terms of the special concepts associated with the present geometrical setting:

<u>Given</u>: (a) The associated quotient space Y_n of an extended cubic lattice X_n, where $Y_n = \{(0,0,1), \ldots, (0,0,n)\}$, and the twenty-six component direction space Ξ' associated with Y_n.

(b) The twenty-six component source vector $N^0(\lambda)$ associated with each point $(0, 0, \lambda)$ of Y_n, $1 \leq \lambda \leq n$.

(c) The Σ and A functions of Y_n.

<u>Required</u>: The resultant twenty-six component specific radiance vector $N(j)$ at each level j in Y_n, $1 \leq j \leq n$

The Scope of the Formulation

An examination of this statement of the problem shows that the general internal-source problem subsumes the complete two-boundary problem considered in reference 5. For, suppose that the elements $(0,0,1)$ and $(0,0,n)$ of Y_n represent the boundaries of Y_n with Σ and A describing the requisite transmittance, reflectance and absorptance properties of these two boundaries. Then the twenty-six-flow problem of reference 5 may be obtained from the present problem by setting $N^0(s) = 0$ (the zero vector) for $2 \leq s \leq n-1$.

To see this in detail, recall (Equations (1) - (6), reference 5) that the specific radiance vector $N(j)$ at level j can be decomposed into three components $N_+'(j)$, $N_0(j)$, and $N_-(j)$, such that:

$$\begin{aligned} N(j) &= [N_+'(j), N_0(j), N_-(j)] \\ &= [N_+(j), N_-(j)] \,. \end{aligned} \tag{1}$$

Here $N_+'(j)$ is the proper upward radiance vector, $N_0(j)$ the horizontal radiance vector, and $N_-(j)$ the downward radiance vector.

In a similar way, the source vector $N^0(s)$, at each level s , $1 \leq s \leq n$ may be decomposed into three components:

$$N^0(s) = [N_+^0(s), N_0^0(s), N_-^0(s)] \,, \tag{2}$$

where $N_+^0(s)'$ is the proper upward component, $N_0^0(s)$ the horizontal component, and $N_-^0(s)$ the downward component of $N^0(s)$. Physically, $N_+^0(s)'$ represents the radiance incident on level s in each of the nine proper upward directions of Ξ_+'; $N_0^0(s)$ represents the radiance incident on level s in each of the eight horizontal directions of Ξ_0; and $N_-^0(s)$ represents the radiance incident on level s in each of the nine downward directions of Ξ_-. Therefore, setting $N^0(s) = 0$, for $2 \leq s \leq n-1$ is equivalent to setting each of the components $N_+^0(s)'$, $N_0^0(s)$, $N_-^0(s)$ to zero at each of these levels.

To obtain the complete two boundary problem exactly, we merely set $N_+^0(1)' = N_0^0(1) = 0$, and let $N_-^0(1)$ be arbitrary at level 1; and set $N_-^0(n) = N_0^0(n) = 0$, and let $N_+^0(n)'$ be arbitrary at level n. Recall that in reference 5 we made the following notational conventions: $N_+^0(n)' \equiv N_+^0(n+1)$, $N_-^0(1) \equiv N_-(0)$. These conventions will be retained in the present work. Thus the problem of reference 5 is a special case of that considered here.

The Physical Significance of the Source Vector

It is essential for a clear comprehension of all that follows, that the reader thoroughly understand the physical significance of the source vector $N^o(s)$ as it is used in the present study. The source vector $N^o(s)$ represents radiance *incident* *on* the layer at level s. The radiant flux of this vector is imagined to originate under some emission process which takes place *outside* the point set of Y_n. That is, the source radiance *incident* *on* level s does *not* come from the layers at levels $s-1$ or $s+1$ just above or below the level s; rather it is envisioned as coming from the *interstices* between these layers. Thus the component $N_-^o(s)$ of $N^o(s)$ represents downward radiance incident on layer s and generated by some emission process *between* layers $s-1$ and $s \geq 1$. Similarly $N_+^o(s)'$ represents upward radiance incident on layer s and generated by some emission process between the layers $s+1$ and $s \leq n$. Finally, $N_0^o(s)$ represents the horizontal radiance incident on the points of a layer s and generated by some emission process within the regions *between* the points of layer s.

One final observation on the physical nature of $N^o(s)$ may be made by noting that it is the discrete-space counterpart to the emission radiance function N_η of the continuous theory which has the units: radiance / length. (See, e.g., reference 7.) Actually, $N^o(s)$ is slightly more general than N_η, since $N^o(s)$ can in addition represent the downward incident radiance on the upper boundary of the medium, thereby being capable of representing natural boundary lighting conditions in addition to internal emission phenomena.

The Role of the Structure of the Local Direction Space

We have specifically limited the present discussion to a cubic lattice with a fixed twenty-six component direction space Ξ'. It might be worthwhile to point out that the actual number of components of the local direction space does not play an essential role in the main outlines of the solution of a transfer problem on a discrete space. This interesting conclusion is illustrated by concrete examples in references 4 and 5. The local direction spaces associated with the two lattices in these papers are quite distinct (two directions in the linear lattice, 26 in the cubic lattice). However, the theories of radiative transfer on these lattices are <u>formally</u> identical, as a comparison of the general partition relations and recurrence relations of each will show. The inessentiality of the structure of the local direction space with respect to the gestalt of the solution equations can be proved by appeal to the general partition relations of reference 2 (Equations (21 - (24)). Thus the solution presented below of the internal-source problem on a cubic lattice with a twenty-six component direction space is, in outline, of the same form as the solution for a cubic lattice with arbitrary direction space.

ANALYSIS OF THE PROBLEM

The Ψ- Operator

The salient structure of the solution of the plane-source problem may be obtained by some general arguments based on the general form of the invariant imbedding relation. In other words, the main topographical features of the final solution may be anticipated by an examination of an invariance relation without going into any detailed study. However, the exact nature of the solution can only be obtained after a direct and detailed appeal to the specific forms of the local interaction principle and its various ramifications such as the principles of invariance and the invariant imbedding relation.

To begin the general analysis of the problem, consider its graphic statement in Figure 1. The source vector $N^0(s)$ at an arbitrary level s, $1 \leq s \leq n$, gives rise to a specific radiance vector $N(j)$ at level j, $1 \leq j \leq n$. On the basis of the general invariant imbedding relation for the cubic lattice (Equation (7), reference 5) we would anticipate the existence of some general linear operator which maps $N^0(s)$ into $N(j)$. For, an examination of the relation shows that: (a) It employs a linear operator $\mathscr{M}$ in the form of an 18 x 26 matrix; (b) The operator $\mathscr{M}$ maps radiance vectors at arbitrary levels $k+1$ and $i-1$ into the radiance vector at level j; and (c) The vectors at levels $k+1$ and $i-1$ may be interpreted as the causes of the resultant radiance vector at level j. Thus the essential feature of the invariant imbedding relation is that it constitutes a linear relation between a radiometric cause at one point in Y_n and its attendant radiometric effect at a generally different point in Y_n .

On these grounds we may anticipate the following form of the solution of the internal source problem: the radiometric cause $N^0(s)$ at level s and its radiometric effect $N(j)$ at level j are related by a linear operator, say $\Psi(s,j)$, which has the property:

$$\boxed{\begin{aligned} &N(j) = N^0(s)\,\Psi(s,j) \\ &1 \leq j, s \leq n \end{aligned}} \qquad (3)$$

$\Psi(s,j)$ is the <u>Ψ-operator</u> associated with the pair of depths (s,j) . Since $N(j)$ and $N^0(s)$ are each twenty-six component vectors, $\Psi(s,j)$ then takes the form of a 26 x 26 matrix. Furthermore, since the main operators and principles of radiative transfer theory are linear, the radiometric effect at some arbitrary level j should be representable as a linear superposition of the set of its radiometric causes distributed throughout Y_n . Thus if a source condition generally exists at every level s , $1 \leq s \leq n$ in Y_n then the effect at level j is representable as:

$$\boxed{\begin{aligned} &N(j) = \sum_{s=1}^{n} N^0(s)\,\Psi(s,j) \\ &1 \leq j \leq n \end{aligned}} \qquad (4)$$

The forms (3) and (4) may be anticipated on even more general grounds by examining the solution of the general transfer problem on an arbitrary discrete space X_n (Equation (31), reference 1)*. As it stands, (4) is the form of the general solution of the present internal-source problem on an extended cubic lattice.

First Decomposition of the Ψ-Operator

If we now take cognizance of the partitioning of the $N(j)$ and $N^0(s)$ vectors in the manner shown in (1) and (2), then the matrix $\Psi(s,j)$ may be analyzed specifically into submatrices which have a simple and useful physical interpretation. Thus by writing (4) as:

$$[N_+(j), N_-(j)] = \sum_{s=1}^{n} [N_+^0(s)', N_0^0(s), N_-^0(s)] \Psi(s,j), \quad (5)$$

there results a natural cleavage of $\Psi(s,j)$ into a six-block matrix as follows:

---

* This equation is the point of departure in the derivation of the Ψ-operator for the point source problem in a subsequent work. The Ψ-operator in the point source context subsumes the present Ψ-operator as a special case.

$$\Psi(s,j) = \begin{array}{cc} \left(\begin{array}{cc} \Psi_{++}(s,j) & \Psi_{+-}(s,j) \\ \Psi_{0+}(s,j) & \Psi_{0-}(s,j) \\ \Psi_{-+}(s,j) & \Psi_{--}(s,j) \end{array}\right) & \begin{array}{c} (9) \\ (8) \\ (9) \end{array} \\ \begin{array}{cc} (17) & \quad (9) \end{array} & \end{array} \qquad (6)$$

The numbers in parentheses along the side and bottom of the matrix indicate the dimensions of the various submatrices. For example, $\Psi_{0+}(s,j)$ is an 8 x 17 matrix.

The physical significance of these submatrices and the role they play in the representation of the internal-source problem becomes clear if we examine any one of the three equations in (5). For example, consider the first component $N_+(j)$ of $N(j)$ in (5):

$$N_+(j) = \sum_{s=1}^{n} \left[N_+^0(s)' \Psi_{++}(s,j) + N_0^0(s) \Psi_{0+}(s,j) + N_-^0(s) \Psi_{-+}(s,j) \right] \qquad (7)$$

To simplify matters seen further, suppose $N^0(s) \neq 0$ at some single level only, then (7) reduces to

$$N_+(j) = N_+^0(s)' \Psi_{++}(s,j) + N_0^0(s) \Psi_{0+}(s,j) + N_-^0(s) \Psi_{-+}(s,j). \qquad (8)$$

This shows that $N_+(j)$, the upward radiance at level j is the sum of three upward radiance distributions: The term $N_+^0(s)' \Psi_{++}(s,j)$

represents the "transmitted" flux from level s to level j and $\Psi_{++}(s,j)$ thus acts as a "generalized transmittance". In a similar way we interpret $\Psi_{-+}(s,j)$ as a "generalized reflectance" operator, and $\Psi_{0+}(s,j)$ is something between a reflectance and a transmittance operator, and is reminiscent of the S-operators used in the derivation of the monolayer equations in reference 5.

All these resemblances of the components of the Ψ-operator to generalized transmittance, reflectance and monolayer operators are not accidental. As they stand, they hint strongly at a possible connection with the complete reflectance and transmittance operators $\mathcal{R}$ and $\mathcal{T}$ associated with general one-parameter carrier spaces[8], and with the general scattering matrices of a monolayer. We now show that these connections exist, and exhibit the exact forms of the $\mathcal{R}$ and $\mathcal{T}$ operators which are involved in such a connection.

The Local Ψ-Operator

If we set $s = j$ in $\Psi(s,j)$, the result $\Psi(s,s)$ is called the <u>local Ψ-operator</u>. According to (3), the physical significance of $\Psi(s,s)$ is read off from that operation by setting $s = j$:

$$N(s) = N^{o}(s)\,\Psi(s,s) \tag{9}$$

Thus $\Psi(s,s)$ maps the source vector at a level s into the resultant specific radiance vector at that same level. Even though the notation shows that the cause and effect are restricted locally to a single monolayer in Y_n, we would expect that $\Psi(s,s)$ takes cognizance of the overall scattering properties of Y_n in some as yet undetermined way. The exact nature of $\Psi(s,s)$, which relates the source activity at the sth monolayer to the resultant light field at that same level, will be worked out in detail below. But in the meanwhile we will continue to analyze the problem into fundamental components as far as we can by using only the general properties of the local interaction principle and its variants. This procedure can then be used in the analysis of the more general point source problem to be considered in a subsequent work.

The Complete Reflectance and Transmittance Operators

In the first decomposition of the Ψ-operator into its six component block matrices, we observed that four of the components (i.e., those with signatures $++$, $+-$, $-+$, $--$) appeared to exhibit the general character of reflectance and transmittance operators. We will now take the first step in the rigorous demonstration of this fact. We begin with the general statement of the invariant imbedding relation given either by (47) of reference 2, or (7) of reference 5. In either case we have the following representation for the twenty-six component vector $N(j) = [N_+(j), N_-(j)]$ at any level j at which there are no sources (i.e., for $j \neq s$):

$$[N_+(j), N_-(j)] = [N_+'(k+1), N_-(i-1)] \begin{pmatrix} \mathcal{T}(k+1, j, i-1) & \mathcal{R}(k+1, j, i-1) \\ \mathcal{R}(i-1, j, k+1) & \mathcal{T}(i-1, j, k+1) \end{pmatrix} \quad (10)$$

Here the levels $k+1$, and $i-1$ within Y_n are completely arbitrary. As a perusal of either of the above references would show, equation (10) relates the radiances impinging on a sublayer bounded by the layers at level i and level k, where $i \leq k$, to the resultant radiance distribution at level j within the layer: $i \leq j \leq k$. This general situation is shown in Figure 2.

Now suppose there is a single internal source condition in Y_n, and that it exists at level s, $1 \leq s < n$. Then set $i-1 = s$, so that $N_-(i-1)$ in (10) becomes $N_-(s)$. Further, since we are at liberty to choose the magnitude of k, we set $k = n$. Therefore, since $N_+'(n+1) = 0$ (by the above hypothesized source condition), the invariant imbedding relation (10) becomes:

$$[N_+(j), N_-(j)] = [0, N_-(s)] \begin{pmatrix} \mathcal{T}(n+1, j, s) & \mathcal{R}(n+1, j, s) \\ \mathcal{R}(s, j, n+1) & \mathcal{T}(s, j, n+1) \end{pmatrix} \quad (11)$$

From this we read off the following relations:

$$N_+(j) = N_-(s)\, \mathcal{R}(s, j, n+1), \quad s < j; \quad (12)$$

$$N_-(j) = N_-(s)\, \mathcal{T}(s, j, n+1), \quad s \leq j. \quad (13)$$

It is important to note that we have derived (12) and (13) for the particular inequality $s < j$. The 9 x 17 matrix $\mathcal{R}(s, j, n+1)$ is the complete reflectance operator for downward radiance of the general theory[8]. The 9 x 9 matrix $\mathcal{T}(s, j, n+1)$ is the complete transmittance operator for downward radiance. Recall that $\mathcal{T}(s, s, n+1) = I$ the identity matrix so that the equality sign is permissible between s and j in (13). Recall further that $\mathcal{R}(s, n+1, n+1)$, the 9 x 17 zero matrix, a fact which is consistent with the boundary condition $N'_+(n+1) = 0$ (the zero vector).

We now complete the description of the relation between $N(j)$ and the radiance distribution at level s by considering the case $s > j$. There is a single internal source condition at level s, $1 < s \le n$ in Y_n. Now set $k+1 = s$ and $i-1 = 0$ in (10). With this choice of k and i, we have $N'_+(k+1) = N'_+(s)$, and $N_-(i-1) = N_-(0) = 0$ (the zero vector) which again follows from the freedom of choice of the i, k values and the present source condition. Hence (10) takes the form:

$$[N_+(j), N_-(j)] = [N'_+(s), 0] \begin{pmatrix} \mathcal{T}(s, j, 0) & \mathcal{R}(s, j, 0) \\ \mathcal{R}(0, j, s) & \mathcal{T}(0, j, s) \end{pmatrix} . \qquad (14)$$

From this we read off the following relations:

$$N_+(j) = N'_+(s)\, \mathcal{T}(s, j, 0) , \quad s \ge j ; \qquad (15)$$

$$N_-(j) = N_+'(s)\,\mathcal{R}(s,j,0), \quad s > j. \qquad (16)$$

The 9 x 17 matrix $\mathcal{T}(s,j,0)$ is the *complete transmittance* operator (for upward radiance), with the property: $\mathcal{T}'(s,s,0) = I$ where $\mathcal{T}'$ stands for the contraction of $\mathcal{T}$ to a 9 x 9 matrix (see reference 5). Further, the 9 x 9 matrix $\mathcal{R}(s,j,0)$ is the *complete reflectance* operator (for upward radiance), with the property: $\mathcal{R}(s,0,0) = O$, the zero matrix.

The net result of the present findings is the knowledge of how to relate, by means of the complete reflectance and transmittance operators of Y_n, the radiance distribution at any level s with every level $j \neq s$. (Equations (12), (13), (15), (16).) This, together with the connection summarized by the local Ψ-operator in (9) will allow, for every pair (s,j) of depths, the complete description of $N(j)$ when $N^0(s)$ is given. We shall now show how this knowledge of $\mathcal{R}$, $\mathcal{T}$ and the local Ψ-operator yields the second and final decomposition of the general Ψ-operator required in the present analysis.

The Second Decomposition of the Ψ-Operator

We now use Equations (9), (12), (13), (15), and (16) to systematically decompose the Ψ-operator into components which then lead to a way of evaluating the operator by means of simple invariance and local interaction considerations. In all that follows, $N^0(s)$ is an arbitrary single source vector at a fixed level s, $1 \leq s \leq n$.

Start with (9) written in component form:

$$[N_+(s), N_-(s)] = [N_+^0(s)', N_0^0(s), N_-^0(s)] \begin{pmatrix} \Psi_{++}(s,s) & \Psi_{+-}(s,s) \\ \Psi_{0+}(s,s) & \Psi_{0-}(s,s) \\ \Psi_{-+}(s,s) & \Psi_{--}(s,s) \end{pmatrix} \quad (17)$$

Consider first the case $s < j$. From (17):

$$N_-(s) = N_+^0(s)' \Psi_{+-}(s,s) + N_0^0(s) \Psi_{0-}(s,s) + N_-^0(s) \Psi_{--}(s,s). \quad (18)$$

Combining (17) and (18):

$$\begin{aligned} N_+(j) = {} & N_+^0(s)' \left[\Psi_{+-}(s,s) \, Q(s,j,n+1) \right] \\ & + N_0^0(s) \left[\Psi_{0+}(s,s) \, Q(s,j,n+1) \right] \\ & + N_-^0(s) \left[\Psi_{-+}(s,s) \, Q(s,j,n+1) \right]. \end{aligned} \quad (19)$$

Observe that (3), written in the first decomposition form, is:

$$[N_+(j), N_-(j)] = [N_+^0(s)', N_0^0(s), N_-^0(s)] \begin{pmatrix} \Psi_{++}(s,j) & \Psi_{+-}(s,j) \\ \Psi_{0+}(s,j) & \Psi_{0-}(s,j) \\ \Psi_{-+}(s,j) & \Psi_{--}(s,j) \end{pmatrix} \quad (20)$$

,

whence

$$N_+(j) = N_+^0(s)' \Psi_{++}(s,j) + N_0^0(s) \Psi_{0+}(s,j) + N_-^0(s) \Psi_{-+}(s,j) \quad (21)$$

Since the vector $N^0(s)$ is arbitrary, we deduce the following operator equalities from (19) and (21):

$$s<j \begin{cases} \Psi_{++}(s,j) = \Psi_{+-}(s,s)\,\mathcal{Q}(s,j,n+1) & (22) \\ \Psi_{0+}(s,j) = \Psi_{0-}(s,s)\,\mathcal{Q}(s,j,n+1) & (23) \\ \Psi_{-+}(s,j) = \Psi_{--}(s,s)\,\mathcal{Q}(s,j,n+1) & (24) \end{cases}$$

which hold for all s and j such that $s<j$.

Finally, the second column of $\Psi(s,j)$ in (20) for the case $s<j$ may be evaluated in a similar way by first using (13) (which actually holds for $s \leq j$) together with:

$$N_-(s) = N_+^0(s)'\,\Psi_{+-}(s,s) + N_0^0(s)\,\Psi_{0-}(s,s) + N_-^0(s)\,\Psi_{--}(s,s), \tag{25}$$

which comes from (17). The result is:

$$N_-(j) = N_+^0(s)'\left[\Psi_{+-}(s,s)\,\mathcal{T}(s,j,n+1)\right] \tag{26}$$

$$+\; N_0^0(s)\left[\Psi_{0-}(s,s)\,\mathcal{T}(s,j,n+1)\right] \tag{27}$$

$$+\; N_-^0(s)\left[\Psi_{--}(s,s)\,\mathcal{T}(s,j,n+1)\right] \tag{28}$$

which, when compared term by term ((26) - (28)) with the equivalent expression from (20), namely:

$$N_-(j) = N_+^0(s)'\,\Psi_{+-}(s,j) + N_0^0(s)\,\Psi_{0-}(s,j) + N_-^0(s)\,\Psi_{--}(s,j), \tag{29}$$

yields the conclusion (since $N^0(s)$ is arbitrary):

$$s \leq j \begin{cases} \Psi_{+-}(s,j) = \Psi_{+-}(s,s)\,\mathcal{T}(s,j,n+1) & (30) \\ \Psi_{0-}(s,j) = \Psi_{0-}(s,s)\,\mathcal{T}(s,j,n+1) & (31) \\ \Psi_{--}(s,j) = \Psi_{--}(s,s)\,\mathcal{T}(s,j,n+1) & (32) \end{cases}$$

We now have deduced exactly half of the required representations of the component matrices of $\Psi(s,j)$. The other half, which goes with the inequalities $s > j$, is derived in a similar manner. The preceding work has set the pattern of the derivations; therefore, the case $s > j$ will not be carried out, and is left for the interested reader. The final results are assembled in TABLE I in the last section of this work.

The next main step in the discussion is the evaluation of the local Ψ-operator and the $\mathcal{R}$ and $\mathcal{T}$ operators in terms of the standard R and T operators of Y_n .

DETAILS OF THE SOLUTION

The preceding section has shown that the internal-source problem on an extended cubic lattice X_n can be represented in terms of a set of general operator relations (4) involving the Ψ-operator $\Psi(s,j)$, $1 \leq s, j \leq n$. The Ψ-operator was then systematically analyzed as far as it was possible to do so without resorting to any particular properties of the associated quotient space Y_n . The resultant operators so found were the local Ψ-operator $\Psi(s,s)$, $1 \leq s \leq n$, and the complete reflectance and complete transmittance operators $\mathcal{R}$ and $\mathcal{T}$ on a general Y_n . In order to make any further progress toward the solution of the internal source problem, explicit cognizance must be made of the particular structure of Y_n . This means that $\Psi(s,s)$, $\mathcal{R}$, and $\mathcal{T}$ must now be expressed in terms of the standard R, S, and T operators on Y_n . Once this is done, the solution will be complete, for the required R, S, and T operators (except for two S-operators given below) were completely determined in terms of the basic Σ-function in reference 5.

Determination of the Monolayer Operators

The next step in the explicit determination of the effects of the internal source distribution $N^{o}(\mathfrak{z})$ directs attention to its component $N_{o}^{o}(\mathfrak{z})$. Recall that $N_{o}^{o}(\mathfrak{z})$ is an eight component source vector where radiance components describe the horizontal irradiation of the points of the monolayer at level $\mathfrak{z}$. The theory of the response of an homogeneous monolayer to an arbitrary plane irradiation was worked out in reference 5. In particular we now use equation (55) of reference 5 to determine the resultant horizontal irradiance $N_{o}(\mathfrak{z})$ induced by the given source component $N_{o}^{o}(\mathfrak{z})$ (considered for the moment as the only source irradiating the monolayer.) The requisite formula governing $N_{o}(\mathfrak{z})$ is:

$$N_{o}(\mathfrak{z}) = N_{o}^{o}(\mathfrak{z})\, S(\mathfrak{z};o;o)\left[I - S(\mathfrak{z};o;o)\right]^{-1} \qquad (33)$$

The definitions of the S-operators appearing in (33) and below are given in full in reference 5. Further, the mathematical and physical background of equation (55) of reference 5 along with instructions on the correct use of the local interaction principle for the present geometry are also given in detail in that reference.

Now the eight component radiance vector $N_{o}^{o}(\mathfrak{z})$ also gives rise to upward $N_{+}(\mathfrak{z})$ and downward $N_{-}(\mathfrak{z})$ radiance flows from the irradiated monolayer at level $\mathfrak{z}$. The principle of local interaction states that the resultant upward radiance is governed by the equation

$$N_+(s) = N_0(s)\, S(s;0;+) + N_0^0(s)\, S(s;0;+). \qquad (34)$$

Further, the principle of local interaction states that the downward resultant radiance is governed by the equation:

$$N_-(s) = N_0(s)\, S(s;0;-) + N_0^0(s)\, S(s;0;-). \qquad (35)$$

Combining equations (33) and (34):

$$\begin{aligned} N_+(s) = {} & N_0^0(s)\, S(s;0;0)\left[I - S(s;0;0)\right]^{-1} S(s;0;+) \\ & + N_0^0(s)\, S(s;0;+). \end{aligned} \qquad (36)$$

We now <u>define</u> an 8 x 17 matrix $S(s,+)$ with the following property:

$$N_+(s) = N_0^0(s)\, S(s,+), \qquad (37)$$

for all $N_0^0(s)$. It follows from (36) that

$$\boxed{S(s,+) = S(s;0;+) + S(s;0;0)\left[I - S(s;0;0)\right]^{-1} S(s;0;+) \qquad (38)}$$

Now combine (33)and (35)

$$\begin{aligned} N_-(s) = {} & N_0^0(s)\, S(s;0;0)\left[I - S(s;0;0)\right]^{-1} S(s;0;-) \\ & + N_0^0(s)\, S(s;0;-). \end{aligned} \qquad (39)$$

We define the 8 x 9 matrix $S(s,-)$ with the following property:

$$N_-(s) = N_0^0(s)\, S(s,-). \qquad (40)$$

It follows from (39) that

$$\boxed{S(s,-) = S(s;0;-) + S(s;0;0)\left[I - S(s;0;0)\right]^{-1} S(s;0;-) \qquad (41)}$$

An examination of the theory of irradiated monolayers established in reference 5 shows that the operator equations (38) and (41) round out the list of possible operators associated with a general monolayer irradiation geometry in Y_n. The operators $S(s,\pm)$ supplement the four R and T operators of the monolayer, developed in reference 5, in the sense that the latter were designed to give the reflectance and transmittance of a monolayer at level s when it was generally irradiated from levels $s-1$ and $s+1$ i.e., from directions properly outside the level s. The situation of internal irradiation did not arise in reference 5; hence there was no need for the operators $S(s,\pm)$ at that time. To gain a complete understanding of the operators $S(s,\pm)$, the reader should compare (38) and (41) with the remaining four R and T monolayer operators developed in reference 5.

Determination of the Local Ψ-Operator

Figure 3 depicts the radiometric elements of the present derivation leading to a determination of the local Ψ-operator $\Psi(s,s)$. The general twenty-six component source vector $N^{o}(s)$ irradiates the monolayer at level s and, for the moment, is the <u>only</u> source within Y_n. The irradiated monolayer gives rise to a multiple scattering process which permeates Y_n with radiant flux. Let $N(s)$, $N(s-1)$, and $N(s+1)$ be the resultant steady-state twenty-six component radiance vectors at levels s, $s-1$, and $s+1$, respectively. Then, using the notation established in reference 5, the principle of local interaction. and the principles of invariance supply the following four statements governing the components of these steady state radiance vectors:

$$N_+(s) = N_+^{o}(s)T(s+1,s) + N_o^{o}(s)\,S(s,+) + \left[N_-^{o}(s) + N_-(s-1)\right]R(s,n+1) \tag{42}$$

$$N_-(s) = \left[N_+^{o}(s)' + N_+'(s+1)\right]R(s,0) + N_o^{o}(s)\,S(s,-) + N_-^{o}(s)\,T(s-1,s) \tag{43}$$

$$N_-(s-1) = N_+'(s)\,R(s-1,0) \tag{44}$$

$$N_+(s+1) = N_-(s)\,R(s+1,n+1) \tag{45}$$

An examination of (42) and (44) shows that if equation (42) is contracted by the contracting matrix C' (equation (24), reference 5) this pair of equations may be solved for $N_+'(s)$. Hence, contracting (42), we have:

$$N'_+(s) = N^0_+(s)'\,T(s+1,s) + N^0_0(s)\,S'(s,+) + \left[N^0_-(s) + N_-(s-1)\right] R'(s,n+1) \,. \tag{46}$$

Substituting (44) in (46), and solving for $N'_+(s)$:

$$N'_+(s) = \left[N^0_+(s)'\,T'(s+1,s) + N^0_0(s)\,S'(s,+) + N^0_-(s)\,R'(s,n+1)\right] \times \left[I - R(s-1,0)\,R'(s,n+1)\right]^{-1}. \tag{47}$$

From this and (44), it follows that $N_-(s-1)$ is determinable. This resultant expression for $N_-(s-1)$ is then plowed back into (42) to determine $N_+(s)$. The reason for this round-about way of finding $N_+(s)$ stems from the fact that its contracted nine component form $N'_+(s)$ is inevitably first in line for determination. The eight components lost by the contraction must then be regained by circling back through the equations. The result may be written:

$$\begin{aligned} N_+(s) = {} & N^0_+(s)'\left[T(s+1,s) + T'(s+1,s)\,A\right] \\ & + N^0_0(s)\left[S(s,+) + S'(s,+)\,A\right] \\ & + N^0_-(s)\left[R(s,n+1) + R'(s,n+1)\,A\right], \end{aligned} \tag{48}$$

Where A is an abbreviation of:

$$A = \left[I - R(s-1,0)\,R'(s,n+1)\right]^{-1} R(s-1,0)\,R(s,n+1) \,. \tag{49}$$

It follows from a comparison of (48) and (17) that

$$\left.\begin{aligned} \Psi_{++}(s,s) &= T(s+1,s) + T'(s+1,s)\,A \\ \Psi_{0+}(s,s) &= S(s,+) + S'(s,+)\,A \\ \Psi_{-+}(s,s) &= R(s,n+1) + R'(s,n+1)\,A \end{aligned}\right\} \qquad (50)$$

The next step is the solution of the system (42) - (45) for $N_-(s)$. From (43) and (45), it follows that:

$$\begin{aligned} N_-(s) = {} & N_+^0(s)'\,R(s,0)\,B \\ & + N_0^0(s)\;S(s,-)\,B \\ & + N_-^0(s)\;T(s-1,s)\,B \end{aligned} \qquad (51)$$

$$B = [I - R'(s+1,n+1)\,R(s,0)]^{-1}$$

It follows from a comparison of (51) and (17) that:

$$\left.\begin{aligned} \Psi_{+-}(s,s) &= R(s,0)\,B \\ \Psi_{0-}(s,s) &= S(s,-)\,B \\ \Psi_{--}(s,s) &= T(s-1,s)\,B \end{aligned}\right\} \qquad (52)$$

The matrices in (50) and (52) are the desired components of the local Ψ-operator. They are assembled for convenient reference in TABLE II in the last section of this work.

Determination of the Complete Reflectance and Transmittance Operators

We begin with the operators for downward flux; i.e., we shall consider the case $s < j$, thereby determining the operators in (12) and (13). Consider Figure [illegible]. As before there is only one arbitrary source in Y_n, and that is on level s. The two main principles of invariance (reference 5) state that for any level $j > s$ within Y_n:

$$N_-(j) = N_-(s)\,T(s,j) + N_+'(j+1)\,R(j,s) \tag{53}$$

$$N_+(j+1) = N_-(j)\,R(j+1,n+1) \tag{54}$$

Solving this set:

$$N_-(j) = N_-(s)\,T(s,j)\left[I - R'(j+1,n+1)\,R(j,s)\right]^{-1} \tag{55}$$

$$N_+(j) = N_-(s)\,T(s,j)\left[I - R'(j+1,n+1)\,R(j,s)\right]^{-1} R(j+1,n+1) \tag{56}$$

It follows from (12) and (56) that

$$Q(s,j,n+1) = T(s,j-1)\left[I - R'(j,n+1)\,R(j-1,s)\right]^{-1} R(j,n+1) \tag{57}$$

And, from (13) and (55):

$$\mathcal{T}(s,j,n+1) = T(s,j)\left[I - R'(j+1,n+1)R(j,s)\right]^{-1} \tag{58}$$

Finally, we consider the case for upward flux in which $s > j$. Thus we may determine the operators in (15) and (16). The two main principles of invariance give the relations:

$$N_+(j) = N'_+(s)\,T(s,j) + N_-(j-1)\,R(j,s) \tag{59}$$

$$N_-(j-1) = N'_+(j)\,R(j-1,0) \tag{60}$$

We first determine $N'_+(j)$: by contracting (59) with the contracting matrix C' (Equation (24), reference 5; see also (46) above). The result is:

$$N'_+(j) = N'_+(s)\,T'(s,j) + N_-(j-1)\,R'(j,s) \tag{61}$$

Replacing $N_-(j-1)$ in (61) by means of (60), and solving for $N'_+(j)$, we have:

$$N'_+(j) = N'_+(s)\,T'(s,j)\left[I - R(j-1,0)\,R'(j,s)\right]^{-1} \tag{62}$$

and also:

$$N_-(j-1) = N'_+(s)\,T'(s,j)\left[I - R(j-1,0)R'(j,s)\right]^{-1} R(j-1,0) \tag{63}$$

Replacing $N_-(j-1)$ of (63) in (59) permits the evaluation of:

$$N_+(j) = N_+'(s)\left\{T(s,j) + T'(s,j)\left[I - R(j-1,0)R'(j,s)\right]^{-1}R(j-1,0)\,R(j,s)\right\} \tag{64}$$

From (63), (64), (15) and (16), we have:

$$\mathcal{Q}(s,j,0) = T'(s,j+1)\left[I - R(j,0)R'(j+1,s)\right]^{-1}R(j,0) \tag{65}$$

$$\mathcal{T}(s,j,0) = T(s,j) + T'(s,j)\left[I - R(j-1,0)R'(j,s)\right]^{-1}R(j-1,0)R(j,s) \tag{66}$$

This completes the detailed evaluation of the complete reflectance operators in particular, and the details of the solution in general. The following section summarizes and organizes the results obtained.

SUMMARY

The problem solved in this work is the general internal-source problem on an extended cubic lattice X_n. Briefly, each monolayer of X_n is irradiated by a twenty-six component radiance source $N^o(s)$, $1 \le s \le n$. It is required to find the resultant twenty-six component radiance vector $N(j)$ at each level j, $1 \le j \le n$. The general form of the solution is (Equation (41)):

$$N(j) = \sum_{s=1}^{n} N^o(s)\,\Psi(s,j) ,$$

where $\Psi(s,j)$ is a 26 x 26 matrix of the form given in (6). The decomposition of the submatrices of $\Psi(s,j)$ into local Ψ-operators and complete reflectance and transmittance operators is given in TABLE I below; the representation of these component matrices are in turn given in TABLE II and III, respectively, by means of the standard R and T operators associated with the quotient space Y_n of X_n. These latter operators are completely described and evaluated in reference 5. Thus the present internal-source problem on an extended cubic lattice is completely solved.

TABLE I

DECOMPOSITION OF Ψ-OPERATOR IN TERMS OF LOCAL Ψ-OPERATOR AND THE COMPLETE REFLECTANCE AND TRANSMITTANCE OPERATORS.

(Refer to Equations (17) - (32) in text.)

$$\Psi_{++}(s,j) = \begin{cases} \Psi_{++}(s,s)\,\mathcal{T}(s,j,0) & s \geq j \\ \Psi_{+-}(s,s)\,\mathcal{R}(s,j,n+1) & s < j \end{cases}$$

$$\Psi_{0+}(s,j) = \begin{cases} \Psi_{0+}(s,s)\,\mathcal{T}(s,j,0) & s \geq j \\ \Psi_{0-}(s,s)\,\mathcal{R}(s,j,n+1) & s < j \end{cases}$$

$$\Psi_{-+}(s,j) = \begin{cases} \Psi_{-+}(s,s)\,\mathcal{T}(s,j,0) & s \geq j \\ \Psi_{--}(s,s)\,\mathcal{R}(s,j,n+1) & s < j \end{cases}$$

$$\Psi_{+-}(s,j) = \begin{cases} \Psi_{++}(s,s)\,\mathcal{R}(s,j,0) & s > j \\ \Psi_{+-}(s,s)\,\mathcal{T}(s,j,n+1) & s \leq j \end{cases}$$

$$\Psi_{0-}(s,j) = \begin{cases} \Psi_{0+}(s,s)\,\mathcal{R}(s,j,0) & s > j \\ \Psi_{0-}(s,s)\,\mathcal{T}(s,j,n+1) & s \leq j \end{cases}$$

$$\Psi_{--}(s,j) = \begin{cases} \Psi_{-+}(s,s)\,\mathcal{R}(s,j,0) & s > j \\ \Psi_{--}(s,s)\,\mathcal{T}(s,j,n+1) & s \leq j \end{cases}$$

TABLE II

THE COMPONENTS OF THE LOCAL Ψ-OPERATOR IN TERMS OF THE STANDARD REFLECTANCE AND TRANSMITTANCE OPERATORS

(Refer to Equations (9), (50), and (52) in text.)

$$\Psi_{++}(s,s) = T(s+1,s) + T'(s+1,s)A$$

$$\Psi_{0+}(s,s) = S(s,+) + S'(s,+)A$$

$$\Psi_{-+}(s,s) = R(s,n+1) + R'(s,n+1)A$$

$$A = [I - R(s-1,0)R'(s,n+1)]^{-1}R(s-1,0)R(s,n+1)$$

$$\Psi_{+-}(s,s) = R(s,0)B$$

$$\Psi_{0-}(s,s) = S(s,-)B$$

$$\Psi_{--}(s,s) = T(s-1,s)B$$

$$B = [I - R'(s+1,n+1)R(s,0)]^{-1}$$

TABLE III

THE COMPLETE REFLECTANCE AND TRANSMITTANCE OPERATORS IN TERMS OF THE STANDARD REFLECTANCE AND TRANSMITTANCE OPERATORS

(Refer to Equations (57) - (66) in text.)

$$\mathcal{R}(s,j,n+1) = T(s,j-1)\left[I - R'(j,n+1)R(j-1,s)\right]^{-1} R(j,n+1) \qquad s < j$$

$$\mathcal{T}(s,j,n+1) = T(s,j)\left[I - R'(j+1,n+1)R(j,s)\right]^{-1} \qquad s \leq j$$

$$\mathcal{R}(s,j,0) = T'(s,j+1)\left[I - R(j,0)R'(j+1,s)\right]^{-1} R(j,0) \qquad s > j$$

$$\mathcal{T}(s,j,0) = T(s,j) + T'(s,j)\left[I - R(j-1,0)R'(j,s)\right]^{-1} R(j-1,0)R(j,s) \qquad s \geqq j$$

REFERENCES

1. Preisendorfer, R. W., Radiative Transfer on Discrete Spaces, SIO Ref. 59-53, University of California, La Jolla, California, (1959).
2. ________________ Principles of Invariance on Discrete Spaces, SIO Ref. 59-55, University of California, La Jolla, California, (1959).
3. ________________ Theory of Polarized Light Fields in Discrete Spaces, SIO Ref. 59-66, University of California, La Jolla, California, (1959).
4. ________________ Radiative Transfer on a Linear Lattice, SIO Ref. 59-67, University of California, La Jolla, California, (1959).
5. ________________ Radiative Transfer on a Cubic Lattice, SIO Ref. 59-68, University of California, La Jolla, California, (1959).
6. Giovanelli, R. G. & Jefferies, J. T., Radiative Transfer With Distributed Sources, Proc. Phys. Soc., B 69 1077-1084 (1956).
7. Preisendorfer, R. W., Unified Irradiance Equations, SIO Ref. 58-43, University of California, La Jolla, California, (1959).
8. ________________ Invariant Imbedding Relation for the Principles of Invariance, Proc. Nat. Acad. Sci., 44, 320-323 (1958).

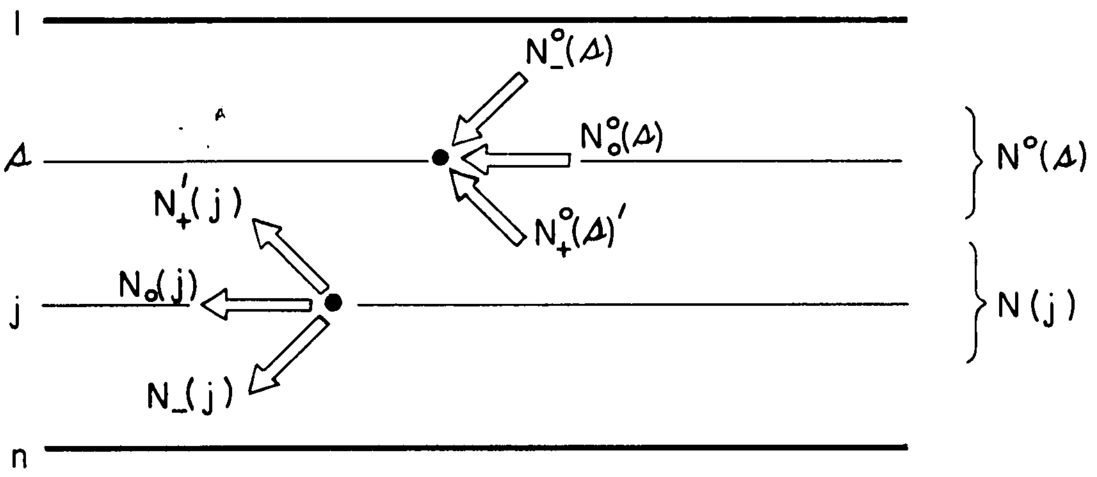

Figure 1

Rudolph W. Preisendorfer

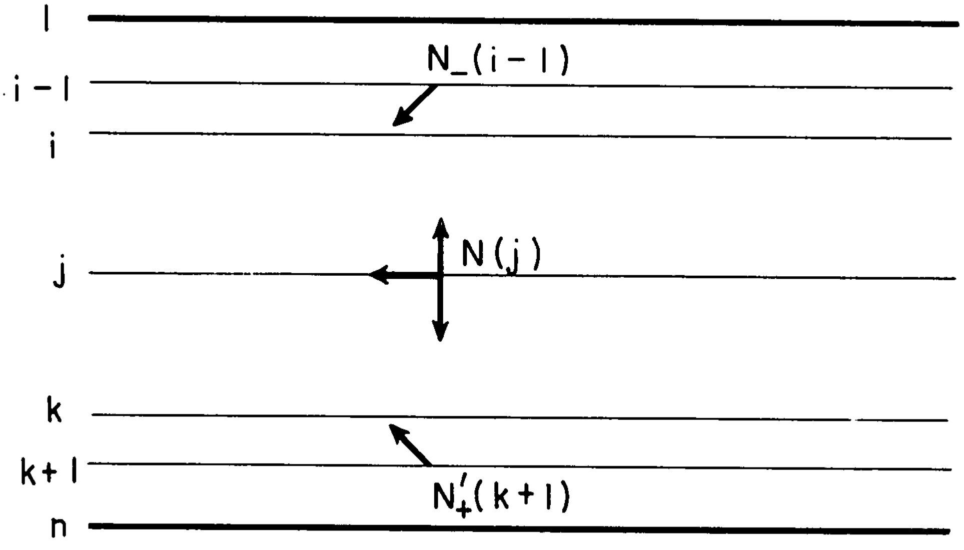

Figure 2

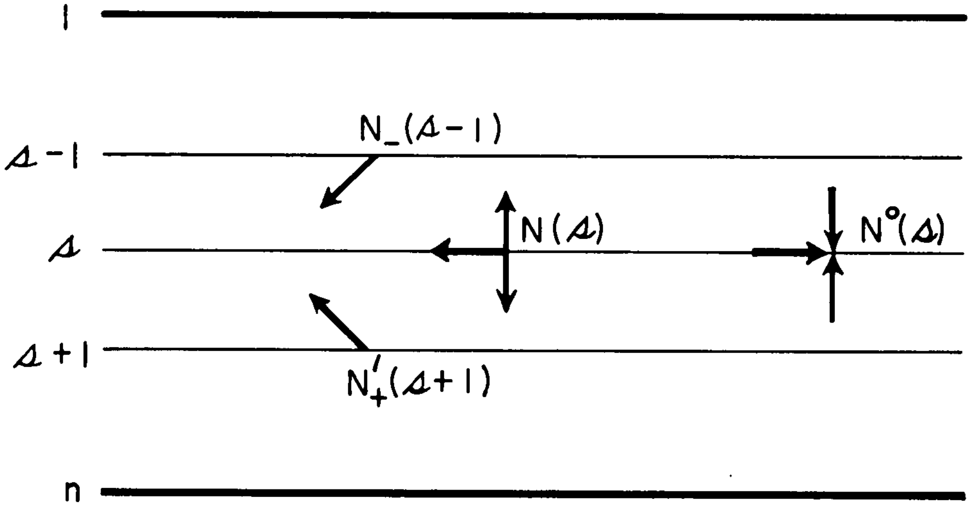

Figure 3

Rudolph W. Preisendorfer