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TWO FUNDAMENTAL METHODS OF SOLVING POINT-SOURCE PROBLEMS
IN DISCRETE-SPACE RADIATIVE TRANSFER THEORY

Rudolph W. Preisendorfer

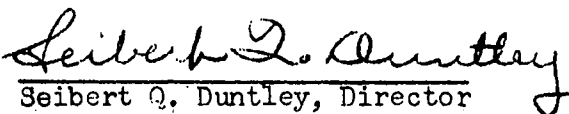
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Two Fundamental Methods of Solving Point-Source Problems
in Discrete-Space Radiative Transfer Theory

Rudolph W. Preisendorfer

INTRODUCTION

The analytical techniques of discrete-space radiative transfer theory¹⁻⁶ are here employed to develop two fundamental methods of determining the light field generated by a point source within a given optical medium. Before outlining the methods of solution it may be helpful for the general reader to briefly review some specific examples of point source problems, and the motivations for studying them on a discrete-space level.

Examples of Point Source Problems and their Physical Origin

Consider a natural aerosol or hydrosol in which there is a highly directional source such as a searchlight. The radiant flux from the searchlight is initially directed in some small solid angle about a given direction. The scattering mechanisms in the medium initiate a multiple scattering process, the end result of which is a diffuse steady state radiance distribution about each point of the medium in which the source is located.

One form of the point source problem requires the detailed calculation of these diffuse steady state radiance distributions generated by the directional point source. Minor variants of this problem allow for the possibility of less directional point sources, such as flares.

The ultimate uses to which the solution of the point source problem may be directed are manifold: the resolution of problems of underwater or aerial visibility, television, and photography, given the inherent optical properties of the source of radiant flux, and the properties of the optical targets. Conversely, the solution of the point source problem may be used to resolve basic research problems which require a determination of the inherent optical properties of a medium, given the radiometric response of the medium to a particular point source.

Another example of the point source problem arises when it is required to determine the transient (non steady state) light field generated by powerful pulse-emitting light sources (either isotropic or anisotropic) within a given medium. Still another example arises in physical and biological radiant emission phenomena occurring in localized portions of the atmosphere and the sea. As a final example which at first sight

may appear not to be of the point source type, we cite the problem of determining the perturbation of the natural or artificial light field in a medium by the introduction of foreign bodies into the field.

Since the formulations below do not commit themselves to any specific type of medium, all of the above transfer problems may be solved within a general scattering-absorbing medium. Furthermore, since the response of a general medium to a set of point sources is the sum of the responses to the individual sources, the present solution methods can in principle be used to determine the effects of extended plane source and volume sources. Thus the present results subsume those of the plane-source methods⁶, which in turn were generalized cubic lattice methods.⁵

Motivations for the Discrete-Space Formulation

The motivations for the discrete-space approach to radiative transfer problems have been stated and illustrated in detail in the earlier papers of this series.¹ However, it would perhaps be of interest to briefly point out the special merits of the discrete-space formulations in the

present case.

The general point source problem in radiative transfer theory is the most difficult of the standard problems encountered in the theory because of the greater number of parameters required in its formulation and solution. Thus, in plane-parallel media with a given incident plane radiance distribution at its upper boundary, three parameters: depth z , and direction angles (θ, ϕ) are needed to specify the radiance distribution. In the case of a general point source within the medium, three parameters (x, y, z) are needed at the very outset to locate a point in the medium relative to the source, and then two more (θ, ϕ) to specify the radiance distribution at that point.

In order to reduce this relatively unmanageable number of parameters, when using the continuous approach, both the optical properties of the medium and the inherent radiance distribution of the point source invariably undergo vitiating assumptions such as the adoption of isotropic sources, isotropic scattering, and homogeneous properties, all resulting in an emasculated radiative transfer problem which is indistinguishable from the more simple varieties of certain heat transfer and neutron transfer problems. Now while such

procedures may point up certain basic processes underlying all transfer problems, they nevertheless do an injustice to the photon transfer process: the very features inherent in it, which distinguish it from these other transfer processes, are indiscriminately hammered smooth in order to obtain a mathematically tractable--but in the last analysis meaningless--formulation.

To be specific, we are referring to such questions as the following which still have not been satisfactorily answered, by the continuous methods, for the photon transfer case: What effect on the light field results from, (a) changing the scattering law from the isotropic extreme to the high forward scattering extreme actually found in natural aerosols and hydrosols? (b) from arbitrarily changing the directional structure of the inherent radiance distribution of the point source? (c) from the introduction of various types of inhomogeneities into the optical medium? (d) from the introduction of foreign objects into the light field? (e) from changes in the shape and size of the medium?

The discrete-space (or finitary) formulations can go beyond the continuous-space formulations by providing relatively simple and direct answers to these questions (See, e.g.

references 1-6). Unlike the continuous space formulation, which must be re-cast with fundamental changes for each new problem, the discrete space formulation uses basically only one equation to answer each of the preceding five quite dissimilar questions (see Equation (2) below); this equation in various simple guises takes the form of an irreducibly simple algorithm suitable for use in large scale automatic computers.

Now in order to avoid misunderstanding it should be pointed out immediately that, in order to reap this seemingly incredible analytical harvest, the discrete (or finitary) theory just as the continuous (or infinitary) theory, must offer up a sacrifice of fidelity somewhere along the way. Just where this sacrifice occurs and what form it takes constitutes the fundamental distinction between the finitary and infinitary formulations. Briefly, the fidelity-reducing procedures in the infinitary theory center on the radiative processes, leaving the underlying space of the optical medium intact; those of the finitary procedures center on the underlying space of the optical medium, leaving intact for exploration the rich variety of radiative processes on the space.

The Basic Ideas Behind the Two Methods

The first step in either method is the transformation of the given continuous geometrical setting X to its discrete counterpart X_n . As will be shown below, every continuous setting X may be replaced by (or imbedded within) a cubic lattice setting X_n , a relatively simple form of discrete space, for which a complete theory has been developed (references 2 and 5).

The first of the two methods presented below is referred to as the Iteration Method; the second is referred to as the Categorical Analysis Method. Each is, in a certain sense, the antithesis of the other.

In the Iteration Method, the imbedding cubic lattice is viewed as an irreducible unit. Equations are drawn from the basic theory of reference 1, which describe the response of X_n at an arbitrary point (x, y, z) to an impinging radiance distribution at another arbitrary point (x', y', z') . The equation governing this method is an elementary matrix-operator form which may be solved by the simple expedient of matrix iteration, i.e., by repeated raising of a single matrix to integral powers and summing over the resultant set of

iterants, a method eminently suitable for use on large-scale automatic computers.

The Categorical Analysis Method, on the other hand, views the imbedding cubic lattice not as an irreducible unit, but rather as a set of plane lattices (the monolayers) arranged in a definite way--much as a crystal lattice can be viewed as an assembly of monomolecular layers. The response of the cubic lattice to an impinging radiance distribution is then reducible to the superposition of the responses of its monolayer components. The Categorical Analysis Method further views each of the monolayers as a reducible collection of linear lattices (the monolines)--again following the crystal lattice analogy which sees a monomolecular layer as a collection of interacting strings or chains of atoms. Thus the response of a monolayer in the cubic lattice may be found by the suitable superposition of the responses of its monoline components. Finally, the monoline response to an incident radiance distribution may be found by using the methods of linear lattices (reference 4) which is developed directly from the given data of its "atomic" (individual point.) components. The response of the monobloc (the imbedding cubic lattice) may then be viewed in the final analysis as the suitable superposition of the responses of its individual component points to incident radiance distributions. The final response formula for this method is

expressible in compact form, suitable for use on large scale automatic computers. Except for a few geometrical details peculiar to the monobloc geometry, the requisite procedures in the Categorical Analysis Method have already been developed in an earlier work of this series (reference 6).

Critique of the Two Methods

The two methods studied below, though based on the same point of departure (Equation (2)), are fundamentally distinct in their practical realizations.

The Iteration Method, as we have seen, attacks the multiple scattering problem directly; and through the brute strength of automatic computers, forges a path from the given source N^o to the required goal N . It may be likened to the sculpturing of a statue from a single monobloc of marble. There is a minimum of preparatory work for the computer; and once the computer is programmed, it methodically hews its way toward the solution until its memory capabilities and free computation channels are saturated. By its very nature (an infinite series of iterations) the iteration method can never be fully completed. However, as will be shown below, the radiometric distance from N^o to N

can essentially be covered in most practical instances; and the residual distance, after the computations have ground to a halt, is estimable in each case.

The Categorical Analysis Method, on the other hand, does not attack the problem directly. There is a relatively intricate and detailed preliminary breakdown of the multiple scattering problem by the analyst before the computer even sees the problem in program form. Furthermore the problem, which is eventually insuperable in monobloc form, is chopped into such sizes and forms (no matrix larger than 9 x 9 to invert, no matter what the size of X_n) that the computer can build each piece in a finite amount of steps and time. In the statue analogy used above, the present method forms the head, body, arms and legs, separately and then assembles them one by one into the final whole, rather than hew them from a single monobloc. Thus the Categorical Analysis Method is fully solvable by present day computer systems. However, the price for this is paid by the relatively large number of preliminary steps of analysis required before the computer is set into action.

FORMULATION AND SOLUTION OF THE GENERAL PROBLEM

The Optical Medium

Let X represent a bounded subset of Euclidean three-space E_3 with an orthogonal cartesian coordinate frame and an arbitrary but given metric on $X \times X$. The associated phase space $\Phi = X \times \Xi$, where Ξ is the unit sphere in E_3 is the domain of definition of the radiance function N . Let α , σ and n be the volume attenuation, volume scattering, and index of refraction functions defined, respectively, on X , $X \times \Xi \times \Xi$, and X , (Both time and wavelength domains of N , α , σ , and n are implicitly carried along in the notation.) The ordered quintuple $(\Phi, N, \alpha, \sigma, n)$ is an optical medium, which forms the starting point of the present discussion.

Let the boundary and initial values of N on Φ be given. Further, an emission radiance function N_η , is presumed given on Φ . If n is not a constant function on X then define a new radiance function $G = N/n^2c$ on Φ , and replace N by G . Because of this, no loss in generality is engendered if, at the outset, we assume n a constant function on X and drop it from notation (see reference

7). Under these given conditions, the problem is to determine N on Φ .

The Associated Discrete Setting

Because of the boundedness of X , there is a parallelepiped in E_3 , of the form:

$$\left\{ (x, y, z) : |x|, |y| \leq c, a \leq z \leq b; a, b, c \text{ integers} \right\}$$

which properly contains X . This parallelepiped may be partitioned into a set of n cubes of unit volume, where $n = (2c+1)^2(b-a+1)$. The set of centerpoints of the cubes of this partition determine the associated discrete-space X_n .

The methods of reference 1 are then ready for application to X_n ; assume a particular set of radiance and eclipse conventions. To fix ideas, assume for the remainder of this work that the radiance and eclipse conventions of reference 5 are in force, and that the Σ and A functions for X_n have been defined on X_n in accordance with the procedures set forth in that reference. The procedure may be generalized to non-stratified media, if necessary, by simply replacing

everywhere the single " z " coordinate by an (x, y, z) triple. Finally, the discrete source functions $N_{k_i}^0$ on $X \times \Xi$ associated with the given continuous emission function N_η , are obtained by setting $N_{k_i}^0 = N_\eta(x_i, \xi_{k_i}) \Delta_k D_i$ where x_i is the center of the i 'th cube in X_η , and ξ_{k_i} , Δ_k , and D_i are defined in reference 5.

For reasons which will become clear in the discussion of the Iteration Method, it will not be necessary to explicitly consider unbounded subsets of E_3 in the present work. We note that the theory of the only important type of unbounded media, namely the stratified plane-parallel type, has already been considered in detail and fully worked out in reference 5.

The general problem has now been formulated and the geometric setting of its discrete counterpart determined. The only remaining preliminary is to formally state the problem and establish the existence of its basic solution in the discrete context. This basic solution will then form the nucleus of the two fundamental methods under present study which show how to realize the basic solution in practical terms.

The Point Source Problem

Given: The discrete optical medium (Φ_n, N_+, A, Σ) where the discrete phase space Φ_n is manufactured from the continuous optical medium $(\Phi, N, \alpha, \sigma)$ (in which n is a constant function on X). Boundary and source conditions on the radiance function N on $\Phi_n = X_n \times \Xi$ are obtained in accordance with the procedures outlined above and are summarized by the discrete source vector N^0 .

Required: The values of the components of the $1 \times n^2$ specific radiance vector $N_+ = [N_{ij} : 1 \leq i, j \leq n]$. Using the basic terminology and concepts set forth in reference 1, the requisite form of the local interaction principle is:

$$N_+ = N_- \Sigma + N^0 \Sigma^0. \quad (1)$$

This determines the required N_+ in terms of the given source vector N^0 . Here Σ is the $n^2 \times n^2$ scattering matrix associated with X_n . In the present general formulation the vector N^0 has $\sum_{i=1}^n \mathcal{A}_i$ components, where $\mathcal{A}_i \geq 0$ is the number of incident source directions at $x_i \in X_n$. Hence Σ^0 is an $(\sum_i \mathcal{A}_i) \times n^2$ matrix whose elements, as those

of Σ , are determined by assigning values to the local scattering function: $\Sigma(x_i; \xi'; \xi)$. The assignment is arbitrary except, of course, that the usual local conservation requirement is satisfied.¹ The present problem is then to determine N_+ (or, equivalently, $N_- = N_+ M$, where M is the permutation matrix of reference 1).

Basic Solution of the Point Source Problem

We recall first of all that the matrix operator Σ is a norm-contracting operator on the vector space V_{n^2} of all ordered n^2 -tuples of real numbers (reference 1). The norm on V_{n^2} is defined by the rule: $|V| = \sum_{i=1}^{n^2} |v_i|$, for each $V = [v_1, \dots, v_{n^2}] \in V_{n^2}$. It is easy to check that the operation $|\cdot|$ (which is a function on V_{n^2} into the set of non-negative real numbers has the properties of a norm (positive-definite, homogeneous; triangle inequality holds). This particular norm defined on V_{n^2} is called the radio-metric norm. The continuous analog of this norm is the operation $\int_{\Phi} [\cdot] d\nu$ on the set of all radiance functions on Φ . The physical significance of the norm is that the number $\frac{1}{c} \int_{\Phi} N d\nu = U(\Phi)$ is the radiant energy content of Φ where c is the speed of light in Φ . An exactly

similar interpretation holds for the discrete form of the radiometric norm.

It may be well to repeat the argument, sketched in reference 1, which shows that Σ on V_{n^2} is norm-contracting, for this fact is crucial in the existence proof of the solution of (1).

Now the $1 \times n^2$ field vector N_- is related to the specific radiance vector N_+ by means of an $n^2 \times n^2$ permutation matrix M (reference 1): $N_- = N_+ M$. It is clear, because of the fact that M simply rearranges components, that M preserves the norm of an element of V_{n^2} , i.e., $|VM| = |V|$. Hence if Σ is norm-contracting, so will $M\Sigma$ be norm-contracting. This fact is useful in the present solution procedure because (1) may then be cast into the following form:

$$N_+ = N_- M (M\Sigma) + N^0 \Sigma^0,$$

by making use of the idempotency of M (reference 1). Further, from the fact that $N_+ = N_- M$ we have

$$N_+ [I - M\Sigma] = N^0 \Sigma^0,$$

where I is an $n^2 \times n^2$ identity matrix.

For the reasons outlined in reference 1, it follows that the matrix $I - M\Sigma$ has an inverse whenever Σ is norm-contracting, and hence that (1) possesses a unique non-trivial solution:

$$N_+ = N^0 \Sigma^0 [I - M\Sigma]^{-1} \quad (2)$$

whenever N^0 is not the zero vector in V_n^2 .

Hence the linear transformation Σ on V_n^2 maps the non-negative cone of V_n^2 (the physical specific and field radiance vectors) into itself, and is norm-contracting thereby assuring the system (1) of a unique generally non-trivial solution (2).

The stage is now set for the presentation of the two fundamental methods. Part A is devoted to the Iteration Method, Part B to the Categorical Analysis Method.

PART A

ITERATION METHOD

THE BASIC ITERATION FORMULA

Starting with the norm-contracting operator $M\Sigma$ we form the $n^2 \times n^2$ matrix $I - M\Sigma$, and recall (reference 1) that its inverse $[I - M\Sigma]^{-1}$ has the series representation:

$$[I - M\Sigma]^{-1} = I + (M\Sigma) + (M\Sigma)^2 + \dots \quad (3)$$

Thus if we set

$$N_+^0 = N^0 \Sigma^0, \quad (4)$$

and in general define

$$N_+^{j+1} = N_+^j (M\Sigma), \quad j = 0, 1, \dots, \quad (5)$$

then (2) may be written:

$$N_+ = \sum_{j=0}^{\infty} N_+^j. \quad (6)$$

Equation (6) is the desired basic iteration formula.

The $1 \times n^2$ radiance vector N_+^j is generally a specific radiance vector and, as such, is therefore subject to the eclipse and local direction space conventions extant in its associated discrete space X_n . The physical significance of the j th iterant N_+^j of N_+^0 is easy to state: N_+^j represents radiance on X_n which, relative to the radiant flux comprising N_+^0 , has been scattered at most j times. Hence (6) represents a scattering-order decomposition of the radiance field over X_n .

Equation (6) forms the core of the Iteration Method. By starting with the given source vector N^0 and the given source scattering matrix Σ^0 , the j th-order scattering component of N_+ is determined from the known $(j-1)$ component according to the rule (5). The required specific radiance vector is then the sum of the infinite series of $1 \times n^2$ vectors, as shown in (6). There are several interesting features inherent in (6) which, if judiciously employed, considerably simplify the practical details of the computation procedure leading to (6). We will now consider the most important of these features.

A TIME-DEPENDENT INTERPRETATION OF THE ITERATION FORMULA

In any discrete space with a reasonable number of points (of the order of two to fifty) the associated scattering matrix Σ , while unwieldy for hand or desk computation, is still within manageable proportions for a large-scale automatic computer. For larger spaces ($n > 50$) the resultant weight of matrix operations will stagger even the best of present day computers. However, there are two features of the Iteration Method which, in many practical situations, effectively reduce the computation load--or at least forestall it for a while. The first of these is, for lack of a more suggestive name, the time-dependent property of the Iteration Method in (6). The second is the convergence property of (6) which can effectively reduce the infinite series to a finite series for many practical situations. These two features will now be illustrated in detail.

For simplicity, and to fix ideas with a minimum of geometric detail, we shall illustrate these features by means of a plane lattice χ_n . The plane lattice will be defined by setting $a = b = c$ in the general expression for a cubic lattice (see above). Hence $n = (2c+1)^2$ in this case. Two types of plane lattice will be used. They may be distinguished by the structure of the local direction space assigned to each. The first type,

which we shall designate by the name square grid, has assigned to each point a four element local direction space Ξ' (Figure 1). In terms of the elements of Ξ in E_3 , these are (in ad hoc notation): $\xi_1 = (1, 0, 0)$, $\xi_2 = (0, 1, 0)$, $\xi_3 = -\xi_1$, $\xi_4 = -\xi_2$. The second special plane lattice will be designated by the name square lattice; it has an eight element local direction space Ξ' assigned to each point (Figure 2):

$$\Xi' = \left\{ \xi_1, (\xi_1 + \xi_2)/\sqrt{2}, \xi_2, (\xi_2 + \xi_3)/\sqrt{2}, \xi_3, (\xi_3 + \xi_4)/\sqrt{2}, \xi_4, (\xi_4 + \xi_1)/\sqrt{2} \right\}$$

The Square Grid

Let the source vector N^o on the square grid have precisely one nonzero component of unit magnitude in the direction ξ_1 , at point $x_1 = (0, 0, 0)$ (Figure 1a). Now by the construction of the local direction space on X_n , and by virtue of the standard restriction convention of reference 1 ((ii) on page 28), it follows that x_1 may interact directly only with its four immediate neighbors $x_1 + \xi_i$, $1 \leq i \leq 4$ in X_n . In general, any point $x_j \in X_n$ may then interact directly only with its four immediate neighbors $x_j + \xi_i$, $1 \leq i \leq 4$. Thus it follows that we may reduce the $1 \times n^2$

vectors N_+ and N_- to $1 \times 4n$ vectors, and Σ reduces to a $4n \times 4n$ matrix. This contraction results from weeding out the permanently zero elements of the vectors N_{\pm} and those of the matrix Σ which result from the adoption of the present restriction convention.

Now by hypothesis, N_+^0 has only one nonzero component, so that the $1 \times 4n$ vector $N^0 \Sigma^0 = N_+^0$ has at most four nonzero components. These are the specific radiance components at point X_1 , and are shown schematically in Figure 1b. The remaining $4n - 4$ components of N_+^0 are zero, and are thus not explicitly shown on the grid.

According to the definition (5), the specific radiance vector N_+^1 is obtained from N_+^0 by multiplication by the $4n \times 4n$ matrix $(M\Sigma)$. It follows from the structures of $(M\Sigma)$, N_+^0 and the present restriction conventions that N_+^1 has at most sixteen additional nonzero components than N_+^0 . These are shown schematically in Figure 1c.

The pattern forming is now clear: Each point of the grid which has nonzero vector components in the j th iterant N_+^j , $j = 0, 1, \dots$, generally "infects" its four immediate neighbors so that they have nonzero radiance components in N_+^{j+1} . The region of radiometrically active points in X_n is then an

"expanding" region with increasing j . The important feature to observe is that no matter how extensive X_n , the iteration scheme initially, and for relatively low order iterants, does not take place over the entire grid. Therefore if (6) is a relatively rapidly converging series, it may be possible in some cases to restrict the iteration to a relatively small region about the irradiated point. The details of the convergent process will be considered below. For the present we continue to explore some further (apparent) time-dependent properties of the iteration scheme.

An examination of the Figures 1a - 1e shows that the disturbance initiated by N^0 , i.e., the subregion of X_n associated with nonzero N_+^j components, "travels" outward from X_1 in the form of a diamond-shaped wave front. By concentrating on only the successive values of N_+^j we are in effect observing the history of a transient light field induced by a unit pulse source on X_n at the time $t = -1$, which is shut off for all $t \geq 0$. In the terminology and general point of view of reference 8, the diamond-shaped region is the characteristic spheroid associated with the local source X_1 . If we define "time" t by the value of j in N_+^j , then the local source initially emits flux at time $t = j = 0$ (after being irradiated at time $t = -1$). In this framework, time increases by unit jumps, so that the speed of light on X_n is taken as

unity. Each point of X_n is eventually (i.e., for some sufficiently large t) a local source and, for all subsequent time, it becomes the center of a characteristic spheroid expanding about it. Thus, any given point of the grid is irradiated again and again during alternate time instants. To round out the geometric description of the expanding light field in X_n , an illustration of the characteristic ellipsoid associated with two points (source-receiver pair) is given in Figure 1f. This shows what subregion of X_n is redirecting scattered flux from the source to the receiver point at any given instant.

By the norm-contracting property of Σ in any non-conservative X_n , we conclude that $|N_+^j| \rightarrow 0$ as $j \rightarrow \infty$. Hence, as the characteristic spheroid of X_1 expands, the residual radiant energy of the initial irradiating pulse distributed within the spheroid monotonically decreases to zero with increasing scattering order j . In fact, according to the results of reference 1 (page 44) $|N_+^j| < \gamma^j |N_+^0|$, so that the radiometric norm of N_+^j decreases toward zero, at least as fast as an exponential function, with "time" j .

Now if the source were to remain steady for all integral $t \geq 0$, then at any time $t = k \geq 0$, the specific radiance

vector on χ_n would be none other than $\sum_1^k N_+^j$, whose radiometric norm builds up eventually to its "steady state" limit $|N_+| = \left| \sum_1^{\infty} N_+^j \right|$. Adopting this time-dependent point of view, we may then postulate several intermediate source possibilities between the two extremes considered so far (unit pulse, steady source). For example, the incident source vector N_+^0 may be of unit magnitude for the times $t = -1, \dots, t = k-1$, and then zero for all $t \geq k$. In such a case the radiance vector on χ_n at integral times $t \geq k$ would be

$$N_+(t) = \left(\sum_{j=0}^k N_+^j \right) (M\Sigma)^{t-k}$$

This follows from the fact that the $1 \times 4n$ vector $\sum_0^k N_+^j$ may serve as a new N^0 for χ_n . Still further possibilities exist, but the preceding examples will suffice for the present purposes of illustrating the time-dependent interpretation of the Iteration Method.

We now go on to consider a final example of the time-dependent interpretation. This example is designed to show the effect of the structure of the local direction space Ξ' on the characteristic spheroids and ellipsoids generated within χ_n . (For a complete discussion of these latter concepts, see reference 8.)

The Square Lattice

The square lattice (Figure 2) occupies the same point set of E_3 as does the square grid defined above. However, the local direction space \equiv' associated with the points of the square lattice has twice as many elements as that of the square grid. Thus the square lattice allows a richer radiometric interaction among its points, which manifests itself in $1 \times 8'$ vectors and $8' \times 8'$ scattering matrices. This may be seen in the series of Figures 2a - 2e. The characteristic spheroid of the local source χ_i , now has the general outlines of a square. A diagram of a typical characteristic ellipsoid is given in Figure 2f. Unlike the square grid, once a point (with the exception of χ_i) in the square lattice is "lit up", it remains "lit up" ever afterward in a radiance field induced by a unit pulse source irradiating χ_i , at time $t = -1$. In the square grid, and under the same source condition, a point alternately assumed the receiver-emitter role.

GENERALIZATIONS OF THE METHOD

True Time-Dependent Scattering Problems

The preceding interpretations can be formally cast into a time-dependent version of the local interaction principle with general time-varying local scattering function $\Sigma(x_i, t; \cdot; \cdot)$ on $\Xi' \times \Xi'$ and time-dependent local absorption function $A(x_i, \cdot)$ on Ξ' for each $x_i \in X_\eta$ and t in some discrete time domain. The time-dependent scattering function now generally defines an $n^2 \times n^2$ time-dependent scattering matrix $\Sigma(t)$ which in all other respects is identical with its steady state counterpart Σ . For example, on a square grid the solution vector $N_+(t)$, $t=0, 1, \dots$, would now be represented by

$$N_+(t) = \sum_{j=0}^t N_+^j, \quad (7)$$

where

$$N_+^{j+1} \equiv N_+^j M\Sigma(j), \quad j=0, 1, \dots \quad (8)$$

Here, for simplicity, the incident unit source N^0 is assumed to be applied only to $x_1 \in X_n$ and only at $t = -1$, thereby giving rise to a characteristic spheroid in X_n with x_1 as center.

By successively using (7) and (8) in some automatic computer programmed to perform and repeat the operations defined in these equations, the successive values of the components of the $1 \times 4n$ vector N_{\downarrow}^j , $j = 0, 1, \dots$, may be obtained and recorded. The range of choices of the elements $\sum(x_i, t; \xi_{\alpha i}, \xi_{i\beta})$ of $\sum(t)$ is practically limitless. Thus on the one hand, X_n may be characterized as conservative, isotropic and homogeneous, with steady state isotropic scattering; on the other hand we may choose the \sum values so as to actually simulate the movement of an opaque, reflecting object in a quasistationary light field within X_n which may now be selected as a non-conservative, anisotropic, inhomogeneous medium with anisotropic time-dependent scattering. Both these extremes, it is quite clear, are handled with equal ease by the pair of equations (7), (8).

Radiative Transfer on Extended Discrete Spaces

The theory of radiative transfer on discrete spaces has been explicitly limited to finite discrete spaces (see reference 1). In such spaces the number of points in X_n is finite and, as a set, X_n is bounded in E_3 . The principal reason for restricting X_n in this way stems from the intuitionistic limitations on the mathematics of general infinite unbounded sets of points (reference 2): Roughly speaking, the intuitionistic mathematics permit only constructable infinities such as those comprising the set of all integers or rationals, or some subsets of the real numbers. Up to the present section of this work, however, even though they were available, there was no need for the extended discrete spaces X_n (the cubic lattice of reference 5 is essentially a finite X_n), that is, spaces which contain a constructable, unbounded, infinity of points. Furthermore, no computer is available which could first of all store the infinity of elements of the scattering matrix Σ associated with X_n , let alone invert the related matrix $I - M\Sigma$ as required in the formal solutions for the specific radiance vector N_+ on X_n .

The observations on the time-dependent character of the scattering-order decomposition, especially those which observed that at any finite stage j of the iterations in (5), there were at most a finite number of components of N_+ which were not zero, lead us to conclude that at any finite stage of the iterations only a finite portion of an extended space is active radiometrically. It follows that only a finite portion of the scattering matrix Σ of the extended space X_n need be used in the computation, and hence that a computer with only a finite memory will be needed.

This observation coupled with the one in reference 1 concerning the estimate of the difference $|N_+ - \sum_{j=0}^k N_+^j|$ (expressions (37), (38) reference 1) leads to an explicit means of carving a finite discrete space out of an extended space which, for all practical purposes, is equivalent to the extended space. Suppose we agree to stop the iteration procedure when the norm of the difference between N_+ and the truncated solution $\sum_{j=0}^k N_+^j$ is less than $\epsilon |N_+^0|$, $0 < \epsilon < 1$. Now if the space X_n is non-conservative with a γ -factor defined by $0 < \gamma = \lim_n \gamma_n < 1$ where γ_n is associated with an ascending sequence $\{X_n\}$ of discrete spaces such that $X_n \rightarrow X$ (the original continuous space) then we can estimate the number of iterations which yield a relative error less than ϵ . For, from (38) of reference 1,

this number is $[k]$, where

$$k = \frac{\ln [\epsilon(1-\gamma)]}{\ln \gamma} - 1, \quad (9)$$

and where $[k]$ means the smallest integer larger than k .

Thus we need only work with a subset of X which has a temporal radius (reference 8) equal at most to $[k]$, which then gives rise to a strictly finite discrete space X_n .

We now conclude the discussion of the Iteration Method with the development of two conservation theorems which are of intrinsic interest in discrete-space theory and supply valuable numerical checks in the computation procedure.

TWO CONSERVATION THEOREMS

Conservation theorems in the present theory are those which may be deduced from the local interaction principle with the explicit help of the local conservation property (equation (11), reference 1). We exhibit two conservation theorems on a general X_n which are perhaps the most important from both a theoretical and computational point of view. These are the local and global divergence theorems. They are precise analogs of those in continuous case, and follow immediately from the local interaction principle. In order to set the stage for their derivation, we introduce the following definitions.

If $N_-(x_i)$ is the local field radiance vector at $x_i \in X_n$ and $\{N_{ki} : 1 \leq k \leq n\}$ the set of its components, then in analogy to the definition,

$$h(x) = \int_{\Xi} N(x, \xi) d\Omega(\xi)$$

of scalar irradiance in the continuous theory, we define:

$$h_i = \sum_{k=1}^n N_{ki} \quad , \quad 1 \leq i \leq n . \quad (10)$$

Now in the continuous theory the scalar irradiance function is related to the radiant density function μ (radiant energy per unit volume) by:

$$h = \mu v,$$

where v is the speed of light. In the present general space

X_n we have set $v=1$. Hence in (10), we have $\mu_i = h_i$ (numerically). In analogy to the continuous case, we set

$$U(X'_n) = \sum_{x_i \in X'_n} \mu_i, \quad (11)$$

i. e., $U(X'_n)$ is the radiant energy content of the subset $X'_n \subseteq X_n$ obtained by summing over the radiant densities associated with each element x_i of X'_n .

Finally, if $N_{ij}: 1 \leq j \leq n$ are now the components of the local specific radiance vector $N_x(x_i)$ at x_i , then we set:

$$h_{*i} = \sum_{j=1}^n N_{ij}, \quad 1 \leq j \leq n, \quad (12)$$

which is clearly analogous to

$$h_{*}(x) = \int_{\equiv} N_{*}(x, \xi) d\Omega(\xi)$$

in the continuous case, where N_{*} is the path function.

Now starting with the local interaction principle:

$$N_{ij} = \sum_{k=1}^n N_{ki} \Sigma(x_i; \xi_{ki}; \xi_{ij}) + \sum_{k=1}^{\Omega_i} N_{ki}^{\circ} \Sigma^{\circ}(x_i; \xi_{ki}^{\circ}; \xi_{ij}),$$

and summing over all j , we have:

$$\begin{aligned} h_{*i} &= \sum_{k=1}^n N_{ki} \left[\sum_{j=1}^n \Sigma(x_i; \xi_{ki}; \xi_{ij}) \right] \\ &+ \sum_{k=1}^{\Omega_i} N_{ki}^{\circ} \left[\sum_{j=1}^n \Sigma^{\circ}(x_i; \xi_{ki}^{\circ}; \xi_{ij}) \right] \\ &= \sum_{k=1}^n N_{ki} S(x_i, \xi_{ki}) + \sum_{k=1}^{\Omega_i} N_{ki}^{\circ} S^{\circ}(x_i, \xi_{ki}^{\circ}), \end{aligned}$$

still following the notation of reference 1. Now if χ_n is isotropic, so that S and S^o do not depend on the direction of incidence ξ_{ki} or ξ_{ki}^o , respectively, the preceding relation may be written:

$$h_{*i} = h_i S(x_i) + h_i^o S^o(x_i) \quad (13)$$

which, in the event that $h_i^o = 0$, the scalar irradiance induced by N_{ki}^o at point x_i , reduces to the useful equation involving h_{*i} , h_i , and $S(x_i)$:

$$\boxed{\frac{h_{*i}}{h_i} = S(x_i)} \quad (14)$$

It may be of interest to compare (14) with its continuous counterpart in reference 9, which states that

$$\frac{h_*(x)}{h(x)} = \Delta(x)$$

where Δ is the volume total scattering function in the continuous context.

Local And Global Divergence Relations

The first conservation theorem is obtained by assuming that X_n is locally isotropic and source-free at x_i and making use of the preceding definitions, and (11) of reference 1. The result is the local divergence relation:

$$\text{I.} \quad h_{*i} - h_i = -A(x_i) h_i, \quad (15)$$

where $A(x_i)$ is the ξ -independent value of the local absorption function at $x_i \in X_n$. This is the discrete counterpart to the continuous-space divergence relation:

$$\nabla \cdot H(x) = -a(x)h(x),$$

The global divergence relation follows from assuming X_n to be isotropic, homogeneous and source-free over some subset X'_n of X_n . From this it follows that, by summing (15) over X'_n and making use of (11),

$$\text{II.} \quad \bar{P}(x'_n, -) = A U(x'_n), \quad (16)$$

where A is the constant value of $A(\cdot, \cdot)$ on $X'_n \times \Xi$, $U(X'_n)$ is the radiant energy content of X'_n , and

$$\bar{p}(x'_n, -) = \sum_{x_i \in X'_n} (h_i - h_{*i}) \quad (17)$$

is the net inward flux to X'_n . The reader may wish to compare this with the continuous-space counterpart in reference 10:

$$\bar{p}(s, -) = a v U(M).$$

This completes the present discussion of the Iteration Method of solving the point source problem.

PART B

THE CATEGORICAL ANALYSIS METHOD

GENERAL OUTLINE OF THE CATEGORICAL ANALYSIS METHOD

There are twelve preparation stages in the solution of the point source problem in an arbitrary discrete space X_n using the Categorical Analysis Method. We now explicitly outline the procedures required to realize each of these stages. The first three stages form the conceptual core of the Categorical Analysis Method and therefore will be developed in minute detail. The remaining nine stages are simply the higher-dimensional analogs of the stages developed in reference 6 for the plane-source problem and are mainly mechanical in nature. For this reason, these latter nine stages are not developed in detail in the present work.

The first stage of the present method is devoted to the derivation of the $\hat{\Psi}$ -operator equation from the basic solution (2). This $\hat{\Psi}$ -operator equation is a three-dimensional analog of the one-dimensional $\hat{\Psi}$ -operator equation of the plane-source method of reference 6. However, since it plays a crucial role in the present method, it will be well to give its formulation in detail.

The second stage of the method defines the required analysis of the imbedding cubic lattice (the monobloc) into its component parts: monobloc into monolayers, monolayers into monolines, monolines into points of X_n . The monobloc, monolayer, and monoline are the catagories of X_n .

The third stage develops the three sets of principles of invariance which govern the various R and T operators associated with each of the three categories. The remaining nine stages are devoted to the methodical repetition of the basic themes of references 5 and 6: representation of the global and local Ψ -operators in terms of the monobloc R and T operators; representation of the monobloc R and T operators in terms of monolayer operators; representation of monolayer R and T operators in terms of monoline operators; representation of monoline R and T operators in terms of the given A and Σ functions on X_n ; and finally, the synthesis of the categorical representations into a compact solution formula.

STAGE ONE: THE Ψ -OPERATOR

Introduction of the Global and Local Ψ -Operators

The starting point for the derivation of the Ψ -operator equation is equation (2), the general representation of the $1 \times n^2$ specific radiance vector N_+ in terms of the $n^2 \times n^2$ matrix of the general discrete optical medium (Φ_n, N_+, A, Σ) , the $1 \times (\sum_{i=1}^n \Delta_i)$ source vector N^o , and the $(\sum_{i=1}^n \Delta_i) \times n^2$

source scattering matrix Σ^0 .

In order to give a concrete instance of the Ψ -operator and also to tie in the results of the present work with those of the earlier papers of this series, we specifically adopt the twenty-six component local direction space and the radiance restriction conventions of reference 5. It follows that, for the remainder of this work, N_+ and N^0 are $1 \times 26n$ vectors, and that Σ^0 and Σ are $26n \times 26n$ matrices, where $n = (2c+1)^2(b-a+1)$. (See the section above, entitled: "The Associated Discrete Setting.") Consequently, the matrix

$$\Psi = \Sigma^0 [I - M\Sigma]^{-1} \quad (18)$$

defined in (2) is a $26n \times 26n$ matrix.

Our immediate task is to reduce the operator equation (2):

$$N_+ = N^0 \Psi \quad (19)$$

to one which relates the 1×26 specific radiance vector $N(x_j)$ associated with point x_j in X_n to the 1×26 source radiance

vector $N^o(x_i)$ irradiating an arbitrary point x_i in X_n , assuming no other sources are present in X_n . The final result of the reduction will then be the basic operator equation:

$$N(x_j) = N^o(x_i) \Psi(x_i; x_j), \quad (20)$$

where $\Psi(x_i; x_j)$ is a 26×26 matrix, the requisite Ψ -operator, and where x_i , and x_j are ordered triples of real numbers denoting points in $X_n \subset E_3$.

A special form of this operator plays an important role in the following theory, and is defined by setting $x_i = x_j$. The resultant operator $\Psi(x_i; x_i)$ is called the local Ψ -operator. Occasionally, when it is required to explicitly distinguish between the two, we call $\Psi(x_i; x_j)$ the global Ψ -operator whenever $x_i \neq x_j$.

Linearity of the global Ψ -operator implies that, for an arbitrary set of sources on X_n at points $x_i, 1 \leq i \leq n$, we may write:

$$N(x_j) = \sum_{i=1}^n N^o(x_i) \Psi(x_i; x_j), \quad (21)$$

Derivation of the Basic Operator Equation

The derivation of (20) from (19) consists simply of a pair of matrix extraction operations. To see this, recall^{*} that the $1 \times 26n$ vectors N_+ and N^o may be expressed in general as an ordered set of n vectors each of twenty-six components:

$$N_+ = [N(x_1), \dots, N(x_j), \dots, N(x_n)], \quad (22)$$

$$N^o = [N^o(x_1), \dots, N^o(x_i), \dots, N^o(x_n)]. \quad (23)$$

In order to extract the 1×26 vectors $N(x_j)$ and $N^o(x_i)$ from N_+ and N^o we introduce the general $26n \times 26$ contracting matrix $C(x_k)$ of the form:

$$C(x_k) = \begin{pmatrix} 0 \\ \mathbf{I} \\ 0 \end{pmatrix}, \quad (24)$$

* Reference 1, pp. 30, 31; note that for simplicity here we have dropped the $+$ subscripts on the component vectors of N_+ ..

where I is a 26 x 26 identity matrix beginning at row number $26(k-1) + 1$ of $C(x_k)$. The remaining entries in $C(x_k)$ are occupied by the 0 scalar. It follows that

$$N(x_j) = N_+ C(x_j) , \quad (25)$$

$$N^o(x_i) = N^o C(x_i) , \quad (26)$$

The contracting matrix $C(x_k)$ is a generalized form of the contracting matrix C' used in reference 5. (See Equation (24) of that reference.)

The present source hypothesis (namely that only the single arbitrary point $x_i \in X_n$ is irradiated by a nonzero radiance vector) is analytically described by:

$$N^o = [0, \dots, 0, N^o(x_i), 0, \dots, 0] , \quad (27)$$

where we have set the 1 x 26 vectors $N^o(x_k) = 0$, for $k \neq i$. It follows from this particular form of N^o , that we may write:

$$N^o \Psi = N^o C(x_i) C^T(x_i) \Psi , \quad (28)$$

where $C^T(x_i)$ is the transpose of $C(x_i)$. Hence, from (19):

$$N_+ = N^0 C(x_i) C^T(x_i) \Psi. \quad (29)$$

Postmultiplying each side of (29) by the contracting matrix $C(x_j)$:

$$N_+ C(x_j) = N^0 C(x_i) C^T(x_i) \Psi C(x_j).$$

This now reduces to (20) by virtue of (25), (26) and the following definition of $\Psi(x_i; x_j)$:

$$\Psi(x_i; x_j) \equiv C^T(x_i) \Psi C(x_j) \quad (30)$$

This completes the derivation of the required basic Ψ -operator equation (20).

First Decomposition of the Global Ψ -Operator

Following the pattern set in reference 6, we now decompose $\Psi(x_i; x_j)$ into a 3 x 2 block matrix. The decomposition is induced by a specific decomposition of $N(x_j)$ and $N^o(x_j)$ into subvectors associated with the partition: $\{ \Xi'_+, \Xi_0, \Xi_- \}$ of the twenty-six component local direction space Ξ' adopted in the present study. See reference 5 for a complete description of Ξ' and this partition.

Thus we first write:

$$N(x_j) = [N_+(x_j), N_-(x_j)] \quad (31)$$

where $N_+(x_j)$ now stands for the 17-component radiance vector associated with Ξ_+ , and $N_-(x_j)$ the 9-component radiance vector associated with Ξ_- .

Furthermore:

$$N_+(x_j) = [N'_+(x_j), N_0(x_j)], \quad (32)$$

where $N_+^1(x_j)$ is the 9-component radiance vector associated with Ξ_+^1 , and $N_o(x_j)$ is the 8-component radiance vector associated with Ξ_o .

A similar partition of $N^o(x_i)$ into subvectors is possible:

$$N^o(x_i) = [N_+^o(x_i)', N_o^o(x_i), N_-^o(x_i)], \quad (33)$$

where the subvectors, from left to right, have 9, 8, and 9 components, and are associated with Ξ_+^1 , Ξ_o and Ξ_- , respectively. It turns out that we require the partition (31) of $N(x_j)$, and the partition (33) of $N^o(x_i)$.

Hence (20) becomes

$$[N_+(x_j), N_-(x_j)] = [N_+^o(x_i)', N_o^o(x_i), N_-^o(x_i)] \Psi(x_i; x_j),$$

which induces the following cleavage of the matrix representation of the global Ψ -operator:

$$\Psi(x_i; x_j) = \begin{matrix} (9) & \left(\Psi_{++}(x_i; x_j) & \Psi_{+-}(x_i; x_j) \right) \\ (8) & \left(\Psi_{0+}(x_i; x_j) & \Psi_{0-}(x_i; x_j) \right) \\ (9) & \left(\Psi_{-+}(x_i; x_j) & \Psi_{--}(x_i; x_j) \right) \end{matrix} \quad (34)$$

(17) (9)

The dimensions of the block matrices are fully defined by those of the subvectors of $N(x_j)$ and $N^0(x_i)$. For example, $\Psi_{0+}(x_i; x_j)$ is an 8 x 17 matrix, and $\Psi_{+-}(x_i; x_j)$ is a 9 x 9 matrix. The dimensions are indicated on the side and bottom of (34). Equation (34) is the required first decomposition of the global Ψ -operator; it begins to show the power inherent in the Categorical Analysis Method: the matrices are dimensionally independent of the size of X_n . It will turn out that 9 x 9 is the upper bound on the sizes of matrices required to be inverted by this method applied to any X_n .

STAGE TWO: CATEGORICAL ANALYSIS OF THE MONOBLOC

The z -Oriented Monobloc

In the section above on the formulation and solution of the general problem, the location space X of the given continuous optical medium was imbedded in a certain parallelepiped. The associated discrete space X_n was formed from this parallelepiped by the set-formation rule:

$$X_n = \{ (x, y, z) : |x|, |y| \leq c, a \leq z \leq b; x, y, z, a, b, c, \text{ integers} \}. \quad (35)$$

X_n is the imbedding cubic lattice, or monobloc for short. When defined in this way with an obvious favoritism shown for the z -axis, we will say that X_n is z -oriented. The two other possibilities, namely an x and a y -oriented monobloc may do just as well for our present purposes. However, we adopt the z -oriented monobloc in order to maintain contact with the notations established earlier in the series and in other works which use the z -axis as the axis along which depth or altitude is customarily measured in real media.

The Monolayers and Monolines

A monolayer in X_n at level z is a subset $X_n(z)$ of X_n defined by

$$X_n(z) = \{(x', y', z') \in X_n : z' = z\}. \quad (36)$$

A monoline in X_n at levels y and z is a subset $X_n(y, z)$ of X_n defined by

$$X_n(y, z) = \{(x', y', z') \in X_n : y' = y, z' = z\}. \quad (37)$$

The generic term covering the notion of monobloc, monolayer and monoline will henceforth be category. A graphical representation of typical categories is shown in Figure 3.

A monolayer at level z is evidently the set union of all monolines at levels y with fixed z :

$$X_n(z) = \bigcup_{|y| \leq c} X_n(y, z),$$

and X_n itself is the set union of all monolayers in X_n :

$$X_n = \bigcup_{z=a}^b X_n(z) .$$

Hence X_n is viewable as the set union of all monolines in X_n :

$$X_n = \bigcup_{z=a}^b \left[\bigcup_{|y| \leq c} X_n(y, z) \right] .$$

A monoline, of course, is simply a form of linear lattice, the general theory for which was worked out in reference 4. It will be possible to characterize the response of a monolayer in terms of the responses of its monoline subsets making use of the principles of invariance on a linear lattice and plane lattice. The response of the entire monobloc, as noted earlier, would then be some suitable superposition of the responses of its monolayer subsets. The form of this superposition is governed basically by the principles of invariance on the monobloc. We now give the detailed statements of the principles of invariance on the various categories.

STAGE THREE: PRINCIPLES OF INVARIANCE ON THE CATEGORIES

The principle of invariant imbedding (Equation (47), reference 2) forms the starting point of derivation of all the forms of the principles of invariance on the present categories. The principles have already been derived in detail several times in simpler contexts. For example, reference 4 gives a derivation of the principles for a linear lattice, and reference 5 gives their derivation for the extended cubic lattice considered there. The pattern of derivation is now so well established that we may safely leave the details of the present derivations to the reader. Thus we will merely state the requisite forms of the two main principles (I, II) for each of the three categories. The remaining two principles (III, IV) in each of the three categorical cases are derived in the usual manner (See, e.g., references 4, 5).

Principles for the Monobloc

Let (x, y, z) denote a point in X_n : let $a-1 \leq z' < z < z'' \leq b+1$, then

$$\begin{aligned} \text{I} \quad N_+(x, y, z) &= N'_+(\bar{x}, \bar{y}, z'') T(\bar{x}, x; \bar{y}, y; z'', z) \\ &+ N_-(\bar{x}, \bar{y}, z-1) R(\bar{x}, x; \bar{y}, y; z, z'') \end{aligned} \quad (38)$$

$$\begin{aligned} \text{II} \quad N_-(x, y, z) &= N_-(\bar{x}, \bar{y}, z') T(\bar{x}, x; \bar{y}, y; z', z) \\ &+ N'_+(\bar{x}, \bar{y}, z+1) R(\bar{x}, x; \bar{y}, y; z, z') \end{aligned} \quad (39)$$

These principles and their operators apply to the set of monolayers in X_n beginning with $X_n(z'+1)$ and ending with $X_n(z''-1)$ (Figure 4). The following summation convention has been adopted: If f and g are real or matrix valued functions defined on X_n , then we write:

$$\sum_{|x| \leq c+1} \sum_{|y| \leq c+1} f(x, y, z_1) g(x, y, z_2) \equiv f(\bar{x}, \bar{y}, z_1) g(\bar{x}, \bar{y}, z_2) \quad (40)$$

Thus the bars over a symbol for a real number denote the summation over the range of that real variable as defined in the formulae (40). Thus the x and y variables range from $-(c+1)$ to $(c+1)$, and z lies in the interval $[a-1, b+1]$. The reason for allowing the range of x , y , and z to exceed the geometric limits of the monobloc by one unit all around the monobloc is to allow a compact formulation of the boundary source conditions within the principles. The precedent for this convention has been established in references 4 and 5, and a study of the convention in these simpler contexts will clarify its usage here.

The R and T operators in (38) and (39) above are the standard operators for the monobloc. Because of the z -orientation convention and the adoption of the twenty-six component local direction space Ξ' , the R and T operators are matrices of the same general dimensions as those in reference 5. For example the matrix $T(x', x; y', y; z', z)$ is the standard transmittance operator for downward flux, since $z' < z$; hence it is a 9×9 matrix. Further $R(x', x; y', y; z', z)$ is the standard reflectance operator for downward flux with dimensions 9×17 . A similar set of observations hold for the other operators R and T for the upward flux.

Finally, the zero and identity operator properties hold for the present R and T operators just as in the extended cubic lattice case. (See equations (16) - (20), reference 5.)

The physical processes summarized by these standard R and T operators are exactly analogous to their extended cubic lattice counterparts developed in reference 5. As an example, consider the 9×9 matrix $T(x', x; y', y; z', z''-1)$, $z' < z''$. Because of the Z -oriented monobloc convention, this matrix refers to the transmittance of the subset (a slab) of X_n consisting of the $z'' - z' - 1$ monolayers beginning with $X_n(z'+1)$ and ending with $X_n(z''-1)$. (See Figure 4.) Thus if this subset could be isolated from the remaining points of X_n , and if it is irradiated by the radiance distribution emitted by point (x', y', z') in $X_n(z')$ i.e., by the 1×9 vector $N_-(x', y', z')$, then the resultant transmitted radiance distribution in the form of the 1×9 vector $N_-(x, y, z''-1)$ emitted by a point on $X_n(z''-1)$ (the lowest monolayer of the slab) would be:

$$N_-(x, y, z''-1) = N_-(x', y', z') T(x', x; y', y; z', z''-1).$$

In general, if all of the points on the upper boundary of this isolated slab are irradiated by downward flux from the points of $X_n(z')$, the resultant transmitted radiance $N_-(x, y, z''-1)$ at $(x, y, z''-1) \in X_n(z''-1)$ would be

$$N_-(x, y, z''-1) = N_-(\bar{x}, \bar{y}, z') T(\bar{x}, \alpha; \bar{y}, \gamma; z', z''-1).$$

This is the first term in (39).

Finally, to complete this example, we observe that the present subslab, consisting of the layers $X_n(z'+1), \dots, X_n(z''-1)$, cannot in practice be isolated from its complement in X_n . Hence the complete response $N_-(x, y, z''-1)$ must take cognizance of the flux impinging on the slab which is emitted by the points in monolayer $X_n(z'')$; this flux is represented by the 1×9 vector $N'_+(x', y', z'')$. When this additional irradiation is taken into account, the result is the main principle of invariance II for the slab comprised of the monolayers $X_n(z'+1)$ to $X_n(z''-1)$.

These remarks and observations should help the reader to form his own additional observations on the remaining operators and principles of invariance above, and to understand the material presented below.

Principles for a Monolayer

To prepare the way for the principles of invariance associated with the monolayer $X_n(z)$, $a \leq z \leq b$, we must introduce several new radiance vectors. These are sub-vectors of the 1×8 vector $N_0(x, y, z)$ defined above (see Equation (32)). Specifically, we require the following partition of $N_0(x, y, z)$:

$$N_0(x, y, z) = [N_{0+}(x, y, z), N_{0-}(x, y, z)], \quad (41)$$

where $N'_{0+}(x, y, z)$ is a 1×5 vector and $N_{0-}(x, y, z)$ is a 1×3 vector. Further, we partition $N_{0+}(x, y, z)$ thus:

$$N_{0+}(x, y, z) = [N'_{0+}(x, y, z), N_{00}(x, y, z)], \quad (42)$$

where $N'_{0+}(x, y, z)$ is a 1×3 vector and $N_{00}(x, y, z)$ is a 1×2 vector. The geometric significance of these vectors is clarified by Figure 5. Their exact definitions may be phrased by means of the elements of $\Xi_0 \subset \Xi'$:

Recall that (reference 5) the subset Ξ_0 of elements of Ξ' consists of the members:

$$\Xi_0 = \{ \xi_{13}, \xi_{12}, \xi_{11}, \xi_{10}, \xi_{14}, \xi_{15}, \xi_{16}, \xi_{17} \}.$$

Then redefine^{*} the old N_0 -vector as follows:

$$N_0(x_i) = [N(x_i, \xi_{13}), N(x_i, \xi_{12}), \dots, N(x_i, \xi_{17})], \quad (43)$$

where the order of components is governed by the order of appearance of the ξ_i in Ξ_0 above, and where $x_i \in X_n(z)$,

* This new arrangement of the components of $N_0(x_i)$ is adopted in order to have (41) and (42) partition into the indicated subvectors. Thus N_0 and N_0+ correspond exactly to N and $N+$ as given in (5) and (6) of reference 5. A simple permutation matrix relates the old and new arrangements of N_0 .

Therefore:

$$N_{0+}(x_i) = [N(x_i, \xi_{13}), N(x_i, \xi_{12}), N(x_i, \xi_{11}), N(x_i, \xi_{10}), N(x_i, \xi_{14})] \quad (44)$$

$$N'_{0+}(x_i) = [N(x_i, \xi_{13}), N(x_i, \xi_{12}), N(x_i, \xi_{11})] \quad (45)$$

$$N_{0-}(x_i) = [N(x_i, \xi_{15}), N(x_i, \xi_{16}), N(x_i, \xi_{17})] \quad (46)$$

$$N_{00}(x_i) = [N(x_i, \xi_{10}), N(x_i, \xi_{14})] \quad (47)$$

Now within $X_n(\bar{z})$, the vectors $N_{0+}(x_i)$ and $N'_{0+}(x_i)$ represent the upward radiance vectors; $N_{0-}(x_i)$ is the downward radiance vector, and $N_{00}(x_i)$ is the horizontal radiance vector. This choice of terminology for $N_{0\pm}(x_i)$ and $N_{00}(x_i)$ attempts to follow the general terminology established earlier in the series, and serves to assign intuitively meaningful names that can be used in following the text. With these notational conventions, the requisite principles of invariance for the monolayer $X_n(\bar{z})$, $a \leq \bar{z} \leq b$, may now be written down. Let $-(c+1) \leq y' < y < y'' \leq (c+1)$, and let $(x, y, \bar{z}) \in X_n$, then:

$$\text{I}_{\bar{z}} \quad N_{0+}(x, y, \bar{z}) = N'_{0+}(\bar{x}, y'', \bar{z}) T_{\bar{z}}(\bar{x}, x; y'', y) \quad (48)$$

$$+ N_{0-}(\bar{x}, y-1, \bar{z}) R_{\bar{z}}(\bar{x}, x; y, y'')$$

$$\text{II}_{\bar{z}} \quad N_{0-}(x, y, \bar{z}) = N_{0-}(\bar{x}, y', \bar{z}) T_{\bar{z}}(\bar{x}, x; y', y) \quad (49)$$

$$+ N'_{0+}(\bar{x}, y+1, \bar{z}) R_{\bar{z}}(\bar{x}, x; y, y')$$

The bars over the α -symbols denote the summation convention adopted above. In practice, when no confusion can result, it is permissible to drop all references to the subset $\bar{\Xi}_0$ of $\bar{\Xi}'$ and to the fixed \bar{z} associated with the monolayer under consideration. Thus, (48) and (49) may be written more succinctly as:

$$\text{I}'_{\bar{z}} \quad N_+(x, y) = N'_+(\bar{x}, y'') T(\bar{x}, x; y'', y) \quad (48')$$

$$+ N_-(\bar{x}, y-1) R(\bar{x}, x; y, y'')$$

$$\text{II}'_{\bar{z}} \quad N_-(x, y) = N_-(\bar{x}, y') T(\bar{x}, x; y', y) \quad (49')$$

$$+ N'_+(\bar{x}, y+1) R(\bar{x}, x; y, y')$$

$$-(c+1) \leq y' < y < y'' \leq (c+1).$$

However, when the R and T operators and radiance distributions in adjacent monolayers $X_n(z \pm 1)$ to $X_n(z)$ are to be explicitly considered, it will be necessary to use the set (48), (49). The principles (48) - (49) and their operators apply to the subset of the monolayer $X_n(z)$ beginning with the monoline $X_n(y'+1, z)$ and ending with $X_n(y''-1, z)$.

For any $X_n(z)$, $a \leq z \leq b$, the operator $T_z(x'; x; y', y)$ where $y' < y$, is the standard transmittance operator for downward flux within the monolayer $X_n(z)$, and is a 3×3 matrix. The operator $R_z(x'; x; y', y)$ where $y' < y$ is the standard reflectance operator for downward flux on within monolayer $X_n(z)$, and is a 3×5 matrix. Similar definitions and dimensions hold for the operators R_z and T_z associated with upward flux. The discussion of the physical significance of these operators is analogous in all respects to that for the standard monobloc operators. For example $R_z(x'; x; y', y)$, $y' < y$ is the standard reflectance operator for the set of $(y - y')$ monolines in $X_n(z)$ beginning with $X_n(y', z)$ and ending with $X_n(y-1, z)$.

Principles for a Monoline

Let $X_n(y, z)$ be a monoline in X_n . Then at any point $(x, y, z) \in X_n(y, z)$ we have two directions of flow within $X_n(y, z)$: in the direction ξ_{10} the upward direction (+), and ξ_{14} the downward direction (-). (See Figure 6.) Using the components of $N_{00}(x_i)$ in (47) which describe the specific radiances at x_i in $X_n(y, z)$, the requisite forms of the two main principles of invariance for the monoline are:

$$I_{yz} N(x, y, z, \xi_{10}) = N(x'', y, z, \xi_{10}) T_{yz}(x'', x) + N(x-1, y, z, \xi_{14}) R_{yz}(x, x'') \quad (50)$$

$$II_{yz} N(x, y, z, \xi_{14}) = N(x', y, z, \xi_{14}) T_{yz}(x', x) + N(x+1, y, z, \xi_{10}) R_{yz}(x, x') \quad (51)$$

When no confusion with other monolines is possible, and the monoline $X_n(y, z)$ under study is clearly identified, all specific references to y , z , ξ_{10} and ξ_{14} may be dropped, so that the principles may take on the relatively compact forms:

$$I'_{yz} \quad N(x, +) = N(x'', +) T(x'', z) + N(x-, +) R(x, x'') \quad (50')$$

$$II'_{yz} \quad N(x, -) = N(x', -) T(x', z) + N(x+, +) R(x, x') \quad (51')$$

where we have set

$$N(x, y, z, \xi_{10}) \equiv N(x, +) \quad (52)$$

$$N(x, y, z, \xi_{14}) \equiv N(x, -) \quad (53)$$

These abbreviated forms for the principles of invariance on $X_n(y, z)$ are now precisely the forms of the principles of invariance for a general linear lattice developed in reference 4. Observe, however, that if the interaction of two adjacent monolines is under consideration, it will be necessary to use the

complete statements (50), (51). The principles (50), (51) and their R and T operators apply to the subset of the monoline $X_n(y, z)$ beginning with the point $(x'+1, y, z)$ and ending with $(x''-1, y, z)$.

On the Use of the Categorical Statements of the
Principles of Invariance

The principles (38), (39) for the monobloc, (48), (49) for a monolayer, and (50), (51) for a monoline are now used to find the associated complete reflectance and transmittance operators from the standard operators for each category. The general outline for this procedure has been given in reference 6 in sufficient detail so that the reader may readily construct the presently required complete operators after the appropriate sections of reference 6 have been studied (See Equations (10) - (16), reference 6.) These complete operators will be required in STAGE FOUR to TWELVE below.

STAGES FOUR TO TWELVE IN THE CATEGORICAL ANALYSIS METHOD

We now outline the procedures that must be followed to realize the last nine stages of the present method. Since every one of these stages has a lower-dimensional analog whose details have been fully covered in preceding papers of this series, there is no longer any need to enter into any further detailed discussion of the stages except to cite the appropriate reference and perhaps to make an occasional remark and establish notation.

STAGE FOUR: REPRESENTATION OF THE GLOBAL Ψ -OPERATOR
IN TERMS OF THE LOCAL Ψ -OPERATOR AND THE COMPLETE
OPERATORS FOR A MONOBLOC

References: (a) Equations (17) - (32), reference 6.

(b) TABLE I, reference 6.

Remarks: (a) The summation convention now replaces the straight matrix multiplication of reference 6.

Notation: (a) Set $a=1$, $b=n$. Complete transmittance operator for upward flux: $\mathcal{T}(x', x; y', y; z'', z, 0)$
Complete reflectance operator for upward flux:

$Q(x', x; y', y, z, z', 0)$; complete reflectance operator for downward flux:
 $Q(x', x; y', y; z, z'', n+1)$; complete transmittance operator for
 downward flux: $\mathcal{T}(x', x; y', y; z', z, n+1)$.

STAGE FIVE: COMPLETE MONOBLOC OPERATORS FROM STANDARD
 MONOBLOC OPERATORS

References: (a) Equations (53) - (66), reference 6,
 (b) TABLE III, reference 6.

STAGE SIX: STANDARD MONOBLOC OPERATORS FROM COMPLETE
 MONOLAYER OPERATORS

References: (a) Equations (45) - (48), reference 5.
 (b) Equations (49) - (78), reference 5.
 (c) Equations (33) - (41), reference 6.

Remarks: (a) The present counterparts to the recurrence relations
 of reference 5 must be developed. These show how the
 Standard Operators for a monobloc are built up start-
 ing from the complete monolayer operator. The model
 for this occurs in equations (45) - (48), reference 5.

- (b) The next step is to link together the complete monolayer operators for adjacent monolayers. The appropriate S -operators for this task must be defined. The model for this procedure occurs in equations (49) - (78), reference 5, and (33) - (41), reference 6.
- (c) Observe how the radiant flux generally fans out from one point (x, y, z) on $X_n(z)$ and strikes the point $(x, y, z \pm 1)$ and its eight neighbors on $X_n(z \pm 1)$, respectively. Thus the reflectance response (say) of a monolayer $X_n(z)$ to the radiance vector $N'_+(x, y, z+1)$ (say) must be synthesized by an appropriate linear combination of (generally) nine complete R operators for $X_n(z)$.

STAGE SEVEN: THE LOCAL Ψ -OPERATOR IN TERMS OF
COMPLETE MONOLAYER OPERATORS

- Reference: (a) Equations (42) - (52), reference 6.
(b) TABLE II, reference 6.

STAGE EIGHT: COMPLETE MONOLAYER OPERATORS FROM
STANDARD MONOLAYER OPERATORS

References: Repeat Stage Five above for the monolayer context.

STAGE NINE: STANDARD MONOLAYER OPERATORS FROM
COMPLETE MONOLINE OPERATORS

References: Repeat Stage Six above for the Monolayer Context.

STAGE TEN: COMPLETE MONOLINE OPERATORS FROM
STANDARD MONOLINE OPERATORS

References: Repeat Stage Eight for the monoline context.

STAGE ELEVEN: DETERMINATION OF THE STANDARD
MONOLINE OPERATORS

References: (a) Equations (32) - (39), reference 4.
(b) Equations (83) - (95), reference 5.

Remark: (a) Observe that these operators are determined by
direct appeal to the Σ and A functions on X_b .

- (b) The Σ and A functions are determined for the present cubic lattice in accordance with the procedure set forth in equations (83) - (95) of reference 5.

STAGE TWELVE: SYNTHESIS OF SOLUTION

Reference: (a) SUMMARY, p 45, reference 6.

Remark: (a) The method of tabular representations established in reference 6 should be adopted in the present method.

THIS COMPLETES THE DISCUSSION OF THE CATEGORICAL ANALYSIS METHOD.

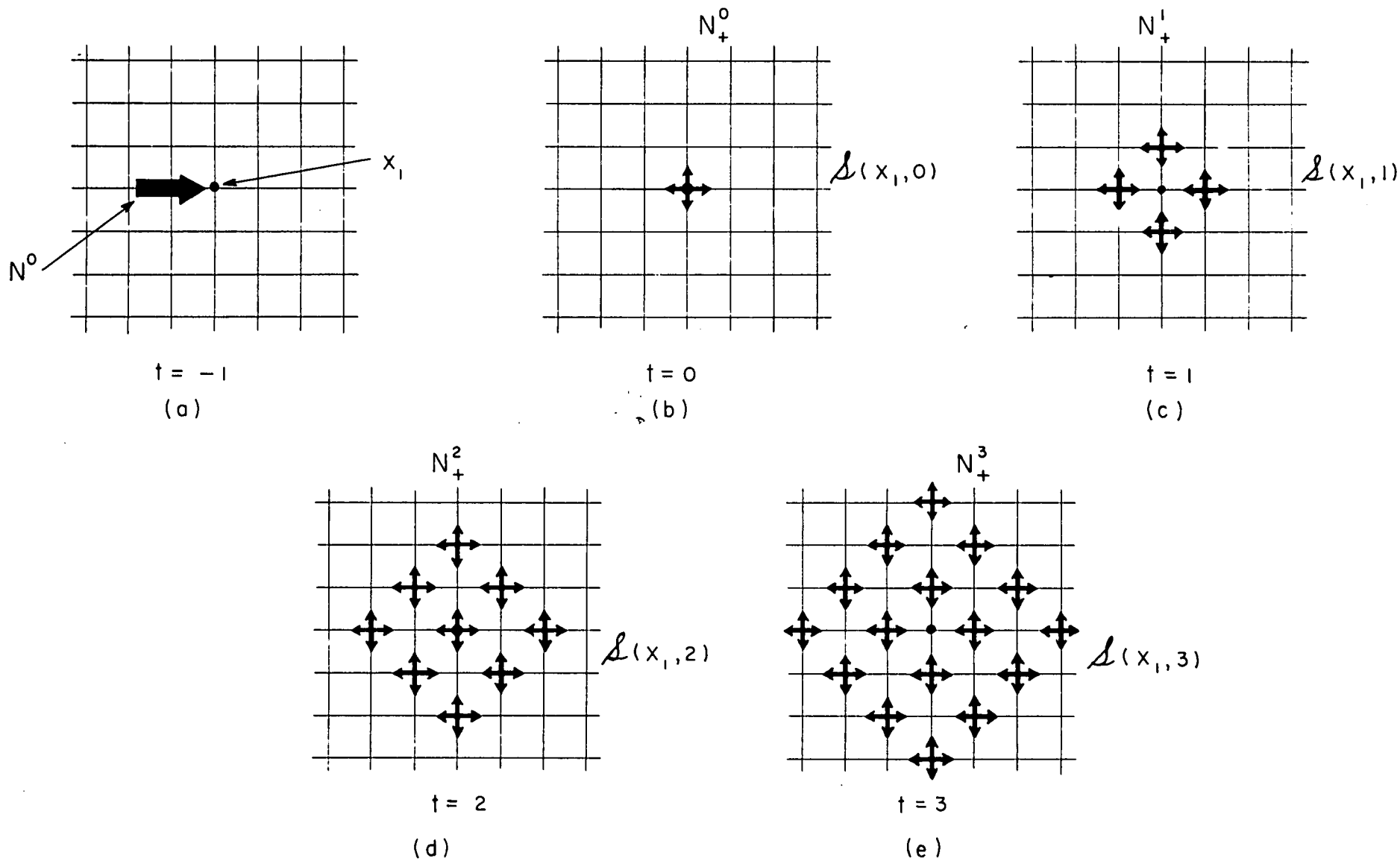
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CHARACTERISTIC SPHEROIDS OF A POINT SOURCE IN A SQUARE GRID



CHARACTERISTIC ELLIPSOIDS OF A SOURCE-RECEIVER PAIR
IN A SQUARE GRID

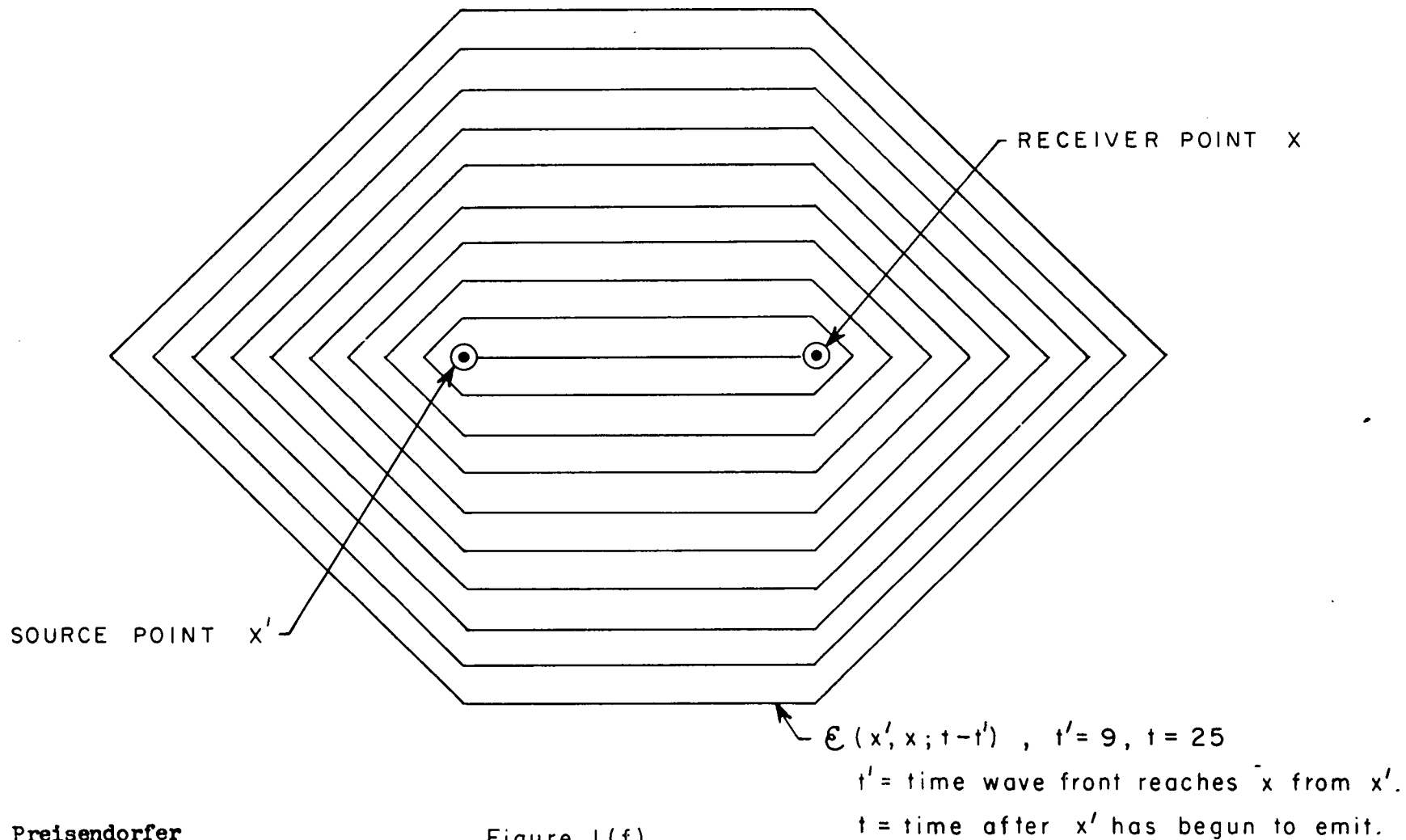
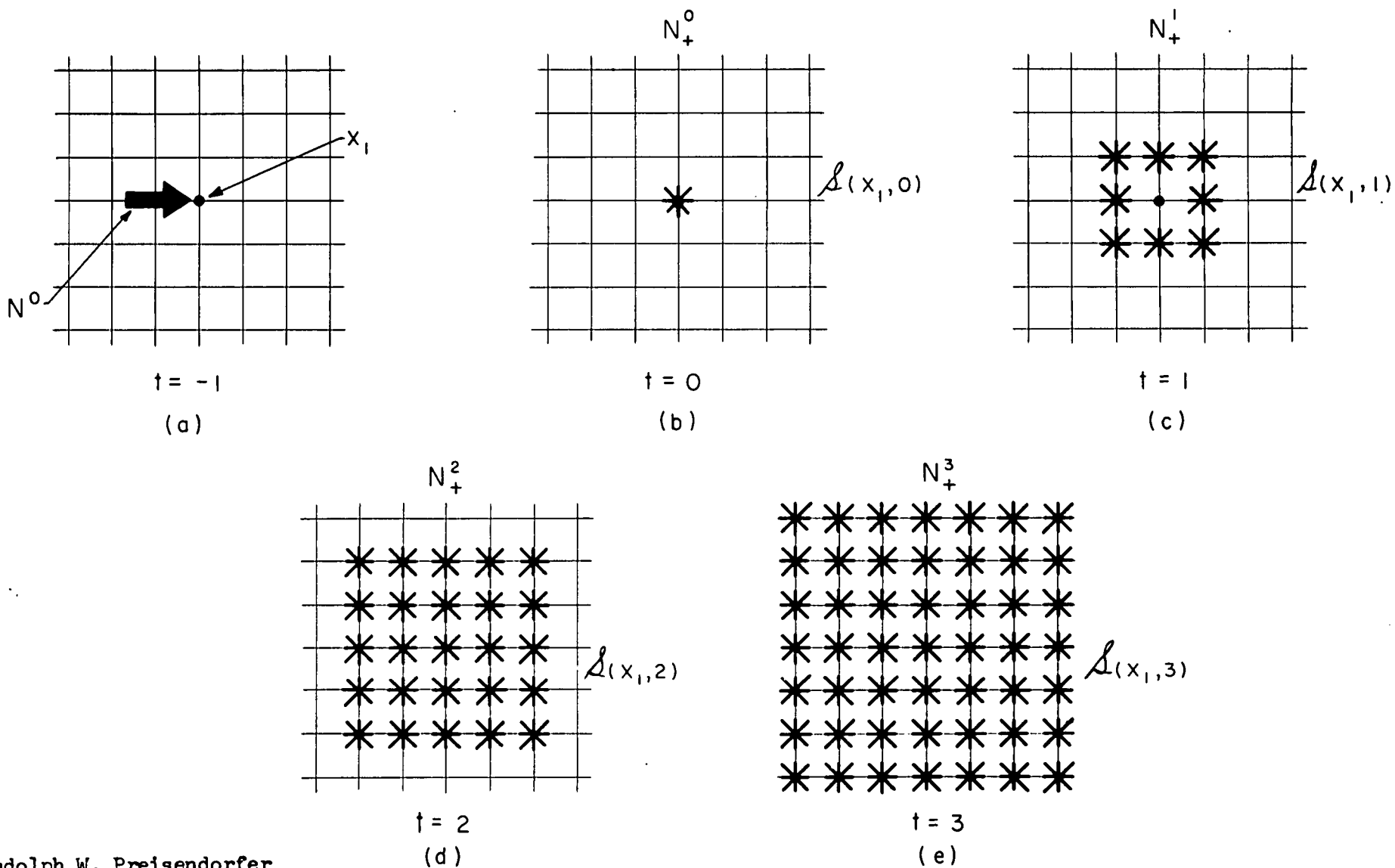


Figure 1(f)

CHARACTERISTIC SPHEROIDS OF A POINT SOURCE IN A SQUARE LATTICE



CHARACTERISTIC ELLIPSOIDS OF A SOURCE-RECEIVER PAIR
IN A SQUARE LATTICE

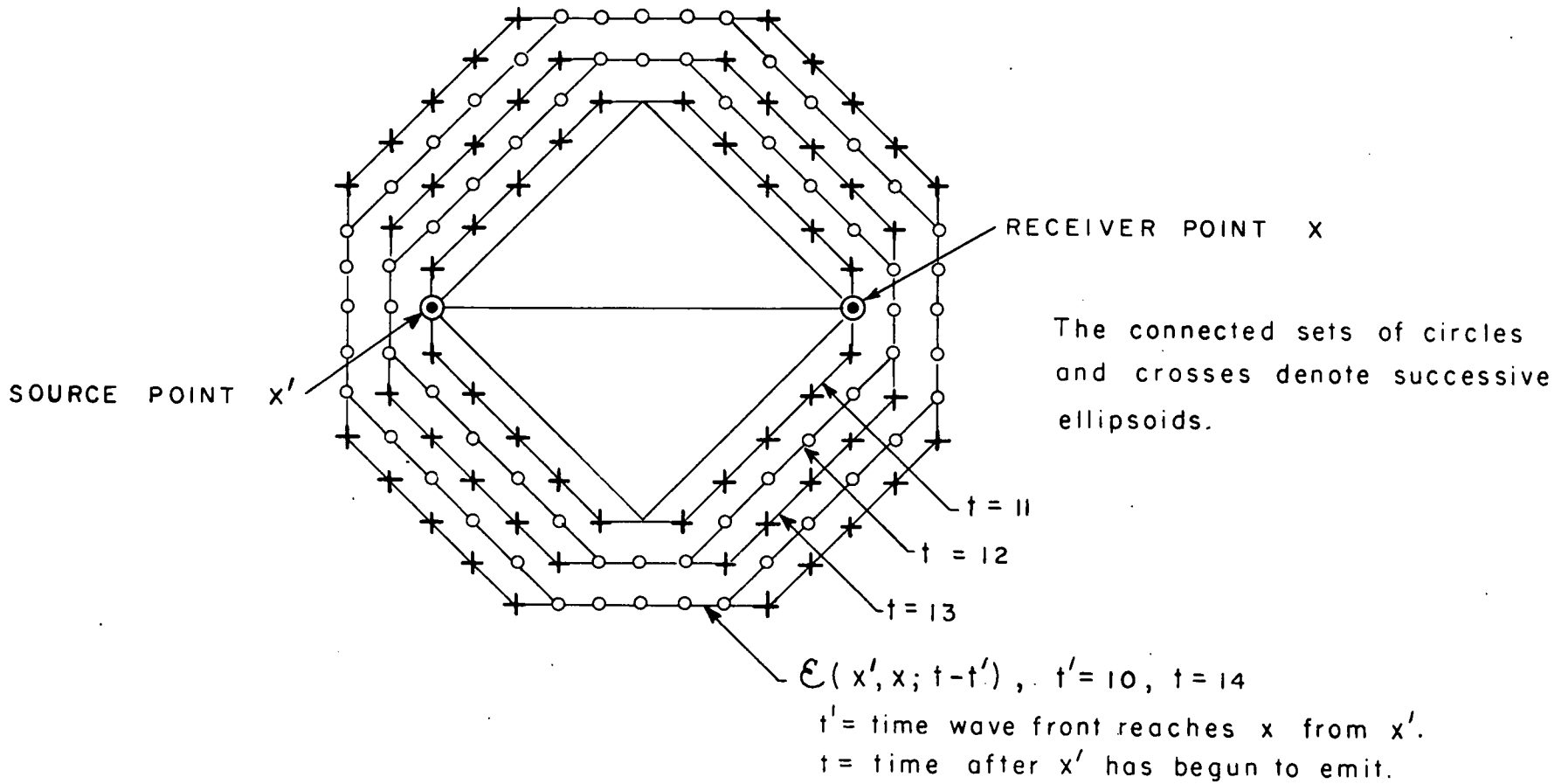
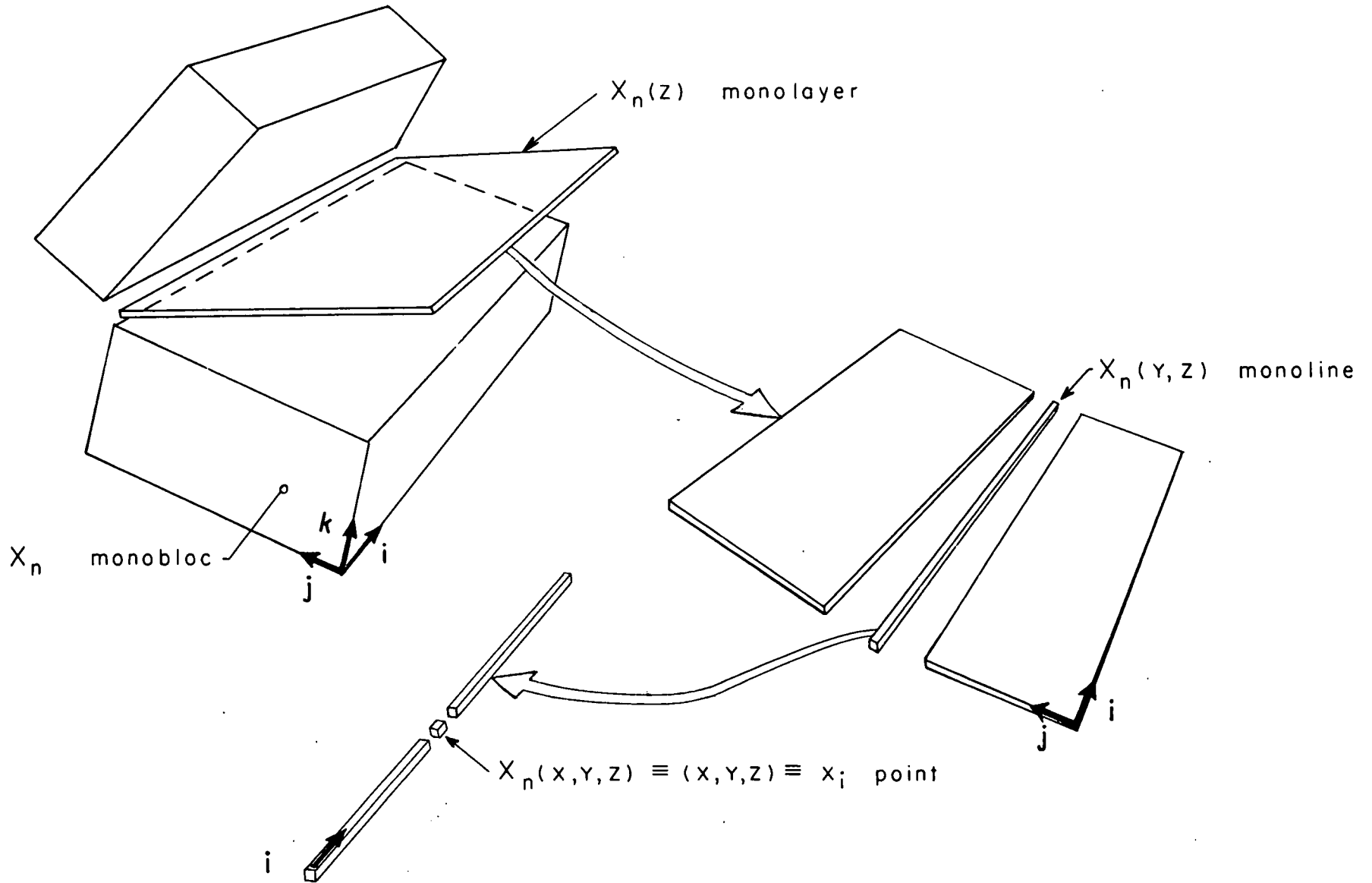


Figure 2(f)

THE CATEGORIES



GEOMETRIC SETTING FOR THE PRINCIPLES OF INVARIANCE
ON THE MONOBLOC

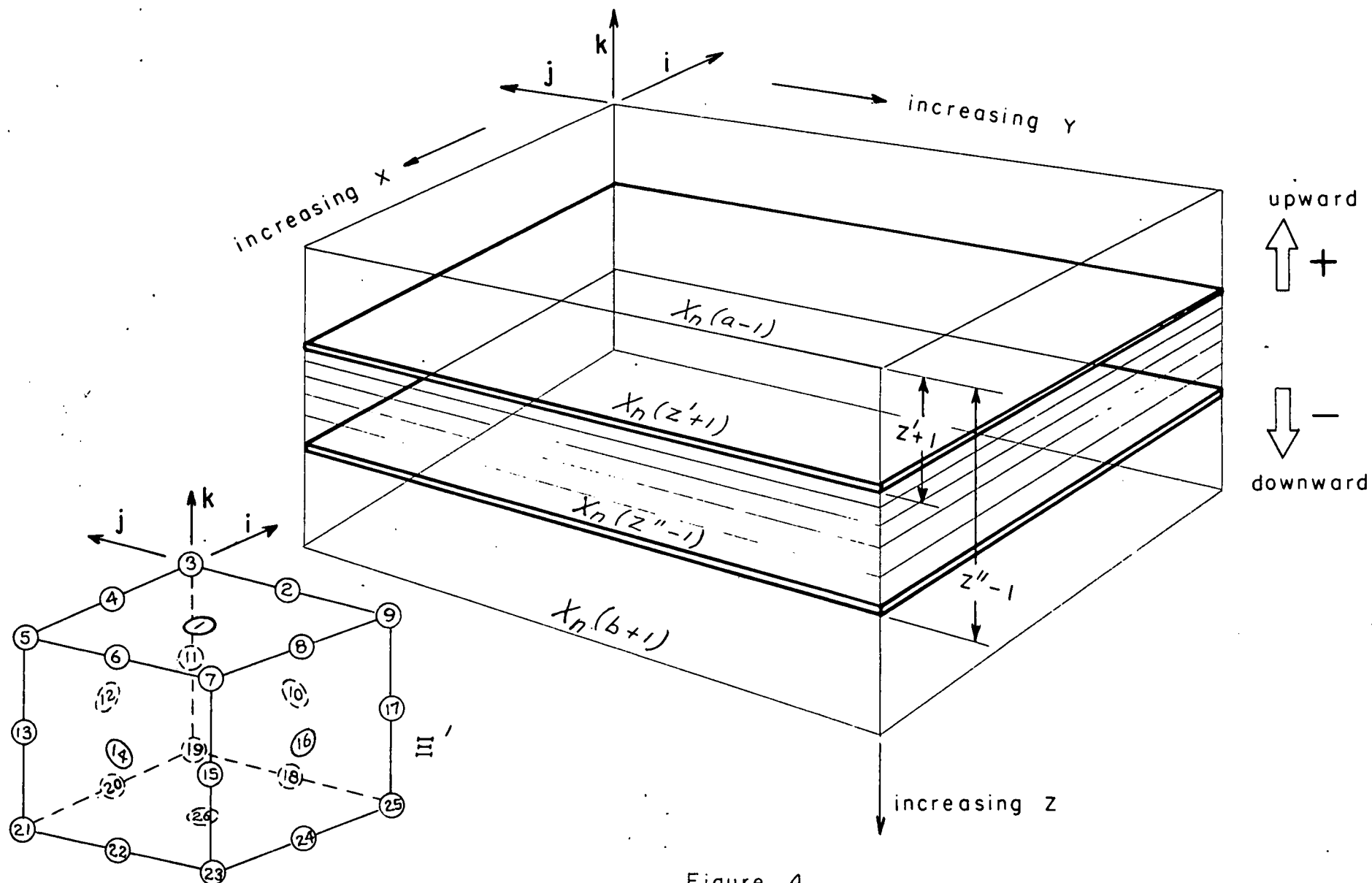


Figure 4

GEOMETRIC SETTING FOR THE PRINCIPLES OF INVARIANCE
IN A MONOLAYER

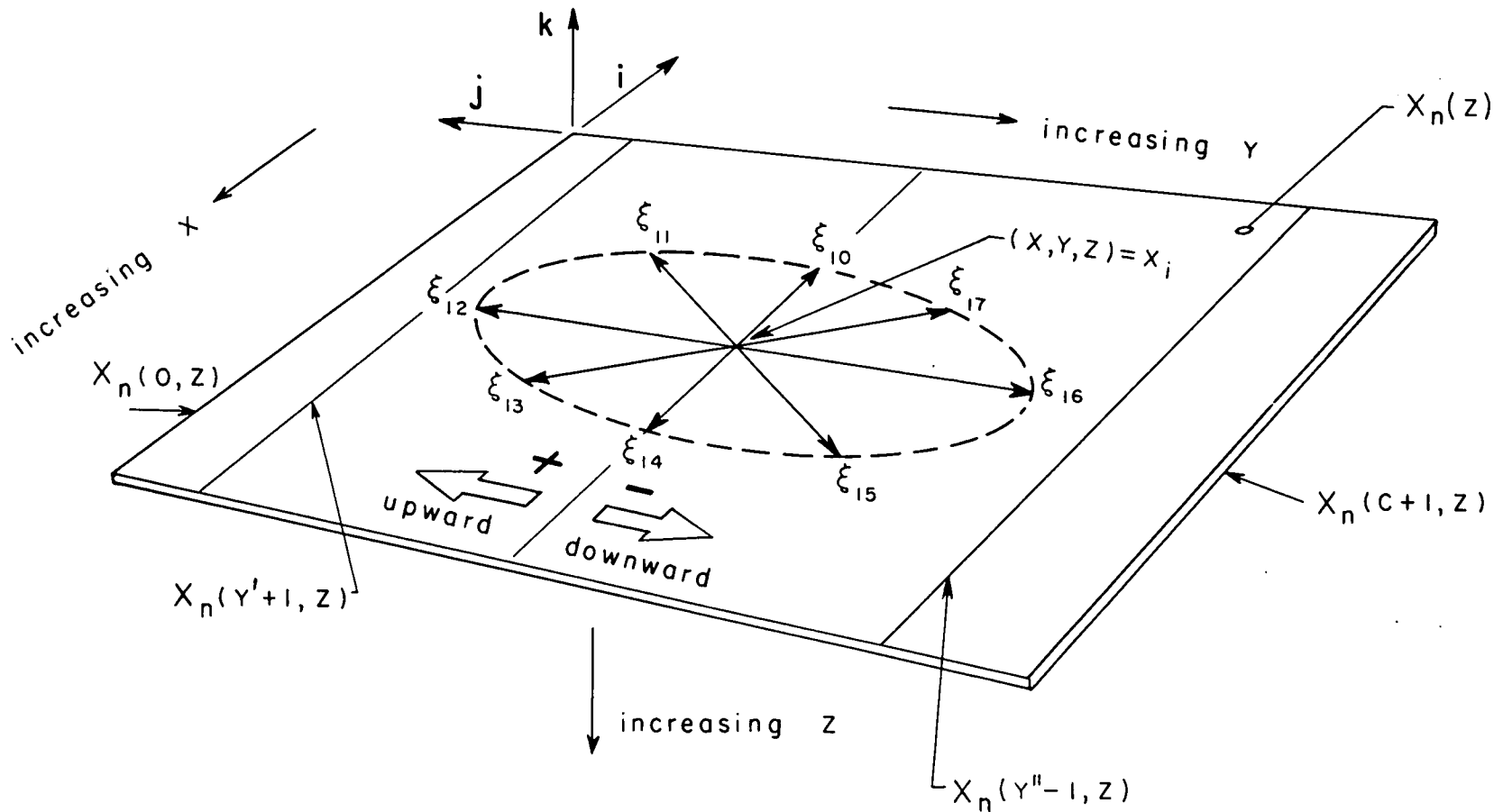


Figure 5

GEOMETRIC SETTING FOR THE PRINCIPLES OF INVARIANCE
IN A MONOLINE

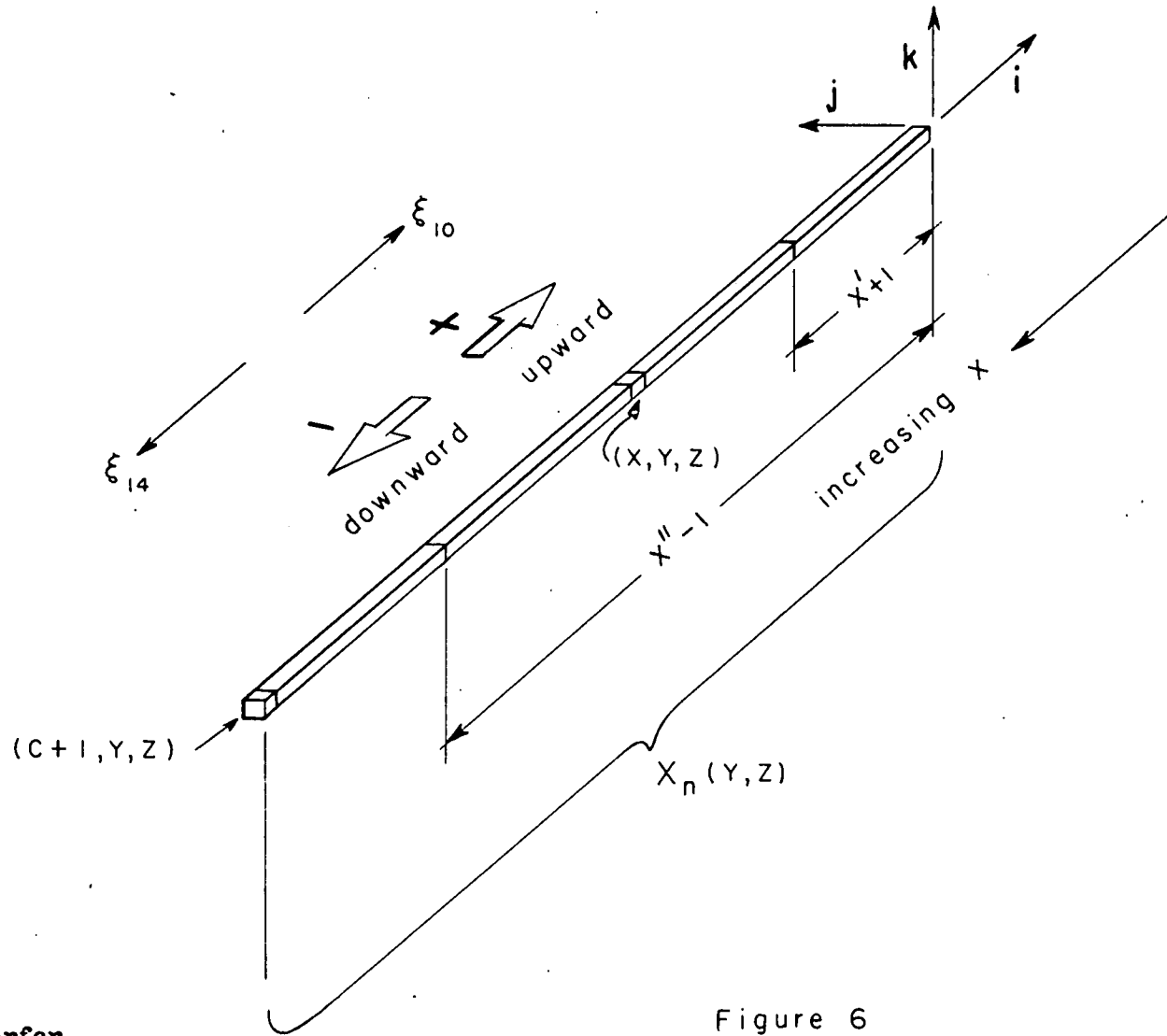


Figure 6