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THE PRINCIPLES OF INVARIANCE FOR DIRECTLY OBSERVABLE IRRADIANCES IN PLANE-PARALLEL MEDIA

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INTRODUCTION

In this work we complete the outline of the basic theory of directly observable irradiances[1] in a plane-parallel medium X by formally deriving the principles of invariance which govern these irradiances and their associated reflectance and transmittance functions over arbitrary subslabs of X. The derivations hold for a completely general plane-parallel medium setting: The medium is arbitrarily stratified with reflecting boundaries, and has a given set of external and internal source conditions.

Some additional results of the present work are: (a) The derivation of the differential equations governing the reflectance and transmittance functions associated with arbitrary subslabs in X for the up-and-downward irradiances, thereby completing the two-flow counterpart to the general radiance setting established earlier[2]; (b) a theorem which establishes the equivalence of the differential equations governing the values of the observable reflectance function $R(z,-)$ for the medium and the values of the reflectance function $R(a,b)$

of an arbitrary subslab of the medium; (c) simple approximate formulae for the empirical determination of the reflectance and transmittance factors associated with an arbitrary subslab of X between depths a and b, $a \leq b$; and finally; (d) a brief discussion which establishes, by means of examples, some connections between the present exact R and T functions and their classical counterparts in the various Schuster two-flow models of the light field.

In this way we complete the basic outlines of the exact theory of the two-flow analysis of the light field begun in reference 3, wherein the exact equations of the two-flow analysis were derived from the basic equation of transfer. The net result is a chain of deductions starting from the basic radiance transfer equation, through the equations governing the observable up-and downwelling irradiances and the differential equations governing their associated reflectance and transmittance functions in an arbitrary real plane-parallel medium.

FORMAL DERIVATION OF THE INVARIANT IMBEDDING RELATION

In an earlier study[4] we established a useful line of approach that may be adopted in the derivation of the principles of invariance for any member of the general class of one-parameter carrier spaces in radiative and neutron transport theory. Thus, in order to establish the principles of invariance in any one-parameter optical medium (such as the present plane-parallel medium) it is necessary only to establish the invariant imbedding relation for that medium. The remaining steps in the explicit derivation of the statements of the principles of invariance are given in general in reference 4.

The present section is devoted to the formal derivation of the invariant imbedding relation governing the pair of irradiance functions $H(\cdot, \pm)$ within an arbitrary subslab $[a, b]$ in an arbitrary plane parallel optical medium over the interval $[0, z_1]$. It must be emphasized that the derivation is <u>formal</u> in the sense that it gives all the manipulative steps that must be covered in the passage from the equations of transfer governing $H(\cdot, \pm)$ to the resultant statements of the principles of invariance. The regularity conditions (such as continuity, differentiability, etc.) on the physical scattering and absorption functions are not given. The emphasis in the present paper is primarily on physical ideas and concepts; hence regularity considerations, which are primarily of mathematical interest, rightfully assume a subordinate role in the following discussions.

Local Forms of the Principles of Invariance

The starting point of the derivations is the set of exact equations[3] governing the up-and downwelling irradiance functions $H(\cdot,\pm)$ over the general depth subinterval $[x,z] \subset [a,b]$ (Figure 1) in the optical medium whose location space extends over the depth interval $[0, z_1]$:

$$\mp \frac{dH(z,\pm)}{dz} = H(z,\pm)t(z,\pm) + H(z,\mp)r(z,\mp) + \omega_\eta(z,\pm). \tag{1}$$

Here we have set

$$\begin{aligned} t(z,\pm) &= -[a(z,\pm) + b(z,\pm)] \\ &= f(z,\pm) - \alpha(z,\pm), \end{aligned} \tag{2}$$

$$r(z,\pm) = b(z,\pm), \tag{3}$$

where the functions $a(\cdot,\pm)$, $b(\cdot,\pm)$, $f(\cdot,\pm)$ and $\alpha(\cdot,\pm)$ on $[0, z_1]$ are completely defined in reference 3. The functions $\omega_\eta(\cdot,\pm)$ are the general emission functions on $[0, z_1]$ which implicitly include the boundary conditions on X . The functions $t(\cdot,\pm)$ on $[0, z_1]$ are the local transmittance functions for the upwelling (+) and downwelling (−) streams (c f. the corresponding functions for the radiance context in reference 2). The functions $r(\cdot,\pm)$ on are the local reflectance functions for their respective streams.

The set of equations (1), written in the indicated form, are called the local forms of the principles of invariance. The reason for the present choice of terminology will become clear after an examination of the (global) statements of the principles of invariance (see Equations (22), (23) below).

The Green's Function Approach

Our present line of approach to the invariant imbedding relation will be through the Green's function associated with the set of equations (1). Toward this end, we introduce the operators

$$D_{\pm} \equiv \frac{d_{\mp} - t(z, \pm)}{r(z, \mp)} , \tag{4}$$

where we have set

$$d_{\mp} = \mp \frac{d}{dz} , \tag{5}$$

To assure unambiguous use of (4), let f be any differentiable function on $[0, z_1]$, then $D_{\pm} f$ is defined by:

$$D_{\pm} f(z) = \frac{d_{\mp} f(z) - t(z, \pm) f(z)}{r(z, \mp)} . \tag{6}$$

With the adoption of the operators $D_{\pm}$, the set (1) may be written (read upper signs together, lower signs together):

$$D_{\pm} H(z, \pm) = H(z, \mp) + \frac{w_{\eta}(z, \pm)}{h(z, \mp)} \tag{7}$$

By means of the set (7), it follows that, for example:

$$D_{+}\left[D_{-} H(z,-)\right] = D_{+} H(z,+) + D_{+}\left[\frac{w_{\eta}(z,-)}{h(z,+)}\right]$$

$$= H(z,-) + \frac{w_{\eta}(z,+)}{h(z,-)} + D_{+}\left[\frac{w_{\eta}(z,-)}{h(z,+)}\right].$$

Hence the set (1) may be reduced to the simple and compact operator form:

$$\boxed{L_{\pm} H(z,\mp) + \Phi_{\pm}(z,\mp) = 0} \tag{8}$$

where

$$L_{\pm} = D_{\pm} D_{\mp} - 1 \tag{9}$$

and

$$-\Phi_{\pm}(z,\mp) = \frac{w_{\eta}(z,\pm)}{h(z,\mp)} + D_{\pm}\left[\frac{w_{\eta}(z,\mp)}{h(z,\pm)}\right]. \tag{10}$$

Now assuming the existence of a Green's function $G_{\pm}(x,\cdot;\cdot,z)$ on $[x,z]\times[x,z]$ for each of the operators $L_{\pm}$ in (8) (which should certainly exist in all physical situations) we may represent the functions $H(\cdot,\pm)$ on $[x,z]$ as:

$$\boxed{\begin{aligned} H(y,\pm) &= \int_x^z \Phi_{\mp}(z',\pm)\, G_{\mp}(x,z';y;z)\, dz' \\ a \le x &\le y \le z \le b \end{aligned}} \tag{11}$$

The Invariant Imbedding Relation

To obtain the requisite form of the invariant imbedding relation we simply adopt a pair of general Dirac-delta source conditions at the arbitrary levels x and z. Thus, $\omega_\eta(\cdot,\pm)$ are chosen so that $\omega_\eta(\cdot,\pm) \equiv 0$ on (x,z), and in particular:

$$\Phi_+(y,-) = \Phi_-(y,+) = H(x,-)\,\delta(y-x) + H(z,+)\,\delta(y-z), \tag{12}$$

where $x \le y \le z$. It follows from (11), with the adopted forms in (12) that

$$\begin{aligned} H(y,-) &= H(x,-)\,G_+(x,x;y,z) \\ &\quad + H(z,+)\,G_+(x,z;y,z), \end{aligned} \tag{13}$$

and

$$H(y,+) = H(x,-)\,G_-(x,x;y,z) + H(z,+)\,G_-(x,z;y,z) \tag{14}$$

The special values of the Green's functions occurring in (13) and (14) define the requisite complete reflectance and transmittance functions over $[x, z]$. Thus, we set:

$$G_+(x,x;y,z) \equiv \mathcal{T}(x,y,z) \tag{15}$$

$$G_+(x,z;y,z) \equiv \mathcal{R}(z,y,x) \tag{16}$$

$$G_-(x,x;y,z) \equiv \mathcal{R}(x,y,z) \tag{17}$$

$$G_-(x,z;y,z) \equiv \mathcal{T}(z,y,x) \tag{18}$$

With these definitions, the pair of values $[H(y,+), H(y,-)]$ is related to the pair $[H(z,+), H(x,-)]$ by means of the linear operator $\mathcal{M}(x,y,z)$:

$$[H(y,+), H(y,-)] = [H(z,+), H(z,-)]\,\mathcal{M}(x,y,z) \tag{19}$$

where

$$\mathcal{M}(x,y,z) = \begin{pmatrix} \mathcal{T}(z,y,x) & \mathcal{R}(z,y,x) \\ \mathcal{R}(x,y,z) & \mathcal{T}(x,y,z) \end{pmatrix} \tag{20}$$

We have now reached the stage represented by Equation (1) in reference 4.

THE PRINCIPLES OF INVARIANCE

Following the methodology established in reference 4, we obtain from (19) the following two main statements of the principles of invariance over the arbitrary subslab $[x, z]$ of $[a,b] \subset [0, z_1]$ (Figure 1):

$$\boxed{\begin{aligned} &\mathrm{I.}\quad H(y,+) = H(z,+)\,T(z,y) + H(y,-)\,R(y,z) \\ &\mathrm{II.}\quad H(y,-) = H(x,-)\,T(x,y) + H(y,+)\,R(y,x) \end{aligned}} \tag{21}$$

Then the same principles for the particular slab $[a, b]$ are:

$$\text{I.} \quad H(y,+) = H(b,+)\,T(b,y) + H(y,-)\,R(y,b) \tag{22}$$

$$\text{II.} \quad H(y,-) = H(a,-)\,T(a,y) + H(y,+)\,R(y,a) \tag{23}$$

And in particular:

$$\text{III} \quad \begin{aligned} H(a,+) &= H(b,+)\,T(b,a) + H(a,-)\,R(a,b) \\ &= H(z,+)\,T(z,a) + H(a,-)\,R(a,z) \end{aligned} \tag{24}$$

$$\text{IV} \quad \begin{aligned} H(b,-) &= H(a,-)\,T(a,b) + H(b,+)\,R(b,a) \\ &= H(x,-)\,T(x,b) + H(b,+)\,R(b,x) \end{aligned} \tag{25}$$

The functions $T(\cdot,\cdot)$ and $R(\cdot,\cdot)$ defined on

$$[a,b] \times [a,b] \subset [0,z_1] \times [0,z_1]$$

are the <u>standard</u> <u>transmittance</u> and <u>reflectance</u> <u>functions</u> for the slab $[a,b] \subset [0, z_1]$.

FUNCTIONAL RELATIONS GOVERNING THE R AND T FUNCTIONS

The functional relations governing the standard R and T functions are obtained by applying suitable differentiation and limiting arguments to the set I - IV and the local forms of the principles of invariance (i.e., the set of equations (1)). Thus, to find the functional relation governing $R(a,b)$ associated with the slab $[a,b] \subset [0,z_1]$ let $H(a,-)$ be arbitrary and set $H(b,+) = 0$; then, differentiate principle I (equation (22)) with respect to y :

$$\frac{dH(y,+)}{dy} = \frac{dH(y,-)}{dy} R(y,b) + H(y,-) \frac{dR(y,b)}{dy} \tag{26}$$

Now let $y \to a$. Then from (1) (with $\omega_\eta(\cdot,\pm) \equiv 0$ on $[a,b]$)

$$\begin{aligned} \lim_{y\to a} \frac{dH(y,+)}{dy} &= \\ &= \lim_{y\to a} -[H(y,+)\,t(y,+) + H(y,-)\,t(y,-)] \\ &= [H(a,+)\,t(a,+) + H(a,-)\,t(a,-)] \end{aligned} \tag{27}$$

where the last equality is obtained by means of III (Equation (24)) once again using the present boundary condition: $H(b,+) = 0$. In a similar way we obtain:

$$\lim_{y \to a} \frac{dH(y,-)}{dy} =$$

$$= \lim_{y \to a} \left[H(y,-)\, t(y,-) + H(y,+)\, r(y,+) \right]$$

$$= H(a,-) \left[t(a,-) + R(a,b)\, r(y,+) \right], \tag{28}$$

Since

$$\lim_{y \to a} \frac{dR(y,b)}{dy} = \frac{\partial R(a,b)}{\partial a}$$

and $H(a,-)$ is arbitrary, the limit operation $\lim_{y \to a}$ applied to (26) yields:

$$\boxed{I'_- \quad -\frac{\partial R(a,b)}{\partial a} = r(a,-) + \left[t(a,-) + t(a,+) \right] R(a,b) + r(a,+) R^2(a,b)} \tag{29}$$

It is informative to compare this with statement I' of reference 2. The similarity is striking. Recall that the statement I' in reference 2 is an <u>operator</u> <u>statement</u>: the quantities appearing in that statement are <u>operators</u> and not functions as in (29) above. Hence the indicated <u>relative</u> <u>positions</u> of the operators in I' of reference 2 are absolutely essential; but in (29) -- since multiplication in the real number system is commutative -- the various factors can be rearranged at will within each term. Thus, if desired, (29) can be put in a form even more closely resembling its operator counterpart in I' of reference 2.

The essential pattern of derivation of the functional relations is now clear: it proceeds in precisely the manner established in reference 2. The only difference worth noting is that we are now working with real-valued functions, instead of function-valued operators. The results are:

$$\boxed{\begin{aligned} &\mathrm{II}' \quad \frac{\partial T(a,b)}{\partial b} = \left[t(b,-) + r(b,-)\, R(b,a) \right] T(a,b) && (30) \\ &\mathrm{III}' \quad \frac{\partial R(a,b)}{\partial b} = r(b,-)\, T(a,b)\, T(b,a) && (31) \\ &\mathrm{IV}' \quad -\frac{\partial T(a,b)}{\partial a} = \left[t(a,-) + r(a,+)\, R(a,b) \right] T(a,b) && (32) \end{aligned}}$$

The set (29) - (32) is associated with downwelling flux and applies to the boundary conditions: $H(a,-)$ arbitrary, $H(b,+) = 0$. Another set follows for upwelling flux and is found by repeating the above steps now for arbitrary $H(b,+)$, and $H(a,-) = 0$.

Observe that the system (29) and (32) supplies just enough relations so that one may, in principle, solve for the R and T functions for both the upwelling and downwelling streams. Thus Equation (29) may be used to determine $R(a,b)$. Then Equation (32) is used to find $T(a,b)$. Knowing $R(a,b)$ and $T(a,b)$ allows the determination of $T(b,a)$ by means of (31) and $R(b,a)$ by means of (30). In practical procedures leading to the determination of the pairs $R(a,b)$, $T(a,b)$ and $R(b,a)$, $T(b,a)$, it is best to establish the appropriate functional relations belonging specifically to the latter pair, the standard factors for the upwelling stream (see, e.g., references 5 and 6 where this procedure was followed for the discrete-space setting.)

One final observation should be made concerning the nature of the present R and T functions. As a preliminary to the observation, recall that R and T entities have been derived for the linear lattice and cubic lattice contexts in discrete optical media[5,6]. In each of the latter cases, these entities are essentially independent of the directional structure of the light field. In the present case, however, the R and T functions are virtually carved out of the living light field, so that a particular pair $R(a,b)$, $T(a,b)$, for example, is in a one to one correspondence with the observable radiance distributions at the fixed levels a and b . While the notation does not explicitly carry along this fact, it may be immediately verified by examining the definitions of the functions $a(\cdot,\pm)$ and $b(\cdot,\pm)$ in terms of which $t(\cdot,\pm)$ and $r(\cdot,\pm)$ are defined.

EQUIVALENCE THEOREM FOR REFLECTANCE EQUATIONS

In an earlier study[7] it was shown that the differential equation governing the <u>observable</u> <u>reflectance</u> <u>function</u> $R(\cdot,-)$ on $[0, z_1]$ defined by:

$$R(z,-) = \frac{H(z,+)}{H(z,-)}$$

at each depth z in $[0, z_1]$, was of the form:

$$-\frac{dR(z,-)}{dz} = b(z,-) - c(z)\,R(z,-) + b(z,+)\,R^2(z,-), \tag{33}$$

where

$$c(z) = a(z,-) + a(z,+) + b(z,-) + b(z,+)$$

Now let $[z, b]$ be an arbitrary variable subinterval of $[0, z_1]$ with b fixed, $0 \le z \le b \le z_1$. Then from a comparison of (33) with (29) and (2), we have the following remarkable

THEOREM: Let X be an arbitrarily stratified source-free plane-parallel optical medium over the depth interval $[0, z_1]$ with arbitrary boundary lighting conditions. Then over a common arbitrary variable subinterval $[z, b] \subset [0, z_1]$ the differential equations governing the observable reflectance function $R(\cdot, -)$ and the standard reflectance function $R(\cdot, b)$ associated with $[z, b]$ are identical.

This theorem, coupled with the results of reference 7, can lead to some interesting practical methods of evaluating the standard reflectance (and hence transmittance) functions for arbitrary subslabs in X. The discussion of these methods here would, however, constitute too great a digression from our present goals. We will be content for the present to derive two simple approximate rules of thumb which may be used to estimate the magnitudes of $R(a,b)$ and $T(a,b)$ for an arbitrary subinterval $[a, b]$ of $[0, z_1]$.

A RULE OF THUMB FOR EMPIRICAL ESTIMATES OF THE STANDARD REFLECTANCE AND TRANSMITTANCE FUNCTIONS

From the statements III and IV (Equations (34) and (35) of the principles of invariance we have, for every subinterval $[a,b]$ of $[0,z_1]$:

$$H(a,+) = H(b,+)\,T(b,a) + H(a,-)\,R(a,b)$$

$$H(b,-) = H(a,-)\,T(a,b) + H(b,+)\,R(b,a)$$

Now, the functions R and T on $[0,z_1]\times[0,z_1]$ generally possess polarity[5,6] as may be seen from general qualitative arguments[6], or directly from (29) - (32). That is, in general, $T(a,b) \neq T(b,a)$ and $R(a,b) \neq R(b,a)$. However, for some practical purposes, the general order of magnitude of $R(a,b)$ and $T(a,b)$ can be estimated by assuming that $R(a,b) = R(b,a)$, and $T(a,b) = T(b,a)$.
In this case, the preceding pair of equations can be solved for $R(a,b)$ and $T(a,b)$. The result is:

$$\boxed{\begin{aligned} R(a,b) &\cong \frac{H(a,-)H(a,+) - H(b,-)H(b,+)}{H^2(a,-) - H^2(b,+)} \qquad (34) \\[2ex] T(a,b) &\cong \frac{H(a,-)H(b,-) - H(a,+)H(b,+)}{H^2(a,-) - H^2(b,+)} \qquad (35) \end{aligned}}$$

How good are these rules? The larger the magnitude of the difference: $|a-b|$, the more accurate each is, since for fixed a , $a \leq b < \infty$ in an infinitely deep medium, we have, respectively from (34) and (35):

$$\lim_{|a-b| \to \infty} R(a,b) = R(a,-) \tag{36}$$

$$\lim_{|a-b| \to \infty} T(a,b) = 0 \tag{37}$$

On the other hand, for small values of $|a-b|$, a fixed, it can be shown from (24) that,

$$\lim_{|a-b| \to 0} \frac{R(a,b)}{|a-b|} = b(a) \tag{28}$$

exists if $b(a,+) = b(a,-) \equiv b(a)$ and $a(a,+) = a(a,-)$. Thus, the estimate (34) of $R(a,b)$ is accurate if the corresponding values of the backward scattering functions for each stream and the absorption functions for each stream are nearly equal at depth a. Finally, it is clear from (35) that, for fixed a ,

$$\lim_{|a-b| \to 0} T(a,b) = 1 , \tag{39}$$

as expected.

CONNECTIONS WITH THE CLASSICAL THEORY

The classical Schuster two-flow theory of the light field describes the irradiances in a boundryless, sourceless, isotropically scattering homogeneous slab over an interval $[0, z_\cdot]$ irradiated at the upper level $(z = 0)$ by a directionally uniform radiance distribution and with $H(z_\cdot, +) = 0$. The theory proceeds on the assumption that $b(z,-) = b(z,+) \equiv b^*$ and $a(z,-) = a(z,+) \equiv a^*$ (i.e. that the backward scattering and absorption functions for each stream are identical and have the constant starred values over the slab.) We can immediately deduce the values $R(0, z)$ and $T(0, z)$ associated with this slab, on the basis of the present general theory. To do this, we merely recall the statement of the equivalence theorem for reflectance equations proved above. This allows us to use the expression for $R(z_2,-)$ given in (7) of reference 7:

$$R(z_2,-) = \frac{R_q - R_\alpha\, C(z_1,-)\, \exp\left\{-\left[c^2 - 4\,b(-)b(+)\right]^{1/2}(z_2 - z_1)\right\}}{1 - C(z_1,-)\, \exp\left\{-\left[c^2 - 4\,b(-)b(+)\right]^{1/2}(z_2 - z_1)\right\}} \qquad (40)$$

Refer to reference 7 for definitions and notation. We need only observe that, under the present setting,

$$z_1 \longleftrightarrow 0$$

$$z_2 \longleftrightarrow z$$

$$R(z_1, -) = 0$$

$$R_\alpha R_q = 1$$

$$R_\alpha + R_q = c/b^*$$

and finally, that

$$[c^2 - 4b(-)b(+)]^{1/2} = 2[a^*(a^* + 2b^*)]^{1/2}$$
$$= 2k$$

where k is the diffuse absorption coefficient of the classical theory.[1]

Hence

$$R(0,z) = \frac{[1 - \exp\{-2kz\}]}{R_\alpha - R_q \exp\{-2kz\}}$$

$$= \frac{b^* \sinh kz}{(a^* + b^*)\sinh kz + k\cosh kz}, \qquad (41)$$

which is the usual form for the reflectance of a slab of depth z over an arbitrary interval $[0, z]$.

The remaining form for $T(0, z)$ can now be deduced immediately from relation IV' (Equation (32)), but the point of this section has essentially been made: the classical two-flow theory is an elementary special case of the present theory of directly observable quantities in real light fields.

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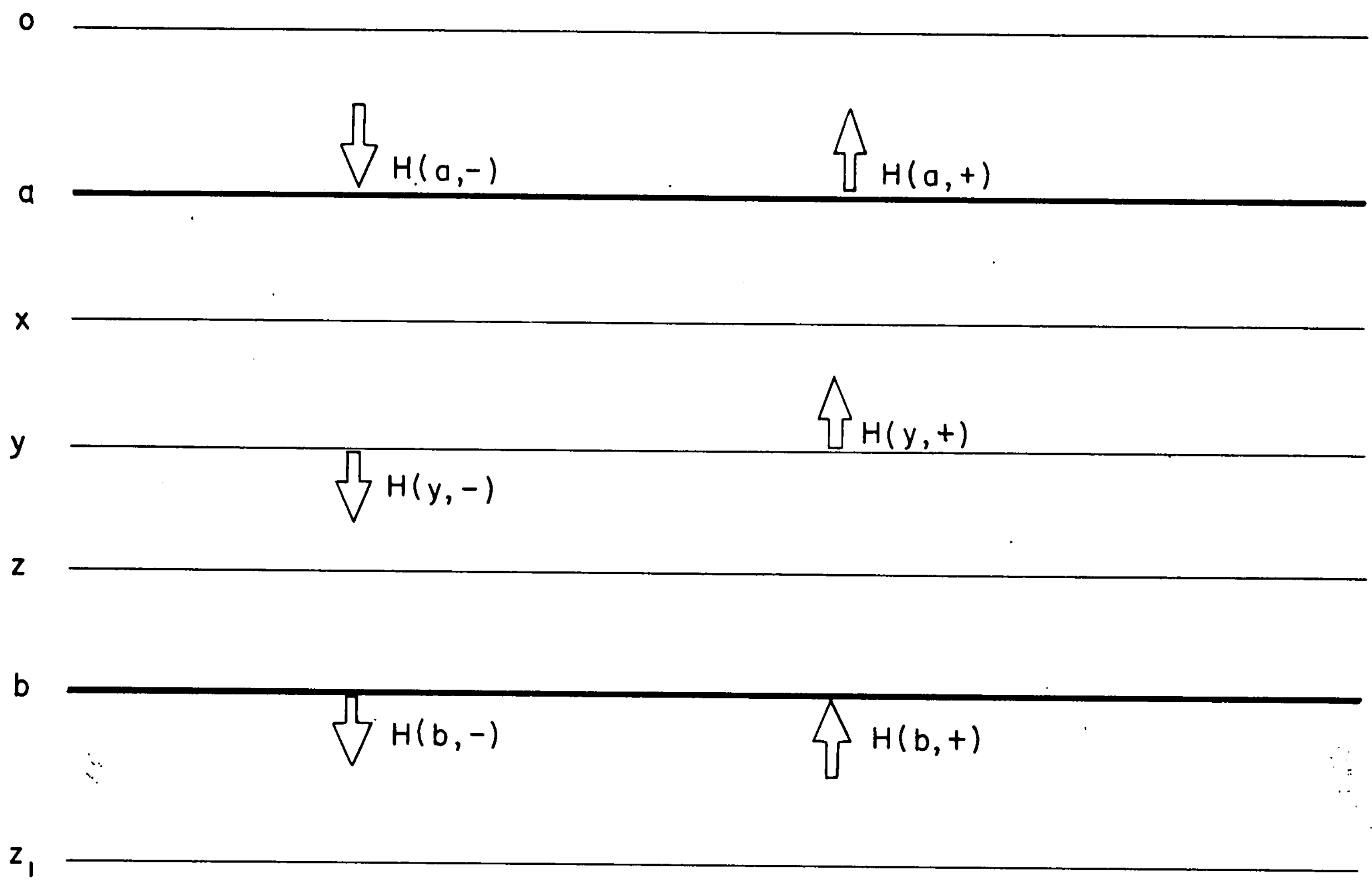

Figure 1

Rudolph W. Preisendorfer