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THE AXIOMATIC BASIS OF THE PRINCIPLES OF RADIATIVE TRANSFER THEORY

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The Axiomatic Basis of the Principles of Radiative Transfer Theory

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INTRODUCTION

Preliminary Remarks

In a recent study [1] the general theory of radiative transfer was given a rigorous mathematical foundation by the simple expedient of showing that the basic equation of the entire theory, namely the general equation of transfer for radiance, could be deduced from three basic structure axioms phrased in the language of modern measure theory. To show that the axiomatic formulation was not empty, the classical equation of transfer, as it is used in the usual Euclidean three-space applications of astrophysics and geophysics, was shown to be a special case of the general equation.

During the three years that have elapsed since the formulation of the foundations of radiative transfer theory, many practical and theoretical results--too numerous to summarize here--have been developed with an unprecedented precision and facility which is due principally to the insight into transfer phenomena which these foundations provide. Thus, as it stands today, the theory of radiative transfer possesses a firm substructure (the axioms) and a rich and relatively complete superstructure (the equation of transfer and its manifold theoretical and practical consequences).

The First Purpose of This Note

There remains, however, one important facet of the theory which has not yet been examined under the penetrating light of the axioms. This is the important and extremely useful set of principles originated and developed by Ambarzumian and Chandrasekhar, namely the principles of invariance. The first purpose of this note is to show that the principles of invariance--in their most general form--can be formally deduced from the axioms of the mathematical foundations.

The groundwork for this task has been developed in some recent papers [2,3,4] which extended the principles of invariance from their simple classical setting of a plane-parallel separable optical medium to general one-parameter carrier spaces. In particular, reference 2 shows that to establish the existence of the principles of invariance with respect to a given space, it is necessary only to establish the existence of the invariant imbedding relation--a relation which is an explicit statement of a general linear operation which maps one general radiance function into another.

The Second Purpose of This Note

The groundwork for the present task, but using a fundamentally different approach, has also been established in still another set of studies [5,6]. In this set the approach to the principles of invariance has been through a detailed study of the general properties of discrete spaces, i.e., spaces consisting of arbitrary bounded finite sets of points in Euclidean three-space. It was possible to establish the principles of invariance on completely arbitrary discrete spaces using this approach. The fundamental starting point was the so-called <u>principle of local interaction</u>. This principle was also the fountainhead of a series of long-sought solutions of very practical transfer problems which are ordinarily intractable in the continuous version of radiative transfer theory. (See reference 7 for a bibliography.) The second purpose of this note is to show that the principle of local interaction is also a simple formal consequence of the axiomatic foundations of radiative transfer theory.

The Net Result of This Note

The net result of this note is the formal completion of the basic mathematical structure of modern radiative transfer theory. Thus, in addition to the basic equation of transfer for radiance, the two fundamental principles of the theory (namely the principles of invariance, and the principle of local interaction) are now deducible from the three axioms C_1, M_1, R_1 of reference 1. Hence these axioms comprise the root of the discipline of radiative transfer theory.

THE INVARIANT IMBEDDING RELATION ON A GENERAL CARRIER SPACE

The starting point of the derivation of the general form of the invariant imbedding relation is section 9 (p. 700) of reference 1. In particular we use the general transition relation for $\mathfrak{G}$-densities:

$$G(t_1,E_1;t_3,x_3) = \int_X G(t_1,E_1;t_2,x_2)\, dG'(t_2,x_2;t_3,x_3), \qquad (1)$$

$$t_1 \leq t_2 \leq t_3 \; ; \quad E_1 \in \underline{S} \; ; \quad x_2, x_3 \in X$$

The meanings of the terms and symbols in (1) are fully described in the reference. We merely observe that the symbol $G(t_1, E_1; t_3, x_3)$ stands for a general radiant density (generalized radiance) at point $x_3 \in X$ at time t_3 in a general carrier space $(X, \underline{S}, \nu)$. This radiance is the result of an initial radiance condition on a measurable subset $E_1 \in \underline{S}$ of X at time $t_1 \leq t_3$.

In order to motivate the following step and to help the reader interpret its physical meaning, we refer him to the Figures of references 2 and 3. These Figures depict a general subset determined by the parameter interval $[x, z]$ of the one parameter carrier space $\Phi = X \times \Xi$ which in turn is defined over the parameter interval $[a, b]$. Figure 1 of reference 2 is for a general space; Figure 1 of reference 3 is for a plane-parallel space. The subset $[x, z]$ is arbitrarily partitioned by the choice of a parameter y, $x \leq y \leq z$. The flow of energy across the "level surface" X_y determined by the parameter y is denoted by the functions $N_+(y)$ and $N_-(y)$ (whose domains of definition are basically $X_y \times \Xi_+$ and $X_y \times \Xi_-$, respectively.

The abstract essence of these geometrical features are then:

(a) The selection of an arbitrary subset E of X (the abstract counterpart of the interval $[x, z]$).

(b) The general partitioning of X (and hence E) into two (disjoint) subsets X_+ and X_- (and hence E into E_+ and E_-). For example, the partitioning of the one-parameter carrier space is into the subsets defined by $[a, y]$, $(y, b]$ and $[x, z]$ is then automatically partitioned into $[x, y]$, $(y, z]$.

(c) The consideration of the radiance function at a point $x \varepsilon X$ and of how it is related to the radiance distributions on the <u>complement</u> of E_+ in X_+ and on the <u>complement</u> of E_- in X_- (the abstract counterpart of how $[N_+(y), N_-(y)]$ is related to $[N_+(z), N_-(x)]$). The schematic representation of this abstract setting is given in Figure 1 below.

With this background in mind, let a general carrier space X be partitioned into two sets X_+, $X_- \varepsilon \underline{S}$. Therefore $X_+ \cup X_- = X$ and $X_+ \cap X_- = \phi$, the empty set in $\underline{S}$ (which is a σ-algebra). Let $E \varepsilon \underline{S}$. Then this partition of X generally induces a partition of E into subsets $E_+ = E \cap X_+$ and $E_- = E \cap X_-$. We may assume that $\nu(E_+)$ and $\nu(E_-)$ are positive; otherwise the problem is trivial. Let E' denote the complement of E in X. Furthermore let $E_+' = E' \cap X_+$ and

$E'_- = E' \cap X_-$ be the complements of E_+ and E_- in X_+ and X_-, respectively. Clearly $E' = E'_+ \cup E'_-$, $E'_+ \cap E'_- = \phi$, and $E \cup E' = X$.

Now let E' be the maximal full support of the radiative measure on $\underline{S}$ at time t. Then at any time $t' \geq t$ we have from (1) for any point $x \varepsilon E$:

$$G(t, E'; t', x) = \int_{E'_+} G(t, E'_+; t, \cdot)\, \gamma'(t, \cdot; t', x)\, d\nu$$

$$+ \int_{E'_-} G(t, E'_-; t, \cdot)\, \gamma'(t, \cdot; t', x)\, d\nu, \qquad (2)$$

where we have made use of the fact that

$$\gamma'(t, \cdot; t', x) = \frac{d\, G'(t, \cdot; t', x)}{d\nu} \qquad (3)$$

is a γ-density of the general theory, which exists by virtue of the fact that the measure $G'(t, \cdot; t', x)$ on $\underline{S}$ is absolutely continuous with respect to the carrier measure ν. (See section 7, reference 1.) We have also made use of the fact that $G(t, E'; t, \cdot) = O\,[\nu]$ on $E = X - E'$, which follows from the hyposthesis that E' is the maximal full support of the radiative measure on $\underline{S}$ at time t .

Let $G_{\pm}(t, E'; t', \cdot)$ denote the restrictions of $G(t, E'; t', \cdot)$ to $E_{\pm}$. Furthermore for each $x \varepsilon X$, let $\gamma'_{\pm}(t, x; t', \cdot)$ denote the restrictions of $\gamma'(t, x; t', \cdot)$ to $E_{\pm}$. Then (2) yields two statements, one for G_+ and one for G_- . For example, the statement for G_+ reads:

$$G_+(t, E'; t', \cdot) = \int_{E'_+} G(t, E'_+; t, x')\, \gamma'_+(t, x'; t', \cdot)\, d\nu(x')$$

$$+ \int_{E'_-} G(t, E'_-; t, x')\, \gamma'_+(t, x'; t', \cdot)\, d\nu(x') \tag{4}$$

A similar equation holds for the restriction $G_-(t, E'; t', \cdot)$.

Now define four integral operators as follows:

$$S_{\underline{\pm}\underline{\pm}}(t, t') = \int_{E'_{\underline{\pm}}} [\cdot]\, \gamma'_{\pm}(t, x'; t', \cdot)\, d\nu(x') \tag{5}$$

and

$$S_{\underline{\pm}\underline{\mp}}(t, t') = \int_{E'_{\underline{\pm}}} [\cdot]\, \gamma'_{\mp}(t, x'; t', \cdot)\, d\nu(x') \tag{6}$$

wherein all upper signs are read together and all lower signs are read together. It follows from (4); its G_- -counterpart; (5); and (6) that:

$$[G_+(t, E'; t', \cdot), G_-(t, E'; t', \cdot)] =$$

$$[G(t, E'_+; t, \cdot), G(t, E'_-; t, \cdot)] \begin{pmatrix} S_{++}(t,t') & S_{+-}(t,t') \\ S_{-+}(t,t') & S_{--}(t,t') \end{pmatrix}, \quad (7)$$

which is the desired general form of the invariant imbedding relation. The operators with mixed signatures $(+-)$, $(-+)$, play the role of generalized reflectance operators, those with unmixed signatures, $(++)$, $(--)$, play the role of generalized transmittance operators.

By introducing the notion of restricted radiative processes (section 16, reference 1) and also the notion of product carrier spaces (section 17, reference 1), equation (7) yields the invariant imbedding relation on the classical product spaces by following the procedures developed in detail in reference 1. By reducing the space to a one parameter carrier space, (7) reduces to the time-dependent version of (1) of reference (2). The remaining steps toward the principles of invariance for the standard reflectance and transmittance operators now proceeds as in references 2, 3, and 4.

THE PRINCIPLE OF LOCAL INTERACTION ON A GENERAL DISCRETE SPACE

The principle of local interaction is the fundamental principle in the radiative transfer theory of discrete spaces. It was motivated and formulated in reference 5, and was shown to yield the principles of invariance and basic discrete-transfer equations in reference 6. We now show how this principle (in fact its general time-dependent form) follows formally from the general transition relation (1) when X is a arbitrary discrete space:

$$X_n = \left\{ \underline{x}_i \in E_3 \ : \ \underline{x}_i = (x_i, y_i, z_i) \ , \ 1 \leq i \leq n < \infty \ ; \ x_i, y_i, z_i \text{ integers} \right\} .$$

We observe first of all that the carrier measure ν on $\underline{S}$ can be defined so as to assign unit measure to the elements $x_i \in X_n \subset \underline{S}$. The radiative measures μ on $\underline{S}$ are then uniquely definable; they retain their absolute continuity with respect to ν ; and are finite on $\underline{\mathfrak{S}}$. The radiative process axiom R_1 remains unchanged for the X_n-context. Then (1) reduces to

$$G(t,E;t',x_k) = \sum_{x_j \in X_n} G(t,E;t,x_j)\,\gamma'(t,x_j;t',x_k) \tag{8}$$

for $t < t'$. With t and E understood, we may abbreviate $G(t,E;t',x_k)$ to $G(t',x_k)$. If sources within some subset E° of X are present at time $t = t'$, then we use (8) once again to find the additional component due to this source. The axiomatic basis for this rests in the local source axiom (p 728, reference 1). Thus, we have:

$$\begin{aligned} G(t',x_k) &= \sum_{x_j \in X_n} G(t,x_j)\,\gamma'(t,x_j;t',x_k) \\ &\quad + \sum_{x_j \in E^\circ} G^\circ(t',x_j)\,\gamma'(t',x_j;t',x_k)\,, \end{aligned} \tag{9}$$

which is the desired general form of the local interaction principle. The γ'-function now plays the role of the $\sum$-function.

As in the case of the principles of invariance, the three-dimensional form of (9) follows by introducing the customary restricted radiative process on the classical product carrier space manufactured from E_3, and also by introducing the various regularity conditions (local conservation property, eclipse conventions, etc.).

SUMMARY

We have shown how the basic principles of radiative transfer theory, namely the principles of invariance and the principle of local interaction can be formally derived from the structure axioms C_1, M_1, R_1 of radiative transfer theory as defined and developed in reference 1. The axioms therefore may form the sole starting point for the formal development of the entire present day theory of radiative transfer on both its applied and abstract levels. This fact is summarized schematically in Figure 2.

REFERENCES

1. Preisendorfer, R. W. A Mathematical Foundation for Radiative Transfer Theory, J. Math. and Mech. 6, 685-730 (1957).

2. ________________ Invariant Imbedding Relation for the Principles of Invariance, Proc. Nat. Acad. Sci., 44, 320-323 (1958).

3. ________________ Functional Relations for the R and T Operators in Plane-Parallel Media, Proc. Nat. Acad. Sci., 44, 323-327 (1958).

4. ________________ Time-Dependent Principles of Invariance, Proc. Nat. Acad. Sci., 44, 328-332 (1958).

5. ________________ Radiative Transfer on Discrete Spaces, SIO Ref. 59-53, Scripps Institution of Oceanography, University of California, La Jolla, California (1959).

6. ________________ Principles of Invariance on Discrete Spaces, SIO Ref. 59-55, Scripps Institution of Oceanography, University of California, La Jolla, California (1959).

7. ________________ Markov Chains and Radiative Transfer, SIO Ref. 59-72, Scripps Institution of Oceanography, University of California, La Jolla, California (1959).

GEOMETRY FOR THE GENERAL INVARIANT IMBEDDING RELATION IN AN ARBITRARY CARRIER SPACE

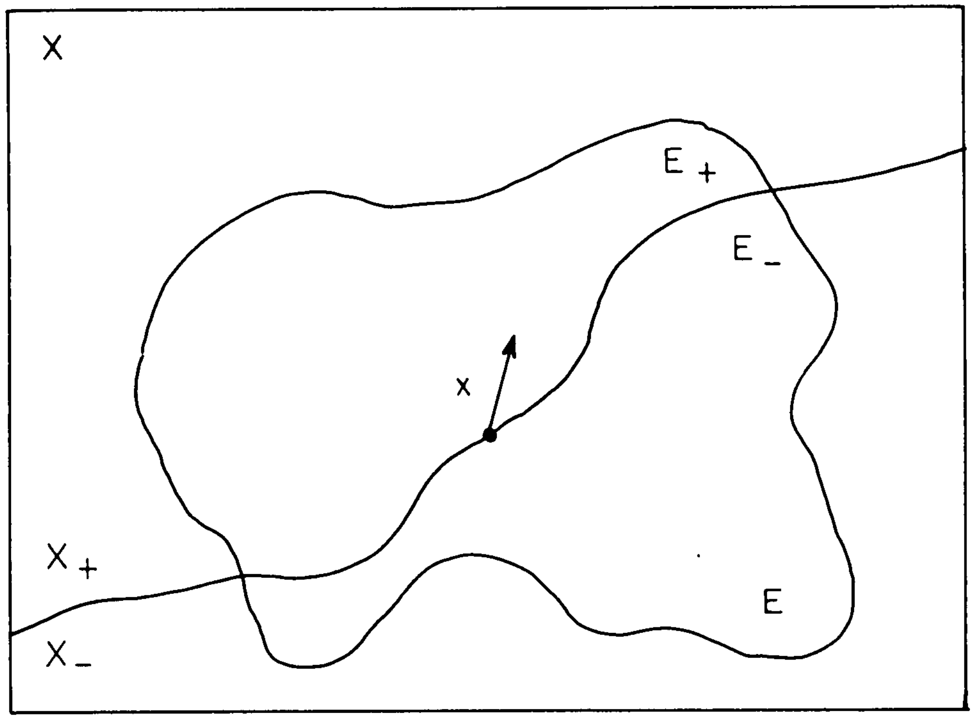

Figure I

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FORMAL CONNECTIONS BETWEEN THE AXIOMS, PRINCIPLES, AND EQUATIONS

OF RADIATIVE TRANSFER THEORY

STRUCTURE AXIOMS OF FOUNDATIONS

C_1 M_1 R_1

(REFERENCE I)

GENERAL SPACE

Equation of Transfer

(p 729, Reference 1)

Special Spaces

Applications

GENERAL SPACE

Invariant Imbedding Relation

(Equation (7) above)

One-Parameter Carrier Spaces

Principles of Invariance

Special Spaces

Applications

DISCRETE SPACE

Local Interaction Principle

(Equation (9) above)

Special Spaces

Applications

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Figure 2