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TRANSIENT RESPONSE AND HUYGEN-FRESNEL

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TRANSIENT RESPONSE AND HUYGEN-FRESNEL

Introduction

Calculations of diffraction patterns and analysis of other propagation problems by Huygen-Fresnel wavefront construction techniques are common content of optical texts.[1] It is generally recognized that the results achieved are approximate, but, for many sets of conditions, sufficiently accurate to make the techniques a valuable engineering tool. In the present paper the basic Huygen-Fresnel formulation is extended from the monochromatic, steady state condition to the transient condition. The steady state condition is a special case derivable from the transient results. My motivation for offering this development is based upon my belief that in some cases the approach yields increased insight into expected results, and offers mathematical simplification as compared to the steady state formulation under some conditions. The approach also yields insight into the nature and magnitude of error which may be attributable to the approximations inherent in the Huygen-Fresnel formulation.

Linearity and Superposition

The development in this paper is based upon concepts of linearity and the resulting superposition theorems. These fundamental notions have been around for a very long time[2], and are briefly reviewed to set the stage for the developments in this paper. The concepts are summarized in Fig. 1. Line (a) depicts a general input to the medium $f(t)$

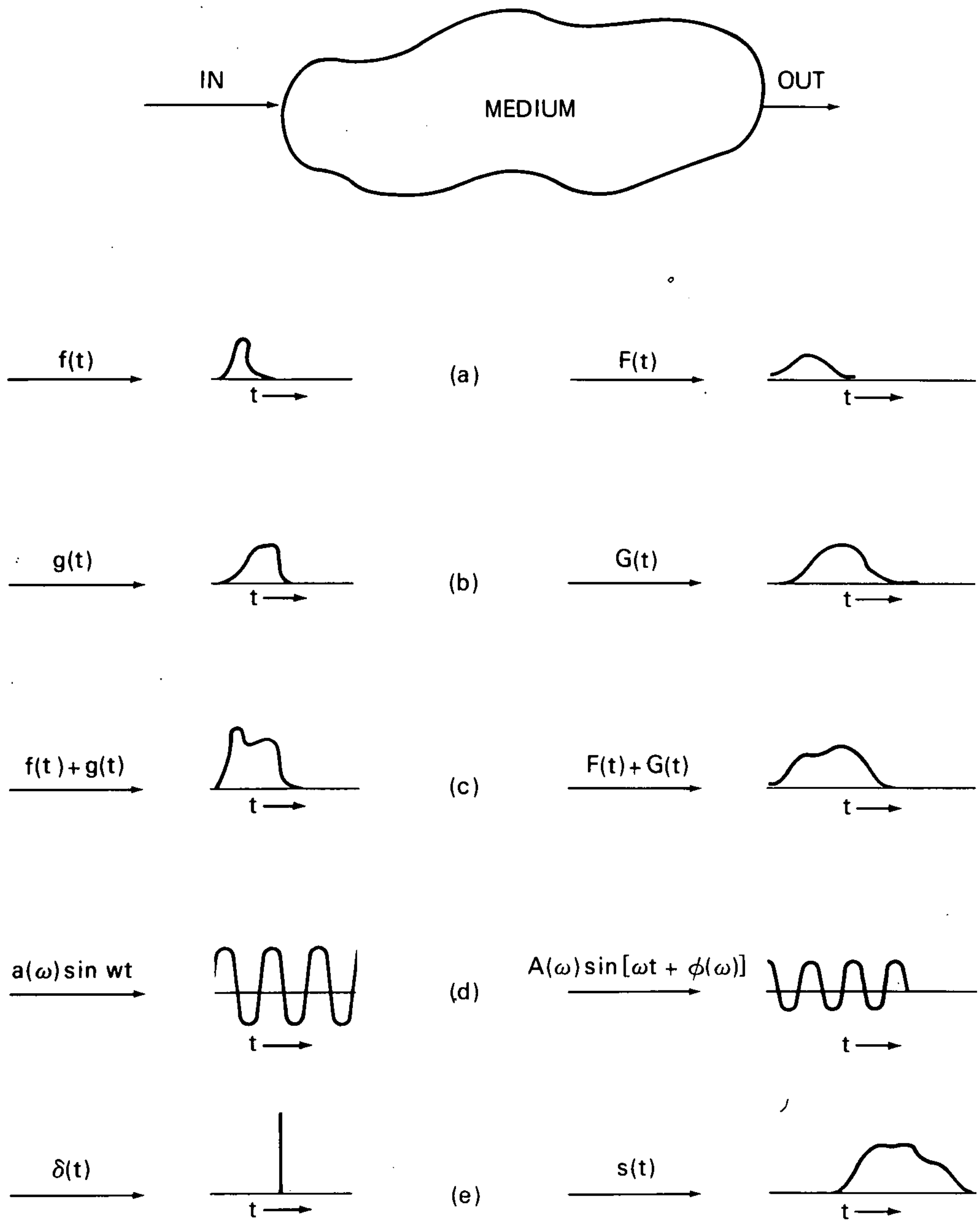
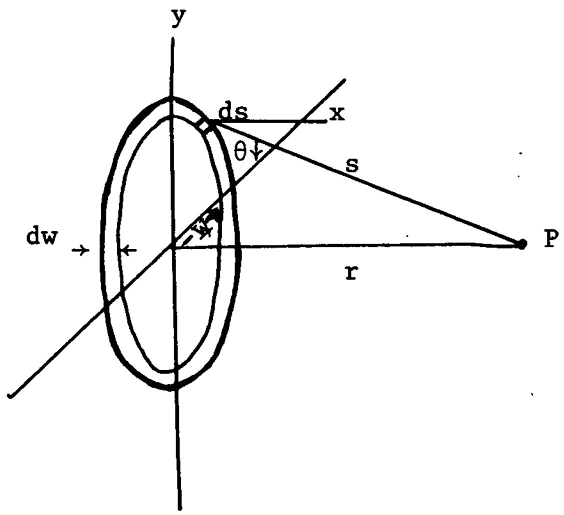


Fig. 1. Concepts of Linearity and Superposition.

which produces an output F(t). It is assumed that the medium is invariant in time over the intervals of interest. This is equivalent to stating that the same input function inserted at a different time, $f(t-\tau)$, will produce an identical output simply delayed in time $F(t-\tau)$. Line (b) depicts another general input, g(t), which produces an output, G(t). The critical test of linearity lies in the superposition indicated in line (c), i.e., for a linear system an input consisting of the sum of two or more functions will produce an output which is the sum of the outputs which would be produced by each of the inputs individually. It is this linearity of superposition which makes it possible to disect the input into arbitrarily chosen components, as for example, individual sinusoidal components obtained by Fourier decomposition of the input as depicted in line (d). Since any input can be synthesized by Fourier analysis, the properties of the linear system can be specified by noting the output amplitude and phase shift for a unit amplitude, zero phase input for each of all possible frequencies in the entire spectrum. Since a Dirac function $\delta(t)$ as shown on line (e) contains all possible frequencies with fixed phase relationship, the output, s(t), from such an input can be Fourier analyzed to determine the transfer function, i.e., the attenuation and phase shift associated with each frequency throughout the spectrum. Since any input function can also be decomposed into an infinite set of Dirac functions of appropriate amplitude and time position, the output from any generalized input will be the convolution of the input with the transient response, s(t).

Propagation of a Plane Wave

The propagation of a plane wave is a simple case to treat and one for which we have strong intuitions as to the proper answer. Consider the following geometry:



In the x-y plane there is a plane monochromatic wavefront. The first step will be to use a Huygen-Fresnel calculation to determine the amplitude and phase of the field at point, P.

The increment of field at P due to the incremental area dS is,

$$dU(P) = -\frac{i}{\lambda} Q(\theta) \frac{A(\mu)}{s} e^{i2\pi \frac{s}{\lambda}} e^{-i[\omega t+\phi(\omega)]} dS. \qquad (1)$$

The term,

$$K(\theta) = -\frac{i}{\lambda} Q(\theta) \quad , \qquad (2)$$

is the inclination factor[3], λ is the wavelength, and w is the angular frequency. $A(\omega)$ is the amplitude of the monochromatic wave in the x-y plane and $\phi(w)$ is its phase in that plane. By noting that,

$$\frac{1}{\lambda} = \frac{\omega}{2\pi C} , \quad (3)$$

where C is the velocity of light, eq.(1) becomes,

$$dU(P) = - \frac{i\,\omega}{2\pi C} \; Q(\Theta) \; \frac{A(\omega)}{s} \, e^{i\omega \frac{s}{c}} \; e^{-i[\omega t + \phi(\omega)]} \, dS \quad . \quad (4)$$

Therefore,

$$U(P) = \int_{\Omega} \frac{Q(\theta)}{2\pi cs} \left\{ -i\omega \; A(\omega) \; e^{-i[\omega(t - \frac{s}{c}) + \phi(\omega)]} \right\} dS \quad . \quad (5)$$

Equation (5) gives the amplitude and phase of the monochromatic wave at point P generated by a monochromatic plane wave in x-y having an amplitude $A(\omega)$ and a phase of $\phi(\omega)$. Assume now that the electromagnetic disturbance in x-y is not a monochromatic wave but instead is a temporal signature, G(t). In terms of monochromatic components,

$$G(t) = \int_{-\infty}^{+\infty} A(\omega) \; e^{-i[\omega t + \phi(\omega)]} \; d\omega \quad . \quad (6)$$

To obtain the temporal signature at point P,

$$V(t) = \int_{-\infty}^{+\infty} U(P) \; d\omega \quad . \quad (7)$$

Substituting eq.(5) into eq.(7) gives,

$$V(t) = \int_{-\infty}^{+\infty} \int_{\Omega} \frac{Q(\theta)}{2\pi cs} \left\{ - i\omega \; A(\omega) \; e^{-[\omega(t - \frac{s}{c}) + \phi(\omega)]} \right\} dS d\omega \quad \cdot \quad (8)$$

Interchanging the order of integration gives,

$$V(t) = \int_{\Omega} \frac{Q(\theta)}{2\pi cs} \int_{-\infty}^{+\infty} -\left\{i\omega\ A(\omega)\ e^{-i[\omega(t - \frac{s}{c} + \phi(\omega)]}\right\} d\omega dS \quad . \qquad (9)$$

The inner integral can be seen to be of the form of eq.(6) except for the presence of the $-i\omega$ and the time delay $\frac{s}{c}$. It can be noted by inspection that,

$$D_t\ G(t - \frac{s}{c}) = \int_{-\infty}^{+\infty} -\ i\omega\ A(\omega)\ e^{-i[\omega(t - \frac{s}{c}) + \phi(\omega)]} d\omega, \qquad (10)$$

where D_t is the derivative with respect to time. Substituting in eq.(9) gives

$$V(t) = \int_{\Omega} \frac{Q(\theta)}{2\pi cs}\ D_t\ G(t - \frac{s}{c})\ dS \quad . \qquad (11)$$

Reversing the order of the derivative and integral yields,

$$V(t) = D_t \left\{\int_{\Omega} \frac{Q(\theta)}{2\pi cs} G(t - \frac{s}{c})\ dS\right\} . \qquad (12)$$

The incremental area dS will be chosen to be a ring of radius W, and thickness dW. Therefore,

$$dS = 2\pi\ W\ dW \quad . \qquad (13)$$

Let,

$$t' = \frac{s}{c} = \frac{(w^2+r^2)^{\frac{1}{2}}}{c} \quad . \qquad (14)$$

Then,

$$dt' = \frac{w\ dW}{c(w^2+r^2)^{\frac{1}{2}}} \tag{15}$$

Combining (15) in (13) gives,

$$dS = 2\pi C\ (w^2+r^2)^{\frac{1}{2}}\ dt' \quad , \tag{16}$$

or,

$$dS = 2\pi c\ s\ dt' \quad . \tag{17}$$

Substituting this result in eq.(12) yields,

$$V(t) = D(t) \int_{r/c}^{\infty} Q(\theta)\ G(t-t')\ dt' \quad . \tag{18}$$

To gain insight, let us start with the simple but invalid assumption that,

$$Q(\theta) \equiv 1 \quad . \tag{19}$$

Let me also define a function

$$F(t') \equiv 0 \quad , \qquad t' < \frac{r}{c} \tag{20}$$

and

$$F(t') \equiv 1 \quad , \qquad t' \geq \frac{r}{c} \tag{21}$$

so that,

$$V(t) = D(t) \int_{-\infty}^{+\infty} F(t')\ G(t-t')\ dt' \quad . \tag{22}$$

Equation (22) is a convolution integral and indicates that to obtain V(t) we first convolve the input function of time G(t) with the step function F(t) and then differentiate the resultant function. For any function G(t) this produces,

$$V(t) = G\left(t - \frac{r}{c}\right) \quad . \tag{23}$$

This result states that a plane wavefront in x-y with any temporal signature will be reproduced at point P with no alteration in either the amplitude or the temporal signature other than the expected delay in arrival due to the finite propagation velocity. This result, although based upon the invalid assumption as to $Q(\theta)$, produces an answer which is in keeping with the intuitive notion as to how plane waves are propagated in a non-dispersive medium.

How would this result be altered by making a "valid" assumption as to to $Q(\theta)$? From reference 3,

$$K(\theta) = -\frac{i}{2\lambda}(1 + \cos\Theta) \quad . \tag{24}$$

From eq.(2) and (24),

$$Q(\theta) = \tfrac{1}{2} + \tfrac{1}{2}\cos\Theta \quad . \tag{25}$$

From the earlier sketch,

$$\cos\Theta = \frac{r}{s} \quad . \tag{26}$$

Let,

$$t_o = \frac{r}{c} \quad , \tag{27}$$

and from (14),

$$t' = \frac{s}{c} \quad , \tag{28}$$

then,

$$\cos\Theta = \frac{t_o}{t'} \quad , \tag{29}$$

so that,

$$Q(\theta) = \tfrac{1}{2} + \frac{t_o}{2t'} \quad . \qquad (30)$$

By analogy to eqs. (20) and 21) I will redefine,

$$F(t') = 0, \qquad t' < \frac{r}{c} \quad . \qquad (31)$$

and

$$F(t') = \tfrac{1}{2} + \frac{t_o}{2t'}, \qquad t' \geq \frac{r}{c} \qquad (32)$$

Returning to eq.(22) with the redefined F(t') let us examine the special case in which,

$$G(t-t') = \delta(t-t') \quad , \qquad (33)$$

where $\delta(t-t')$ is a Dirac function having value only at $t - t'$. Since,

$$\int_{-\infty}^{+\infty} F(t')\ \delta(t-t')\ dt' = F(t) \quad , \qquad (34)$$

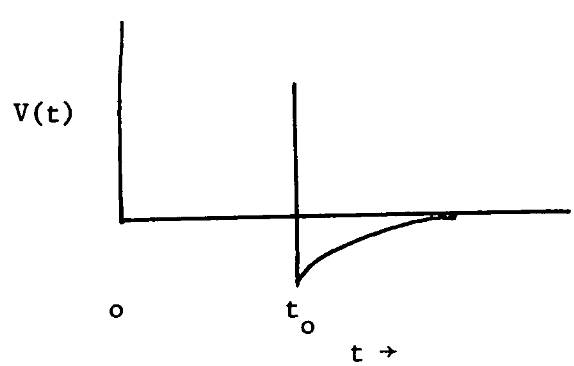
then,

$$V(t) = D(t)\ F(t) \quad . \qquad (35)$$

Therefore

$$V(t) = \delta(t_o) - \frac{t_o}{2t^2} \quad , \qquad t \geq t_o \quad . \qquad (36)$$

A sketch of V(t) is as follows:



This result indicates that a Dirac function plane wavefront will not remain a Dirac function after propagation, i.e., there is a transient response associated with the propagation process. Note that the Dirac function portion of V(t) is the result achieved when Q(θ) was assumed to be unity.

The Fourier transform of V(t) gives the steady state condition in terms of the amplitude and phase for all possible monochromatic plane waves. The exact form of this Fourier transform is unimportant to the present arguments. Certain assymptotic properties of the transform are important and can be deduced easily. To do so, consider first the Dirac function portion of V(t). The Fourier transform of the Dirac function is a flat spectrum of unity amplitude with a phase shift,

$$\phi = 2\pi \frac{ct_o}{\lambda} = \omega\ t_o \quad , \tag{37}$$

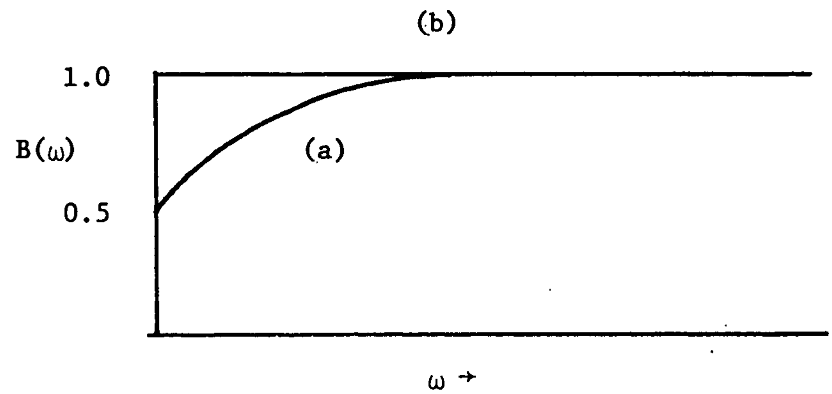
where w is the angular frequency. Consider now the portion of V(t) involving the inverse square of t'. At $\omega = o$ the contribution to the spectrum from this component will be,

$$B(\omega = o) = -\int_{t_o}^{\infty} \frac{t_o}{2t^2}\ dt \quad , \tag{38}$$

or,

$$B(\omega = o) = \frac{t_o}{2t} \Big|_{t_o}^{\infty} = -\tfrac{1}{2} \quad . \tag{39}$$

At very high frequencies this portion of V(t) will have a spectrum which approaches zero. The combined spectrum is therefore as shown labeled (a) in the sketch,



whereas the line labeled (b) is the result when $Q(\theta)$ is assumed to be unity.

As a practical matter, it is important to know the conditions under which (a) and (b) become approximately equal. This will occur when,

$$t_o >> \frac{2\pi}{\omega} \quad . \tag{40}$$

This is equivalent to stating that,

$$\frac{r}{c} >> \frac{\lambda}{c} \quad , \tag{41}$$

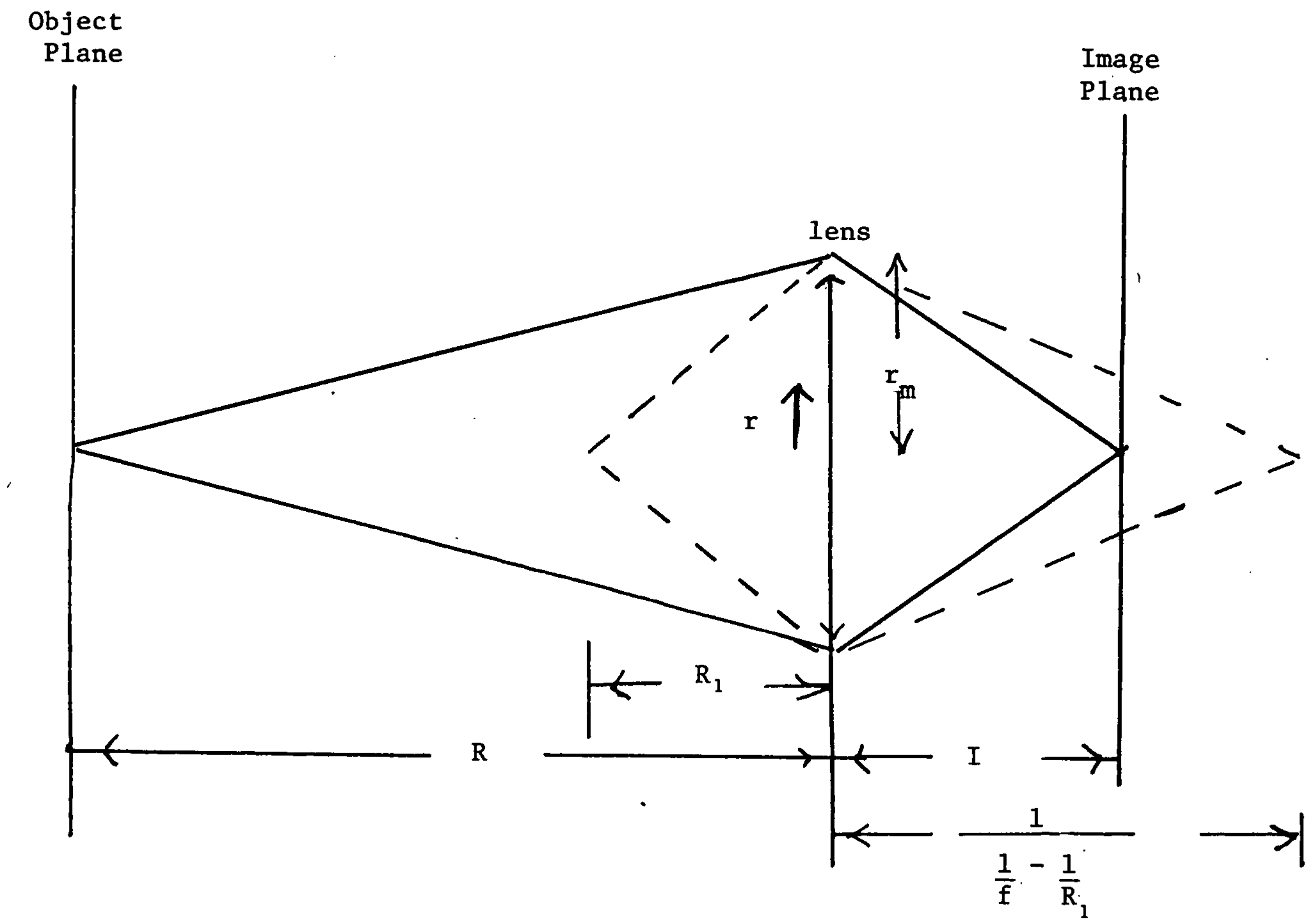
or,

$$r >> \lambda \quad . \tag{42}$$

Therefore when $r >> \lambda$ the result will be numerically similar for the "valid" and "invalid" choices of $Q(\theta)$.

Transient Response of a Simple Lens

To illustrate the transient response approach, a simple, thin lens will be used. The geometry is as follows:



A thin lens designed to image in a plane at distance I an object in a plane at distance R must have a time delay $\tau(r)$, a function of the lens radius r such that

$$\tau(r) + \frac{1}{c}\left\{\sqrt{R^2+r^2} + \sqrt{I^2+r^2}\right\} = \frac{1}{c}\left\{\sqrt{R^2+r_m^{\ 2}} + \sqrt{I^2+r_m^{\ 2}}\right\} \qquad (43)$$

or

$$\tau(r) = \frac{1}{c}\left\{\sqrt{R^2+r_m^{\ 2}} + \sqrt{I^2+r_m^{\ 2}} - \sqrt{R^2+r^2} - \sqrt{I^2-r^2}\right\} \quad . \tag{44}$$

For simplicity assume that $I >> r_m$ and $R >> r_m$ so that by binomial expansion,

$$\tau(r) = \frac{1}{c}\left\{R + \frac{r_m^{\ 2}}{2R} + I + \frac{r_m^{\ 2}}{2I} - R - \frac{r^2}{2R} - I - \frac{r^2}{2I}\right\} \quad , \tag{45}$$

or,

$$\tau(r) = \left\{\frac{1}{2c} \quad r_m^{\ 2}\left(\frac{1}{R} + \frac{1}{I}\right) - r^2\left(\frac{1}{R} + \frac{1}{I}\right)\right\} \quad , \tag{46}$$

or

$$\tau(r) = \frac{1}{2c}\left(r_m^{\ 2}-r^2\right)\left(\frac{1}{R} + \frac{1}{I}\right) \quad . \tag{47}$$

Since,

$$\frac{1}{R} + \frac{1}{I} = \frac{1}{f} \quad , \tag{48}$$

where f is the lens focal length, then,

$$\tau(r) = \frac{1}{2cf}\left(r_m^{\ 2}-r^2\right) \quad . \tag{49}$$

In contrast to the case of the propagation of a plane wave, the properties of the simple lens are such that the on-axis imaging of a point source Dirac function the F function is,

$$F(t') = \delta(t_o) \quad , \tag{50}$$

and

$$F(t') = 0 \quad , \qquad t \neq t_o \tag{51}$$

When convolved with the input time function here assumed to be a Dirac function and differentiated in the manner of eq.(22), the resultant transient response is a differentiated Dirac function, i.e., the derivative operator. In other words, a Dirac function source produces an image which is not a Dirac function but the derivative of a Dirac function.

The power spectrum of this on axis image is found by Fourier transforming the transient response and squaring the amplitude at each frequency. The derivative operator has a Fourier transform which is of the form,

$$|A(f)| = Kf, \tag{52}$$

where f is frequency and K is a constant. The power spectrum is therefore of the form,

$$|A(f)| = K^2 f^2 . \tag{53}$$

In terms of wavelength,

$$|A(f)|^2 = K \frac{C^2}{\lambda^2} . \tag{54}$$

This displays the inverse square of wavelength which we recognize as characteristic of the on-axis imagery of a point source by a diffraction limited optical system.

Consider now a point in the image plane off-axis in the X direction by a distance ΔX. The distance between that point and any point x',y' in the lens plane is

$$P = \sqrt{(x'-\Delta x)^2 + y'^2 + I^2} . \tag{55}$$

If I>> $(x'-\Delta x)$ and y' then,

$$P = I + \frac{(x'-\Delta)^2+y'^2}{2I} \quad . \tag{56}$$

The total transit time from an on-axis Dirac function point source is therefore,

$$\tau = \frac{1}{c}\left\{ R + \frac{(x'^2+y'^2)}{2R} + \frac{1}{2f}\left[r_m^{\ 2}-(x'^2+y'^2)\right] + I + \frac{(x'-\Delta x)^2+y'^2}{2I} \right\} . \tag{57}$$

Expanding, collecting terms and simplifying gives,

$$\tau = \frac{1}{c}\left\{R + I + \frac{r_m^{\ 2}}{2f}\right\} + \frac{1}{c}\left\{\frac{\Delta x^2}{2I}\right\} - \frac{1}{c}\left\{\frac{x'\Delta x}{I}\right\} \quad . \tag{58}$$

The equation has been arranged into three terms because each has special significance. The first term is the transit time from the point source, through the center of the lens to the center of the image plane. The second, is a term which accounts for the quadratic phase shift in the image plane familiar to conventional optics. The third term is the only one containing the spatial variables of the lens, i.e., x'. The first two terms, therefore, represent only time delays because they are independent of lens plane coordinates. The third term is a transient response because each x' position on the lens has a different arrival time at the image plane location Δx,o. Specifically in a time interval $d\tau$ there will be a return from a region of lens such that,

$$d\tau = -\frac{\Delta x}{CI}\, dx' \tag{59}$$

The signal will be received in the image plane at Δx,o within a time interval,

$$\tau = \frac{1}{c}\left\{R + I + \frac{r_m^2}{2f}\right\} + \frac{1}{c}\left\{\frac{\Delta x^2}{2I}\right\} \pm \frac{1}{c}\left\{\frac{r_m \Delta x}{I}\right\}. \tag{60}$$

Since dx' is independent of x', the amplitude of the return will be determined by the height of the lens at each x' position. For the circular lens,

$$y' = \sqrt{r_m^2 - x'^2} \tag{61}$$

The amplitude of the signal is therefore proportional to,

$$y'dx' = \sqrt{r_m^2 - x'^2}\ \frac{CI}{\Delta x}\ d\tau \quad . \tag{62}$$

For simplicity let the sum of the first two terms of eq.(58) be τ_o, then,

$$\tau = \tau_o - \frac{1}{c}\left\{\frac{x'\Delta x}{I}\right\} \quad , \tag{63}$$

or,

$$x' = \frac{CI}{\Delta x}\{\tau_o - \tau\} \ . \tag{64}$$

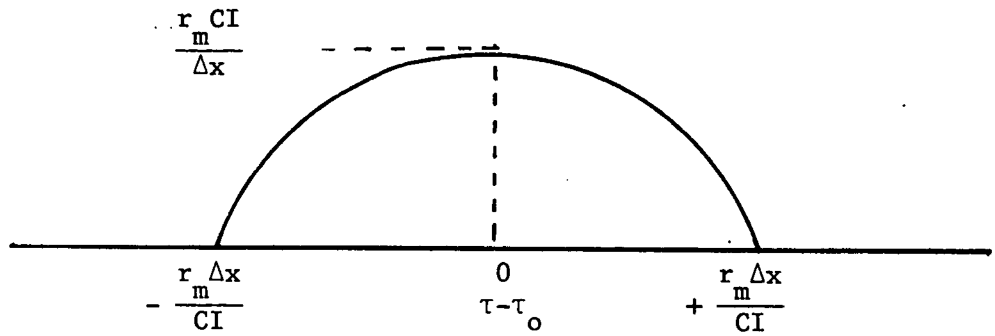
Substitution in eq.(62) gives,

$$y'dx' = \sqrt{r_m^2 - \left(\frac{CI}{\Delta x}\right)^2 \{\tau_o - \tau\}^2}\ \frac{CI}{\Delta x}\ d\tau \quad . \tag{65}$$

From eq.(60), the range of τ is,

$$\tau = \tau_o \pm \frac{1}{c}\left\{\frac{r_m \Delta x}{I}\right\} \ . \tag{66}$$

A sketch of eq.(65) is as follows:



The Fourier transform of this function is the modified Bessel function, i.e.,

$$G = K\pi r_m^{\ 2} \frac{J_1\left(\frac{2\pi\Delta x r_m f}{IC}\right)}{\left(\frac{2\pi\Delta x r_m f}{IC}\right)} \quad . \tag{67}$$

where f is the temporal frequency, and K is a constant of proportionality.

As described earlier in this writing, the derivative of eq.(25) is the transient response. A derivative in the time domain is equivalent to multiplication by frequency coupled with a 90° phase shift. Therefore,

$$G' = iK_1 f\ r_m^{\ 2} \frac{J_1\left(\frac{2\pi\Delta x r_m f}{IC}\right)}{\left(\frac{2\pi\Delta x r_m f}{IC}\right)} \quad . \tag{68}$$

The following observations may be made with respect to eq.(68). For a fixed f, i.e., a fixed wavelength, the amplitude as a function of Δx, will be a modified Bessel function therefore predicting the classic monochromatic diffraction image of a point source by a circular lens. It can also be noted that the first null of the modified Bessel function occurs when the argument is equal to 1.22π so that,

$$2\pi \frac{\Delta x r_m f}{IC} = 1.22\pi \tag{69}$$

For Δx<<I, the angular position of Δx in the image plane is,

$$\alpha = \frac{\Delta x}{I} , \tag{70}$$

and since,

$$\lambda = \frac{c}{f} , \tag{71}$$

then,

$$\frac{\alpha r_m}{\lambda} = 0.61 \tag{72}$$

or,

$$\alpha = 0.61 \frac{\lambda}{r_m} , \tag{73}$$

which is the familiar resolving power limitation associated with a diffraction limited circular aperture.

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