Conservation of Energy Across an Air-Water Surface
As Expressed in Terms of Irradiance

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Hydrolight users occasionally question whether HydroLight conserves energy across the air-water surface. The source of their confusion is usually the observation that the downwelling and upwelling irradiances do not appear to add up correctly just above and below the surface. These notes clarify this matter and show that HydroLight exactly conserves energy across the sea surface.

First, note that it's the Law of Conservation of *Energy*, not the law of conservation of irradiance or radiance. We must express the law of conservation of energy in terms of irradiance to understand the problem. Figure 1 shows the needed quantities.

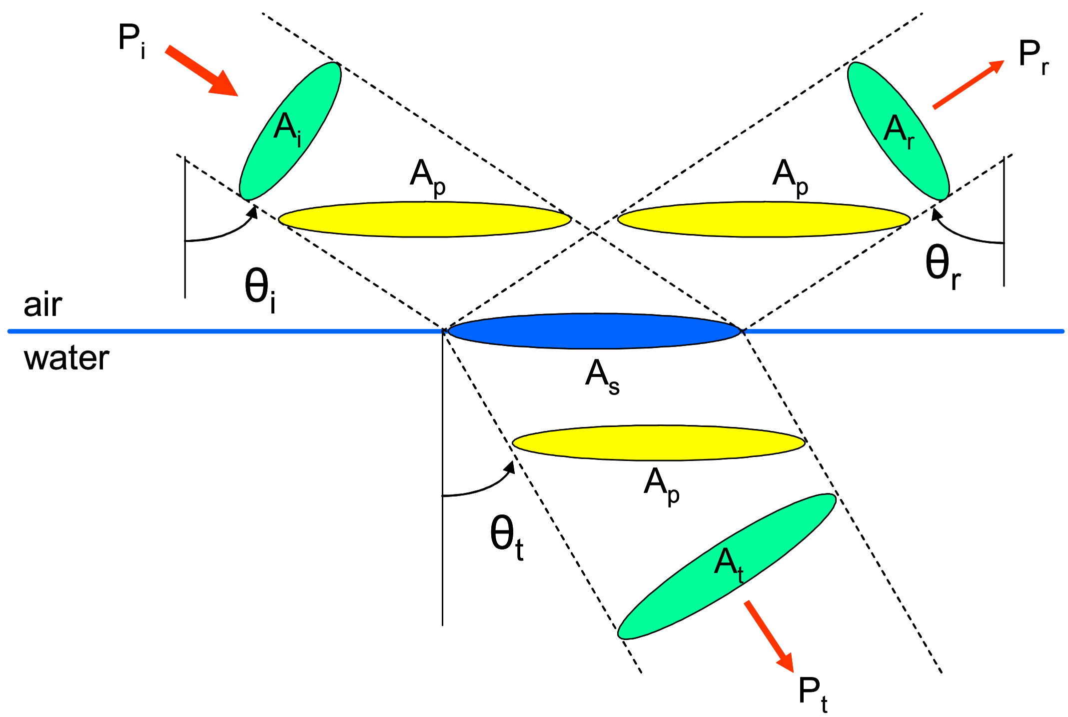

Fig. 1. Geometry needed for expressing the law of conservation of energy in terms of irradiances.

Conservation of energy (or equivalently, power) across the surface requires that

$$P_i = P_r + P_t \qquad (1)$$

where, P_i is the power in an incident collimated beam of cross section A_i, P_r is the power in the reflected beam of cross section A_r, and P_t is the power in transmitted beam of cross section A_t. These beam cross section areas are shown in green in Fig. 1. The beam of incident power illuminates a horizontal area A_s of the level sea surface (shown in blue).

We can express Eq. (1) in terms of the incident (downwelling), reflected (upwelling), and transmitted (downwelling) scalar irradiances by using the definition of irradiance as power per unit area. Thus E_{od}(incident) = P_i / A_i (using the usual oceanographic notation for downwelling scalar irradiance), and likewise for the reflected and transmitted irradiances. Thus (1) becomes

$$A_i\, E_{od}(\text{incident, in air}) = A_r\, E_{ou}(\text{reflected, in air}) + A_t\, E_{od}(\text{transmitted, in water})$$

or

$$E_{od}(\text{incident, in air}) = E_{ou}(\text{reflected, in air}) + E_{od}(\text{transmitted, in water})\, A_t / A_i \qquad (2)$$

after noting that $A_r = A_i$ by the law of reflection. By geometry, the ratio of areas A_t / A_i is just $\cos\theta_t / \cos\theta_i$. θ_t can be written in terms of θ_i using Snell's law, $\sin\theta_i = n \sin\theta_t$, where n is the real index of refraction of the water. Equation (2) then becomes

$$E_{od}(\text{incident, in air}) = E_{ou}(\text{reflected, in air}) + E_{od}(\text{transmitted, in water})\, \frac{1}{\cos\theta_i} \sqrt{1 - \frac{\sin^2\theta_i}{n^2}} \qquad (3)$$

Equation (3) is conservation of energy expressed in terms of *scalar* irradiance.

Likewise, conservation of energy can be expressed in terms of plane irradiances using the horizontal projections of the beam cross section areas. Thus Eq. (1) becomes

$$A_p\, E_d(\text{incident, in air}) = A_p\, E_u(\text{reflected, in air}) + A_p\, E_d(\text{transmitted, in water})\ ,$$

where A_p is shown in yellow in Fig. 1. Since each beam has the same horizontally projected area, this equation reduces to just

$$E_d(\text{incident, in air}) = E_u(\text{reflected, in air}) + E_d(\text{transmitted, in water}) \qquad (4)$$

Equation (4) is conservation of energy expressed in terms of *plane* irradiance.

To test HydroLight output against Equations (3) and (4), I did a run with the sun at a zenith angle of θ_i = 60 deg in a black sky, with no scattering in the water (and an absorption coefficient of 0.001 m^{-1}), and a level sea surface. (These runs used IOP model "abconst"and the "idealized sky model").

The computed irradiances just above and below the surface are then

	Eou	Eod	Eo	Eu	Ed
in air	1.2787E-01	2.0076E+00	2.1355E+00	6.3691E-02	1.0000E+00
in water	0.0000E+00	1.2269E+00	1.2269E+00	0.0000E+00	9.3631E-01

The upwelling irradiances in the water are zero because there was no scattering by the water and the bottom was infinitely deep. These numbers satisfy Equations (3) and (4).

I then did a run with a realistic sky having the sun at 60 deg and a 50% overcast to generate a more diffuse sky radiance pattern, and the wind speed was 10 m/s to generate a non-level sea surface. The irradiances are then

	Eou	Eod	Eo	Eu	Ed
in air	1.6209E-01	1.1939E+00	1.3560E+00	3.8328E-02	5.6929E-01
in water	0.0000E+00	6.9430E-01	6.9430E-01	0.0000E+00	5.3096E-01

As before, the plane irradiances satisfy Eq. (4). However, the scalar irradiances do not satisfy Eq. (3). The reason is that Eq. (3) must be applied for each particular incident direction θ_i of the diffuse sky radiance, in order to correctly compute the angle factor seen in Eq. (3). If this is done one direction at a time, each direction satisfies Eq. (3), as in the black-sky example above. Again, energy in conserved across the surface even if it is not obvious from the scalar irradiances.

A final complication occurs when the water body has scattering. To illustrate this, I set the absorption to 0.1 m^{-1} and the scattering to 0.4 m^{-1} (with a Petzold phase function) in the previous run. The results are

	Eou	Eod	Eo	Eu	Ed
in air	1.7686E-01	1.1939E+00	1.3708E+00	4.5542E-02	5.6929E-01
in water	5.1695E-02	7.3465E-01	.8634E-01	1.9674E-02	5.4341E-01

Now even the plane irradiance appears to violate conservation of energy, Eq. (4). The reason is that E_{ou} and E_u in air contain both surface-reflected radiance and radiance transmitted upward through the sea surface. Likewise, E_{od} and E_d in the water contain upwelling radiance that has been reflected downward by the surface. This makes it appear at first glance that the irradiances do not add up correctly, even though energy actually is conserved.

Note, finally, that Hydrolight exactly conserved energy within the water column if inelastic scatter is included in the run, because the invariant imbedding formalism computes all orders of multiple scattering.