Ocean Optics Summer Class

Calibration and Validation for
Ocean Color Remote Sensing

The Radiative Transfer Equation

Curtis Mobley

Delivered at the Darling Marine Center
July 2011

SEQUOIA

HYDROLIGHT

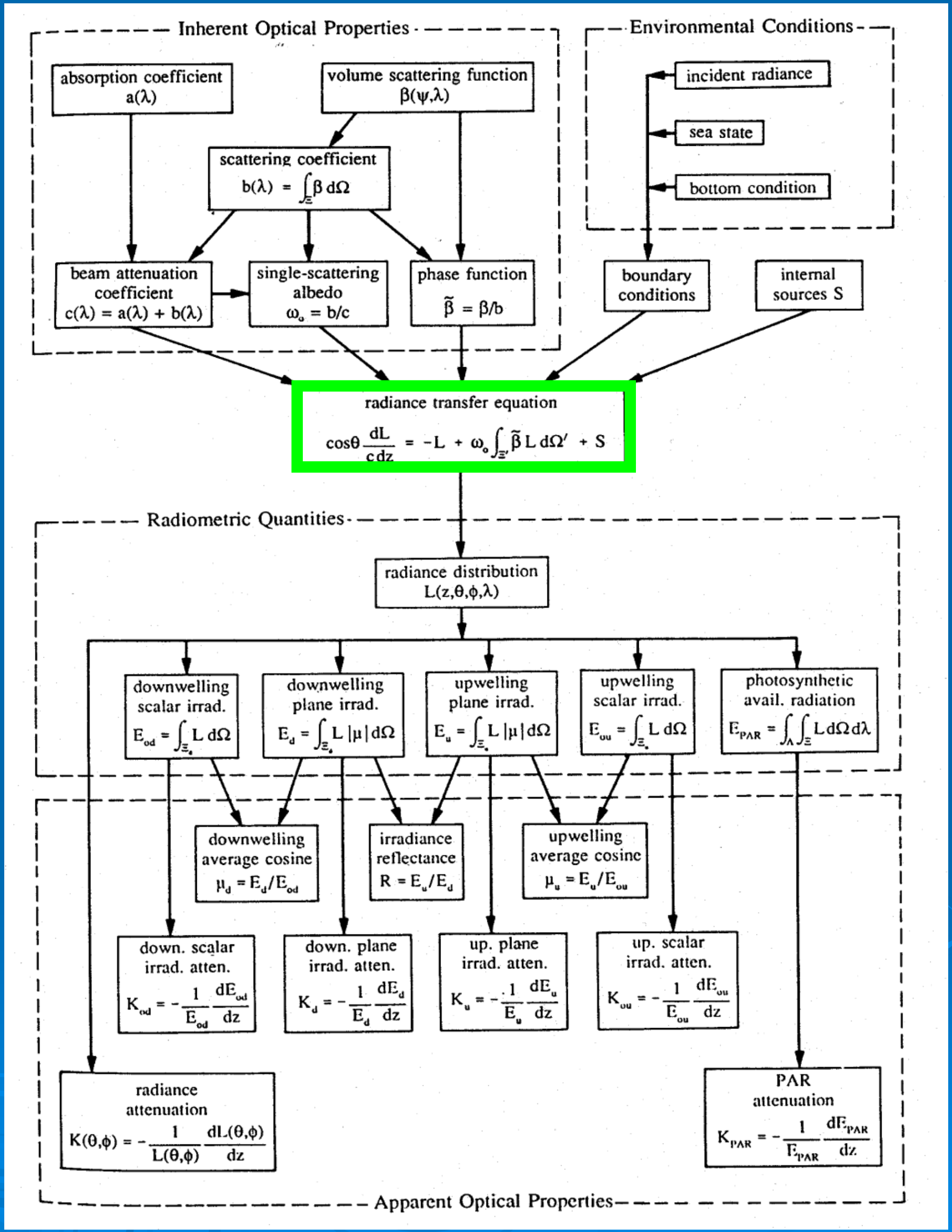
The Radiative Transfer Equation (RTE)

- expresses conservation of energy in terms of the radiance

- connects the IOPs, boundary conditions, and light sources to the radiance

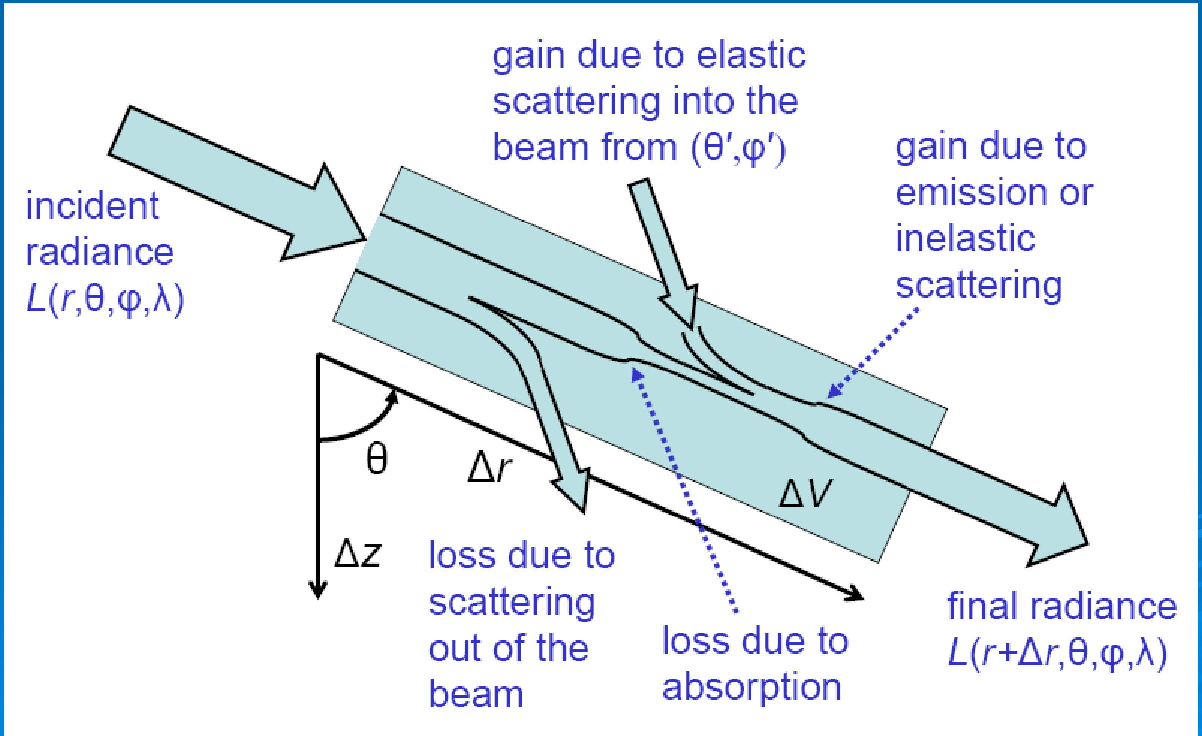
All other radiometric variables (irradiances) and AOPs can be derived from the radiance.

If you know the radiance, you know everything there is to know about the light field



Derivation of the RTE

To derive the time-independent RTE for horizontally homogeneous water, we consider the radiance at a given point *r* (at depth *z)*, traveling in a given direction (θ, ϕ), at a given wavelength λ. We then add up the various ways the radiance $L(r,\theta,\phi,\lambda)$ can be created or lost in a distance Δr along direction (θ, ϕ), going from point *r* to $r+\Delta r$ (at depth *z* to $z+\Delta z$).



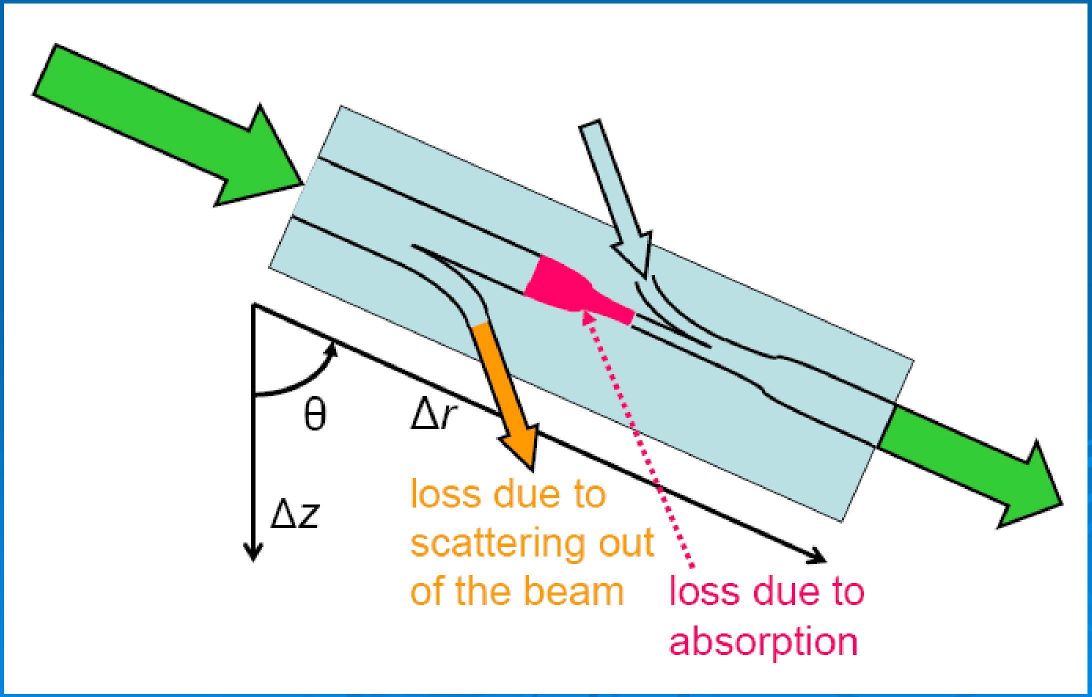
Losses of Radiance

The loss due to absorption is proportional to how much radiance there is:

$$\frac{dL(r,\theta,\phi,\lambda)}{dr} = -\, a(r,\lambda)\, L(r,\theta,\phi,\lambda)$$

Likewise for loss of radiance due to scattering out of the beam:

$$\frac{dL(r,\theta,\phi,\lambda)}{dr} = -\, b(r,\lambda)\, L(r,\theta,\phi,\lambda)$$

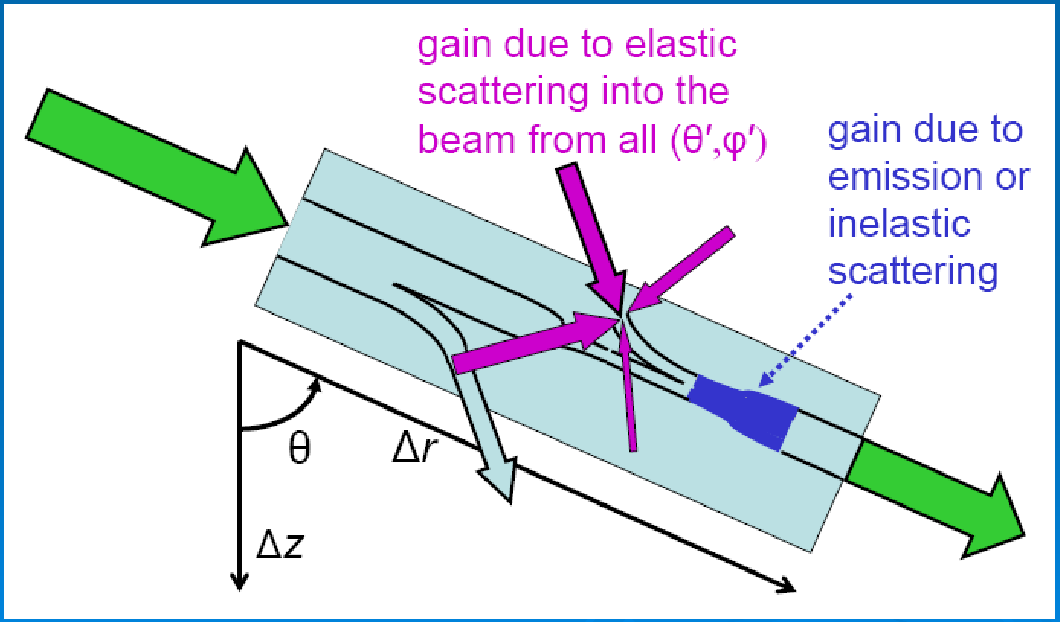


Sources of Radiance

Scattering into the beam from all other directions increases the radiance:

$$\frac{dL(r,\theta,\phi,\lambda)}{dr} = \int_{4\pi} L(r,\theta',\phi',\lambda)\, \beta(r;\, \theta',\phi' \rightarrow \theta,\phi\, ;\lambda)\, d\Omega'$$

See the www.oceanopticsbook.info page on “radiative transfer theory, deriving the radiative transfer equation” for a full development of this equation



There can be internal sources of radiance $S(r,\theta,\phi,\lambda)$, such as bioluminescence

$$\frac{dL(r,\theta,\phi,\lambda)}{dr} = S(r,\theta,\phi,\lambda)$$

Add up the Losses and Sources

$$\frac{dL(r,\theta,\phi,\lambda)}{dr} = -\,a(r,\lambda)\,L(r,\theta,\phi,\lambda)$$

$$-\,b(r,\lambda)\,L(r,\theta,\phi,\lambda)$$

$$+\int_{4\pi} L(r,\theta',\phi',\lambda)\,\beta(r;\,\theta',\phi'\rightarrow\theta,\phi;\,\lambda)\,d\Omega'$$

$$+\,S(r,\theta,\phi,\lambda)$$

Finally, note that $a + b = c$ and that $dz = dr\cos\theta$ to get

The 1D RTE, Geometric-depth Form

$$\cos\theta \, \frac{dL(z,\theta,\phi,\lambda)}{dz} = - c(z,\lambda)\, L(z,\theta,\phi,\lambda)$$

$$+ \int_{4\pi} L(z,\theta',\phi',\lambda)\, \beta(z;\, \theta',\phi' \rightarrow \theta,\phi;\, \lambda)\, d\Omega'$$

$$+ S(z,\theta,\phi,\lambda)$$

This is the RTE that HydroLight solves.

The VSF $\beta(z;\, \theta',\phi' \rightarrow \theta,\phi;\, \lambda)$ is usually written as $\beta(z,\, \psi,\, \lambda)$ in terms of the scattering angle ψ, where

$\cos\psi = \cos\theta' \cos\theta + \sin\theta' \sin\theta \cos(\phi' - \phi)$

The 1D RTE, Optical-depth Form

Define the increment of dimensionless optical depth ζ as $d\zeta = c\,dz$ and write the VSF as *b* times the phase function, $\tilde{\beta}$, and recall that $\omega_o = b/c$ to get

$$\cos\theta \, \frac{dL(\zeta,\theta,\phi,\lambda)}{d\zeta} = -\,L(\zeta,\theta,\phi,\lambda)$$

$$+\ \omega_o \int_{4\pi} L(\zeta,\theta',\phi',\lambda)\ \tilde{\beta}(\zeta;\ \theta',\phi' \to \theta,\phi;\ \lambda)\ d\Omega'$$

$$+\ S(\zeta,\theta,\phi,\lambda)/c(\zeta,\lambda)$$

Can specify the IOPs by *c* and the VSF β, or by ω_o and the phase function $\tilde{\beta}$ (and also *c*, if there are internal sources)

Note that a given geometric depth *z* corresponds to a different optical depth $\zeta(\lambda) = \int_0^z c(z',\lambda)\, dz'$ at each wavelength

The 1D RTE, Geometric-depth Form

$$\cos\theta \, \frac{dL(z,\theta,\phi,\lambda)}{dz} = - \, c(z,\lambda) \, L(z,\theta,\phi,\lambda)$$

$$+ \int_{4\pi} L(z,\theta',\phi',\lambda) \, \beta(z;\, \theta',\phi' \rightarrow \theta,\phi;\, \lambda) \, d\Omega'$$

$$+ \, S(z,\theta,\phi,\lambda)$$

NOTE: The RTE has the TOTAL *c* and TOTAL VSF. Only oceanographers (not photons) care how much of the total absorption and scattering are due to water, phytoplankton, CDOM, minerals, etc.

The RTE is a linear (in the unknown radiance), first-order (only a first derivitive) integro-differential equation. Given the green (plus boundary conditions), solve for the red. This is a two-point (surface and bottom) boundary value problem.

The Source Terms

The source terms for inelastic scatter (Raman, Chl or CDOM fluoresecence) or bioluminescence are similar to the elastic scatter path function, except that there is now an integral over wavelength:

$$S(z,\theta,\phi,\lambda) = \int_0^{\lambda}\int_0^{\pi}\int_0^{2\pi} L(z,\theta',\phi',\lambda')\,\beta(z;\theta',\phi'\rightarrow\theta,\phi;\lambda'\rightarrow\lambda)\,\sin\theta'\,d\theta'\,d\phi'\,d\lambda'$$

$[\mathrm{W/(m^3\ sr\ nm)}]$

Recall that for Chl fluoresecence we had

$$\begin{aligned}\beta^C(z;\psi;\lambda'\rightarrow\lambda) &= b^C(z,\lambda')\,f^C(\lambda'\rightarrow\lambda)\,\tilde{\beta}^C(\psi)\\ &= Chl(z)\,a^*_{\phi}(\lambda')\,\Phi^C(\lambda')\,\frac{\lambda'}{\lambda}\,g^C(\lambda')\,h^C(\lambda)\,\frac{1}{4\pi}\end{aligned}$$

The Source Terms

$$\int_0^{\lambda}\int_0^{\pi}\int_0^{2\pi} L(z,\theta',\phi',\lambda')\, Chl(z)\, a_\phi^*(\lambda')\, \Phi^C(\lambda')\, \frac{\lambda'}{\lambda}\, g^C(\lambda')\, h^C(\lambda)\, \frac{1}{4\pi}\, \sin\theta'\, d\theta'\, d\phi'\, d\lambda'$$

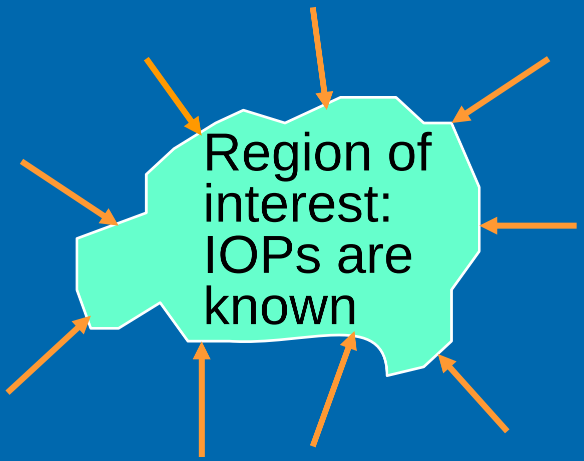
$$= \frac{Chl(z)\, h^C(\lambda)}{4\pi\,\lambda} \int_0^{\lambda} a_\phi^*(\lambda')\, \Phi^C(\lambda')\, \lambda'\, g^C(\lambda') \left[\int_0^{\pi}\int_0^{2\pi} L(z,\theta',\phi',\lambda')\, \sin\theta'\, d\theta'\, d\phi' \right] d\lambda'$$

$$= \frac{Chl(z)\, \Phi^C\, h^C(\lambda)}{4\pi\,\lambda} \int_{370}^{\lambda} a_\phi^*(\lambda')\, \lambda' E_o(z,\lambda')\, d\lambda'$$

This is how HydroLight includes Chl fluoresence. Raman, CDOM, and bioluminescence are handled in the same way, but with different VSFs for each process (bioluminescence at λ does not depend on inputs from shorter wavelengths).

Solving the RTE

A unique solution of the RTE requires:

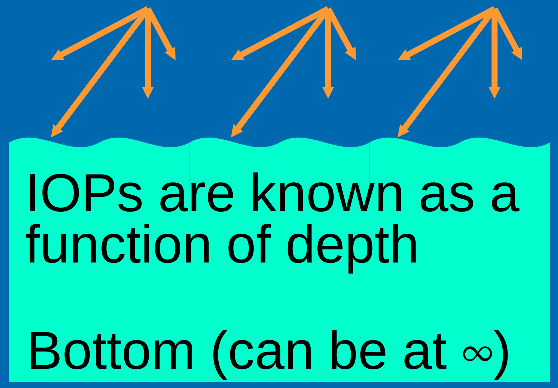


Radiance incident onto all boundaries from outside the region is known

A 3-D problem

Stretch out the region to make a horizontally homogeneous ocean

Radiance incident onto sea surface is known



A 1-D problem

Theorem: Given the IOPs within a region and the incident radiances, there is a unique solution for the radiance within and leaving the region

Solving the RTE: The Lambert-Beer Law

A trivial solution:

- homogeneous water (IOPs do not depend on z)
- no scattering (VSF β = 0, so c = a + b = a)
- no internal sources (S = 0)
- infinitely deep water (no radiance coming from the bottom boundary, so L → 0 as z → ∞)
- incident radiance *L*(z=0) is known just below the sea surface

$$\cos\theta \, \frac{dL(z,\theta,\phi,\lambda)}{dz} = -a(\lambda) L(z,\theta,\phi,\lambda)$$

$$\int_{L(z=0,\theta,\phi,\lambda)}^{L(z,\theta,\phi,\lambda)} \frac{dL}{L} = -\int_0^z \frac{a\,dz}{\cos\theta}$$

$$L(z,\theta,\phi,\lambda) = L(z=0,\theta,\phi,\lambda)\, e^{-az/\cos\theta}$$

Note that this *L* satisfies the RTE, the surface boundary condition, and the bottom boundary condition *L*(z →∞) → 0.

Solving the RTE: Gershun's Law

Start with the 1D, source-free, RTE.

$$\cos\theta \frac{dL(z,\theta,\phi,\lambda)}{dz} = -c(z,\lambda)\,L(z,\theta,\phi,\lambda) + \int_0^{2\pi}\int_0^{\pi} L(z,\theta',\phi',\lambda)\,\beta(z,\theta',\phi' \rightarrow \theta,\phi,\lambda)\sin\theta' d\theta' d\phi'$$

Integrate over all directions. The left-hand-side becomes

$$\int_0^{2\pi}\int_0^{\pi} \cos\theta \frac{dL(z,\theta,\phi)}{dz} d\Omega(\theta,\phi) = \frac{d}{dz}\int_0^{2\pi}\int_0^{\pi} L(z,\theta,\phi)\cos\theta d\Omega(\theta,\phi) = \frac{d}{dz}[E_{\mathrm{d}}(z) - E_{\mathrm{u}}(z)]$$

Solving the RTE: Gershun's Law

The – *cL* term becomes

$$\iint -c(z)L(z,\theta,\phi)d\Omega(\theta,\phi) = -c(z)\iint L(z,\theta,\phi)d\Omega(\theta,\phi)$$
$$= -c(z)E_o(z)$$

The elastic-scatter path function becomes

$$\iint \left[\iint L(z,\theta',\phi')\,\beta(z,\theta',\phi' \to \theta,\phi)d\Omega(\theta',\phi')\right] d\Omega(\theta,\phi)$$
$$= \iint L(z,\theta',\phi')\left[\iint \beta(z,\theta',\phi' \to \theta,\phi)d\Omega(\theta,\phi)\right] d\Omega(\theta',\phi')$$
$$= b(z)\iint L(z,\theta',\phi')d\Omega(\theta',\phi')$$
$$= b(z)E_o(z)$$

Solving the RTE: Gershun's Law

Collecting terms,

$$\frac{d}{dz}\left[E_{\mathrm{d}} - E_{\mathrm{u}}\right] = -cE_{\mathrm{o}} + bE_{\mathrm{o}}$$

or

$$a(z,\lambda) = -\frac{1}{E_{\mathrm{o}}(z,\lambda)}\frac{d}{dz}\left[E_{\mathrm{d}}(z,\lambda) - E_{\mathrm{u}}(z,\lambda)\right]$$

Gershun's law can be used to retrieve the absorption coefficient from measured in-water irradiances (at wavelengths where inelastic scattering effects are negligible).

This is an example of an explicit inverse model that recovers an IOP from measured light variables.

Water Heating and Gershun's Law

The rate of heating of water depends on how much irradiance there is and on how much is absorbed:

$$\frac{\partial T}{\partial t} = \frac{1}{c_v \rho} a_o E_o = -\frac{1}{c_v \rho} \frac{\partial (E_d - E_u)}{\partial z} \qquad \left[\frac{\text{deg } C}{\text{sec}}\right]$$

c_v = 3900 J (kg deg C)$^{-1}$ is the specific heat of sea water
ρ = 1025 kg m^{-3} is the water density

This is how irradiance is used in a coupled physical-biological-optical ecosystem model to couple the biological variables (which, with water, determine the absorption coefficient and the irradiance) to the hydrodynamics (heating of the upper ocean water)

Solving the RTE

Exact analytical (i.e., pencil and paper) solutions of the RTE can be obtained only for very simple situations, such as no scattering. There is no function (that anyone has ever found) that gives

$L(z,\theta,\phi,\lambda) = f(a, VSF$, sun angle, bottom reflectance, etc.)

even for very simple situations such as homogenous water with isotropic scattering. Even the extremely simple geometry of an isotropic point light source in an infinite homogeneous ocean is unsolved (a very complicated solution for $E_o(r)$ around a point source with isotropic scattering does exist). This is because of the complications of scattering (which don't exist for problems like the gravitational field around a point mass).

Solving the RTE

Approximate analytical solutions can be obtained for idealized situations such as single scattering in a homogeneous ocean. (This is where $R_{rs} = b_b/(a + b_b)$ comes from.)

I have included a set of notes on the single-scattering approximation (SSA) and related approximations in the papers directory. However, we don't have time to discuss these approximate solutions, and they are not very useful anyway.

Solving the RTE

The solution of the RTE for any realistic conditions of scattering or geometry must be done numerically. Three widely used exact numerical methods are seen in the literature (in RT theory, “exact” means that we don’t make approximations such as single scattering. Given enough computer time, you can get the correct answer as closely as you wish.)

- **Discrete ordinates**: often used in atmospheric optics
  - highly mathematical
  - difficult to program
  - doesn’t handle highly peaked phase functions well
  - most codes need a level sea surface
  - models the medium as homogeneous layers
  - fast for irradiances and homogeneous systems
  - slow for radiances and inhomogeneous systems
  - therefore, not much used in oceanography

Solving the RTE

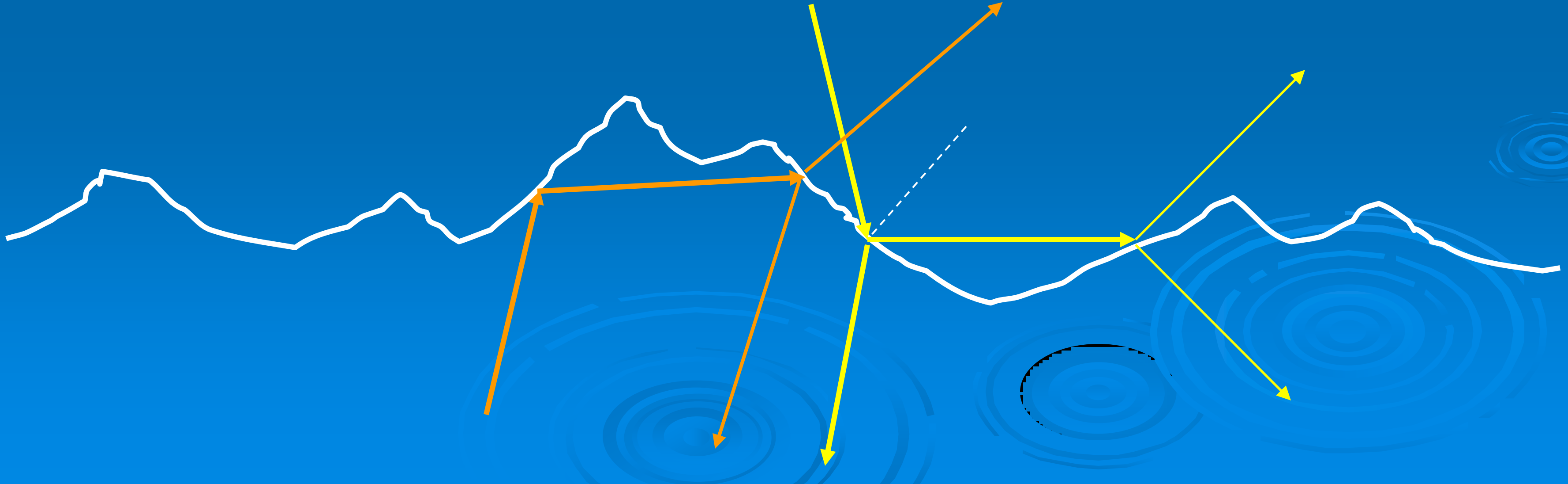
- **Monte Carlo**: widely used
  - simple math, easy to program
  - can solve 3D problems
  - run time increases exponentially with optical depth
  - have to trace many photons to get accurate radiance estimates (solutions have statistical noise)
  - very long run times for radiances and/or great depths
  - more useful for irradiance computations and/or shallow depths

- **Invariant Imbedding**: what Hydrolight uses
  - highly mathematical (see *Light and Water*, Chaps 7 and 8)
  - difficult to program
  - 1D (depth dependence) problems only
  - run time increases linearly with optical depth
  - computes radiances accurately (no statistical noise)
  - extremely fast and accurate even for radiances and large depths

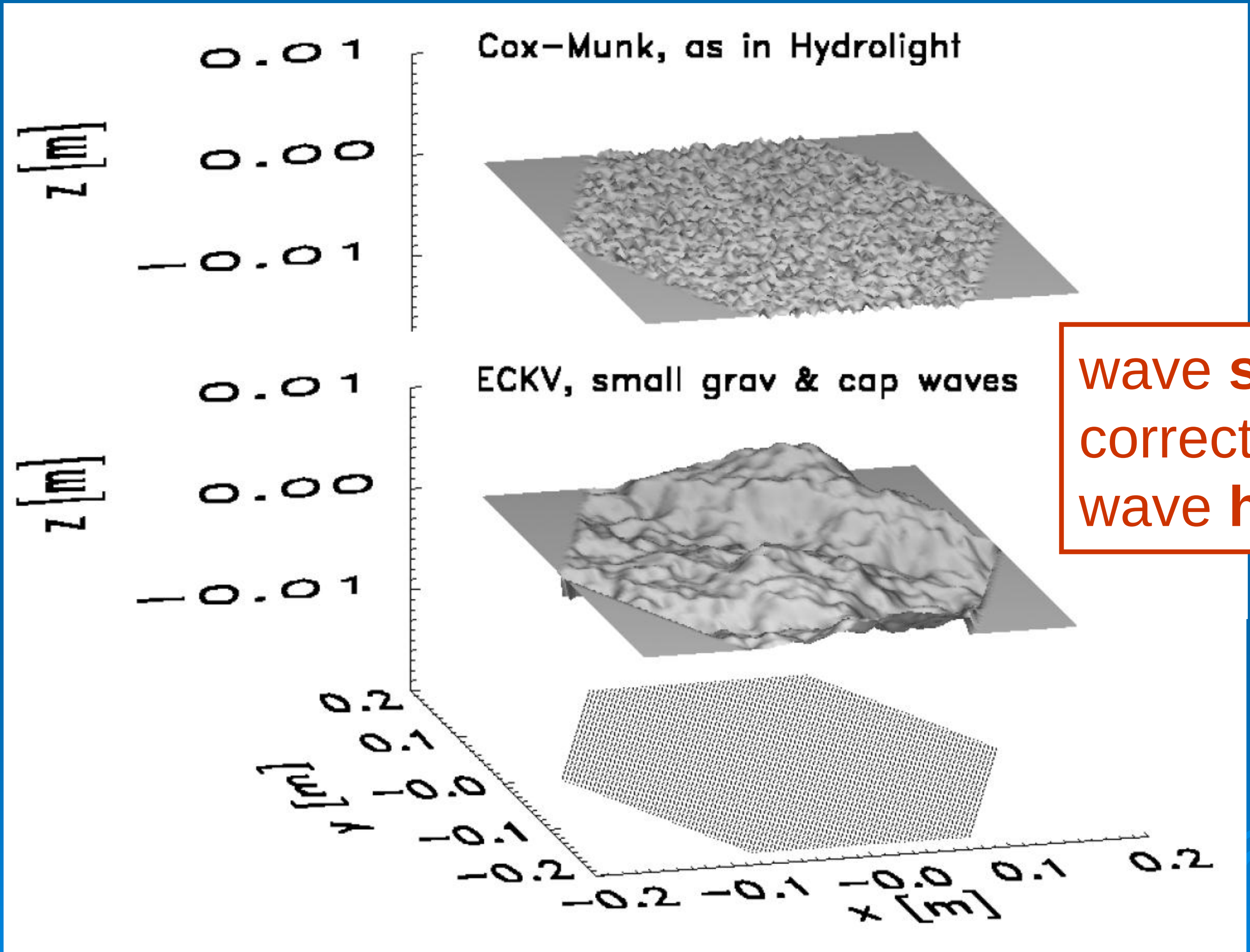
Boundary Conditions at the Sea Surface

Must know how radiance coming from any direction above or below the surface is reflected and transmitted by the surface into any other direction
.
Requires 4 bi-directional surface reflectance and transmittance functions. Depend strongly on wind speed (i.e., on wave slopes); only weakly on wavelength (via the water index of refraction)

Pre-computed by Monte Carlo simlations of random sea surfaces (>10^5 surface realizations) to get time-averaged functions. Details in *Light and Water* Chapter 4.



H Uses Cox-Munk Wind Speed-Wave Slope Statistics to Model the Sea Surface (C-M Includes Both Capillary and Gravity Waves)



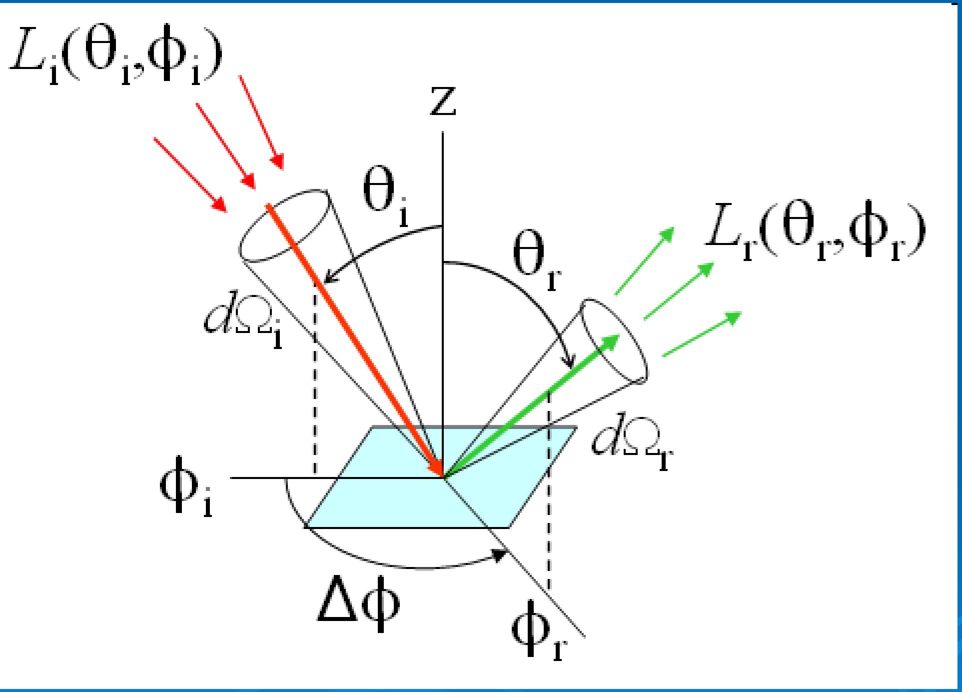
wave **slopes** are correct in HydroLight, wave **heights** are not

See HydroLight Tech Note 1 for more discussion (HE5/Documents/HTN1_SurfaceWaves.pdf)

Boundary Conditions at the Sea Bottom

Must know how the bottom reflects radiance from any downward direction into any upward direction.

Described by the bottom bi-directional reflectance distribtution function, BRDF(θ_i, φ_i, θ_r, φ_r, λ) [units of 1/steradian]



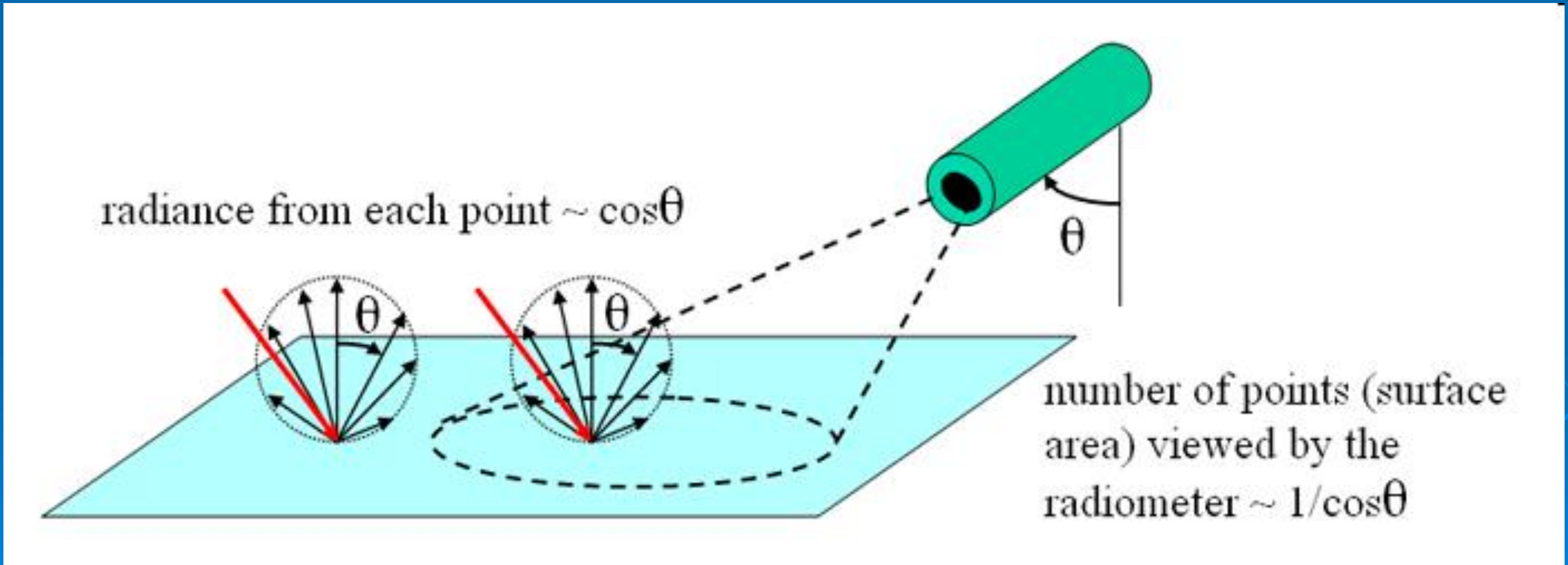
See
www.oceanopticsbook.info/view/radiative_transfer_theory/level_2/the_brdf

Lambertian BRDFs

Physical bottoms are usually assumed to be *Lambertian:*

Each **point** of the surface reflects radiance in a cosine pattern

Viewing many points then gives **radiance reflected equally in all directions**



See
www.oceanopticsbook.info/view/radiative_transfer_theory/level_2/the_lambertian_brdf

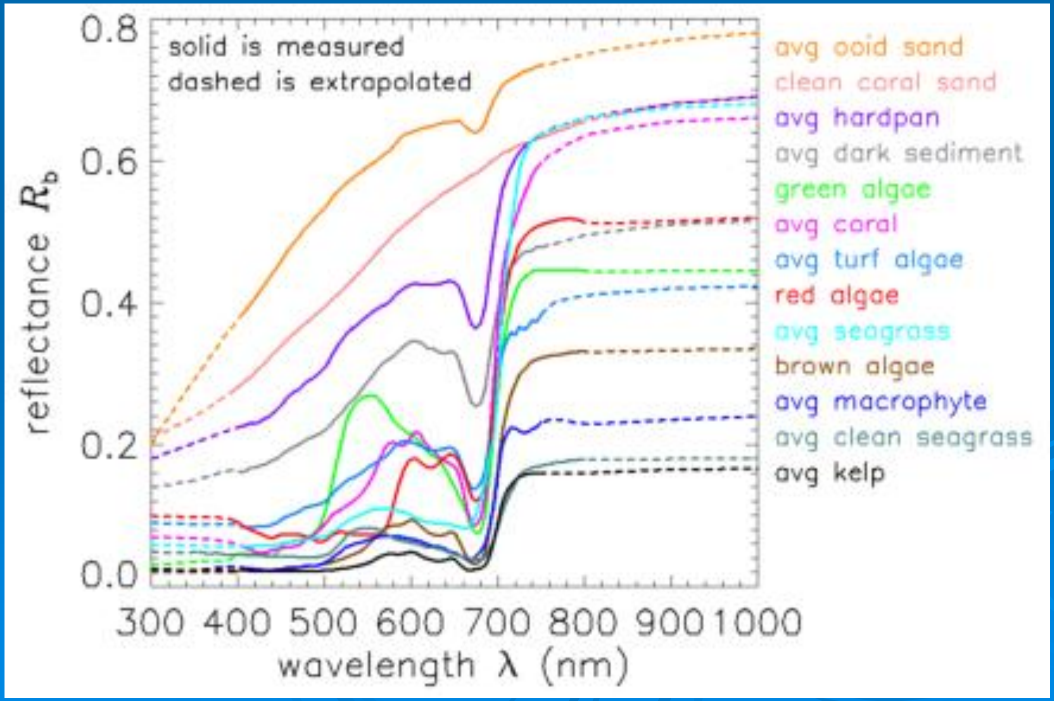
Lambertian BRDFs

The BRDF of a Lambertian reflector is fully specified by it *reflectivity* ρ, which equals the irradiance reflectance $R = E_u/E_d$:

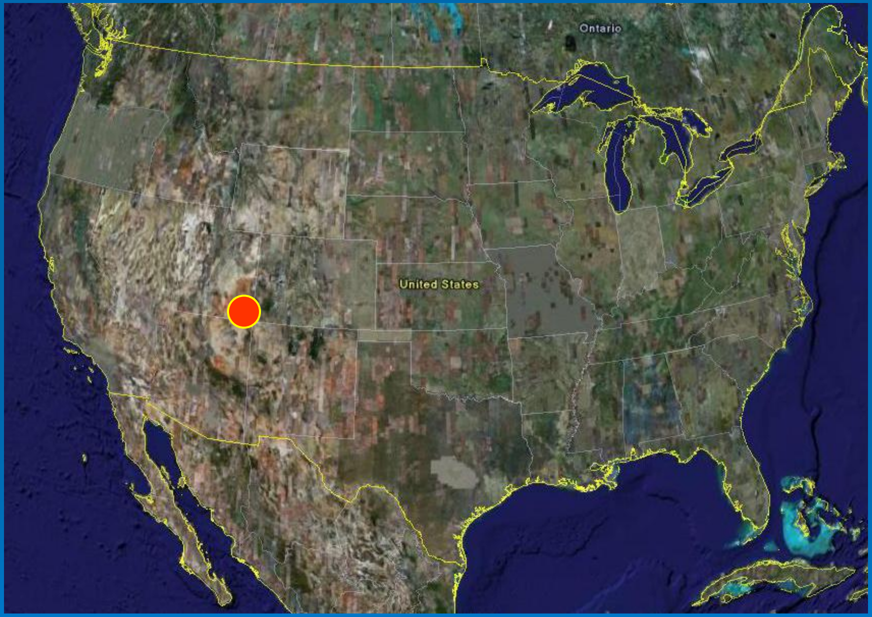
$BRDF(\theta_i, \varphi_i, \theta_r, \varphi_r, \lambda) = \rho(\lambda)/\pi$

ρ = 0 for a “black” surface; ρ = 1 for a “white” surface

The default in HydroLight is to specifiy a bottom reflectance (really $\rho = E_u/E_d$), and H then assumes that the bottom is Lambertian.



It’s always good to go mountain biking in the desert in the springtime after a long, cold, dark, rainy or snowy winter



Approaching Hurrah Pass,
SW of Moab, Utah
photo by Curtis Mobley