Analytical Solution of the RTE:

The Successive-Order-of-Scattering Method
and its baby, the SSA,
and its grandchild, the QSSA

Curt Mobley

At last, we finally get to
do a little bit of math,
instead of just looking at it

Analytic Solution Methods

<u>Predictive models</u>

plug in IOP's, get out *approximate* radiances using simple formulas

<u>Advantages</u>:

physical insight – identify and isolate dominant processes

guidance – in concept formulation and data analysis

simple math – easy to program

adaptable to statistical analysis – e.g. of radiance fluctuations due to surface wave effects

sanity checks on exact numerical models

<u>Disadvantages</u>:

simple situations only – homogeneous water, sun in a black sky, level water surface, etc

inaccurate – factor of 2 to 10 accuracy

From the ***draft*** introduction of "Phase function effects on oceanic light fields" by Mobley, Sundman, and Boss, *Applied Optics* 41, 1035-1057, 2001.

Consider a collimated radiance incident onto the sea surface at polar angle θ_s, which approximates the sky radiance distribution for the sun at zenith angle θ_s in a clear sky. Let θ_w be the angle of the incident (sun's) radiance after refraction through a level sea surface, and let $E_d(0)$ be the plane irradiance just beneath the sea surface. Single-scattering theory[3] then shows that in homogeneous water the upwelling radiance in direction (θ,ϕ) at depth z within the water is given approximately by

$$L_u(\theta,\phi,z) \approx E_d(0)\,\omega_o\,\tilde{\beta}(\theta_w,\phi_w \rightarrow \theta,\phi)\,\frac{1}{\cos\theta_w - \cos\theta}\,\exp\left(-cz/\cos\theta_w\right). \tag{2}$$

Here ϕ is azimuthal direction measured from some convenient reference direction, and θ is measured from 0 at the nadir direction; thus $\theta > 90$ deg for upwelling radiance. The albedo of single scattering, $\omega_o = b/c$, gives the probability of scattering (rather than absorption) in any interaction of a photon with the medium. In ocean waters, ω_o can vary from 0.2 to 0.9 depending on wavelength and water composition. The angle arguments of the phase function denote scattering from the direction of the sun's in-water downwelling beam into the upwelling direction of interest.

The quasi-single-scattering approximation (QSSA) treats forward scattered light as unscattered, in which case[3] $\omega_o = b_b / (a + b_b)$. Here b_b is the backscatter coefficient, which is the integral of the VSF over the hemisphere of backscattering directions, $\psi \geq 90$ deg. For upwelling radiance at $\theta = 180$ deg (the nadir-viewing radiance), Eq. (2) then gives

$$\frac{L_u(z=0)}{E_d(0)} \approx \frac{b_b}{a + b_b}\,\frac{\tilde{\beta}(\psi = 180 - \theta_w)}{\cos\theta_w + 1}. \tag{3}$$

The remote-sensing reflectance R_{rs} is the ratio of the water-leaving radiance to the downwelling plane

H. GORDON in SPINRAD, CARDER, PERRY

where θ_0 and ϕ_0 are the solar zenith and azimuth angles, respectively, and δ is the Dirac delta function. F_0 is the solar irradiance on a plane normal to the sun's rays. The equation for $L^{(0)}$ can be solved immediately, yielding

$$L^{(0)}(z,\theta,\phi) = F_0\delta(\cos\theta - \cos\theta_0)\,\delta(\phi - \phi_0)\exp(-cz/\cos\theta_0) \qquad (1.30)$$

The equation for $L^{(1)}$ then becomes

$$\cos\theta\,\frac{dL^{(1)}}{c\,dz} = -L^{(1)} + \int PL^{(0)'}\,d\Omega' = -L^{(1)} + P(\theta_0,\phi_0\rightarrow\theta,\phi)F_0\exp(-cz/\cos\theta_0) \qquad (1.31)$$

The solution to this equation is

where does this come from??

$$\left\{\begin{aligned} \frac{L_u^{(1)}(z,\theta,\phi)}{E_d(0)} &= \frac{P(\alpha)}{\cos\theta_0 - \cos\theta}\exp(-cz/\cos\theta_0) \\ \frac{L_d^{(1)}(z,\theta,\phi)}{E_d(0)} &= \frac{P(\alpha)}{\cos\theta_0 - \cos\theta}[\exp(-cz/\cos\theta_0) - \exp(-cz/\cos\theta)] \end{aligned}\right. \qquad (1.32)$$

where

$$E_d(0) = F_0\cos\theta_0 \qquad (1.33)$$

and

$$\cos\alpha = \cos\theta\,\cos\theta_0 + \sin\theta\,\sin\theta_0\,\cos\phi \qquad (1.34)$$

The subscripts u and d in Eq. (1.32) mean "up" ($\theta > 90°$) and "down" ($\theta < 90°$), respectively. From these we can compute the first-order or single scattering approximation to the apparent optical properties. As an example we compute $K_d(0)$, the downwelling irradiance attenuation coefficient just beneath the surface. For $cz \ll 1$,

The SUCCESSIVE-ORDER-OF-SCATTERING SOLUTION METHOD $\rightarrow$ SSA $\rightarrow$ QSSA

Consider

- homogeneous water
- plane parallel geometry
- level surface
- point sun in a black sky
- no internal sources

The RTE is then (LEW Eq. 5.24)

$$\mu \frac{dL(\zeta,\mu,\phi)}{d\zeta} = -L(\zeta,\mu,\phi) + \omega_0 \int_{\Xi} \tilde{\beta}(\mu',\phi' \to \mu,\phi) L(\zeta,\mu',\phi')\, d\Omega'$$

$$\zeta \equiv \int_0^z c(z')\,dz' = cz \; ; \; \omega_0 = \frac{b}{c} \; ; \; \mu = \cos\theta \; ; \; d\Omega' = d\mu' d\phi'$$

Assume that we can write

$$L(\zeta,\mu,\phi) = \underbrace{L^{(0)}(\zeta,\mu,\phi)}_{\text{unscattered photons}} + \omega_0 \underbrace{L^{(1)}(\zeta,\mu,\phi)}_{\text{singly-scattered photons}} + \omega_0^2 \underbrace{L^{(2)}(\zeta,\mu,\phi)}_{\text{twice scattered}} + \cdots$$

This will be the SSA solution

Substitute series expansion into RTE:

$$\mu\left[\frac{dL^{(0)}}{d\zeta} + \omega_0 \frac{dL^{(1)}}{d\zeta} + \omega_0^2 \frac{dL^{(2)}}{d\zeta} + \cdots\right] = -\left[L^{0} + \omega_0 L^{(1)} + \cdots\right]$$

$$+ \omega_0 \int_{\Xi} \tilde{\beta}\left[L^{(0)} + \omega_0 L^{(1)} + \omega_0^2 L^{(2)} + \cdots\right] d\Omega'$$

And equate terms with equal powers of ω_0:

$$\mu \frac{dL^{(0)}}{d\zeta} = -L^{(0)} \qquad (0)$$

$$\mu \frac{dL^{(1)}}{d\zeta} = -L^{(1)} + \int_{\Xi} \tilde{\beta} L^{(0)} d\Omega' \qquad (1)$$

$$\mu \frac{dL^{(2)}}{d\zeta} = -L^{(2)} + \int_{\Xi} \tilde{\beta} L^{(1)} d\Omega' \qquad (2)$$

and so on.

We can solve Eq (0) [Beer's law]. The boundary conditions are (collimated beam, black sky)

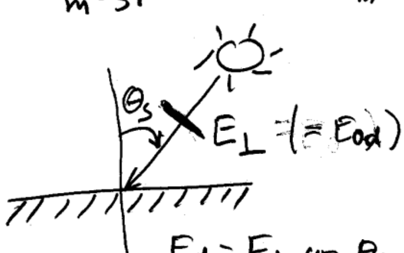
$$L^{(0)}(0,\mu,\phi) = E_\perp(0) \underbrace{\delta(\mu-\mu_s)\,\delta(\phi-\phi_s)}_{\delta(\hat{\xi}-\hat{\xi}_s)} \qquad (BC1)$$

$\frac{W}{m^2 sr}$ $\qquad$ $\frac{W}{m^2}$

↑ element of solid angle centered on $\hat{\xi}_s$

so units are $\frac{1}{sr}$

$\int f(x)\,\delta(x-x_0)\,dx = f(x_0) \Rightarrow [\delta] = \frac{1}{[x]}$



$E_d = E_\perp \cos\theta_s$

and

$$L^{(0)}(\varsigma,\mu,\phi) \rightarrow 0 \text{ as } \varsigma \rightarrow \infty \qquad \text{(BC2)}$$

Rewrite Eq (0) as

$$\frac{dL^{(0)}}{L^{(0)}} = -\frac{d\varsigma}{\mu}$$

$$\ln L^{(0)} \Big|_{L^{(0)}(\varsigma=0)}^{L^{(0)}(\varsigma)} = -\int_0^{\varsigma} \frac{d\varsigma'}{\mu} = -\frac{\varsigma'}{\mu}\Big|_0^{\varsigma}$$

$$\boxed{\begin{aligned} L^{(0)}(\varsigma,\mu,\phi) &= \underbrace{L^{(0)}(0,\mu,\phi)}_{\text{(BC1)}} e^{-\varsigma/\mu} \\ &= \overbrace{E_0(0)\,\delta(\mu-\mu_s)\,\delta(\phi-\phi_s)}\, e^{-\varsigma/\mu} \end{aligned}}$$

Note that $L^{(0)}(\varsigma,\mu,\phi)=0$ if $\mu \neq \mu_s$ or $\phi \neq \phi_s$

Note that $L^{(0)}(\varsigma,\mu,\phi) \rightarrow 0$ as $\varsigma \rightarrow \infty$

So both Bnd Conds are satisfied.

[This solution describes the contribution of unscattered light to the total radiance.]

- Now we can evaluate the scattering term in Eq (1):

$$\int_{\Xi} \tilde{\beta} L^{(0)} d\Omega' = \int_0^{2\pi} \int_{-1}^{1} \tilde{\beta}(\mu', \phi' \to \mu, \phi) \times$$

$$E_d(0)\, \delta(\mu' - \mu_s)\, \delta(\phi' - \phi_s)\, e^{-\zeta/\mu}\, d\mu'\, d\phi'$$

recall that (L&W p21)

$$\int_{-1}^{1} f(\mu)\, \delta(\mu - \mu_0)\, d\mu = f(\mu_0) \text{ etc}$$

so we get

$$\int \tilde{\beta} L^{(0)} d\Omega' = E_d(0)\, \tilde{\beta}(\mu_s, \phi_s \to \mu, \phi)\, e^{+\zeta/\mu_s}$$

= how much of the solar beam reaches depth ζ and then gets scattered into the direction of interest (μ, ϕ)

= a local (at ζ) <u>source term</u> for singly-scattered radiance

So Eq (1) is now

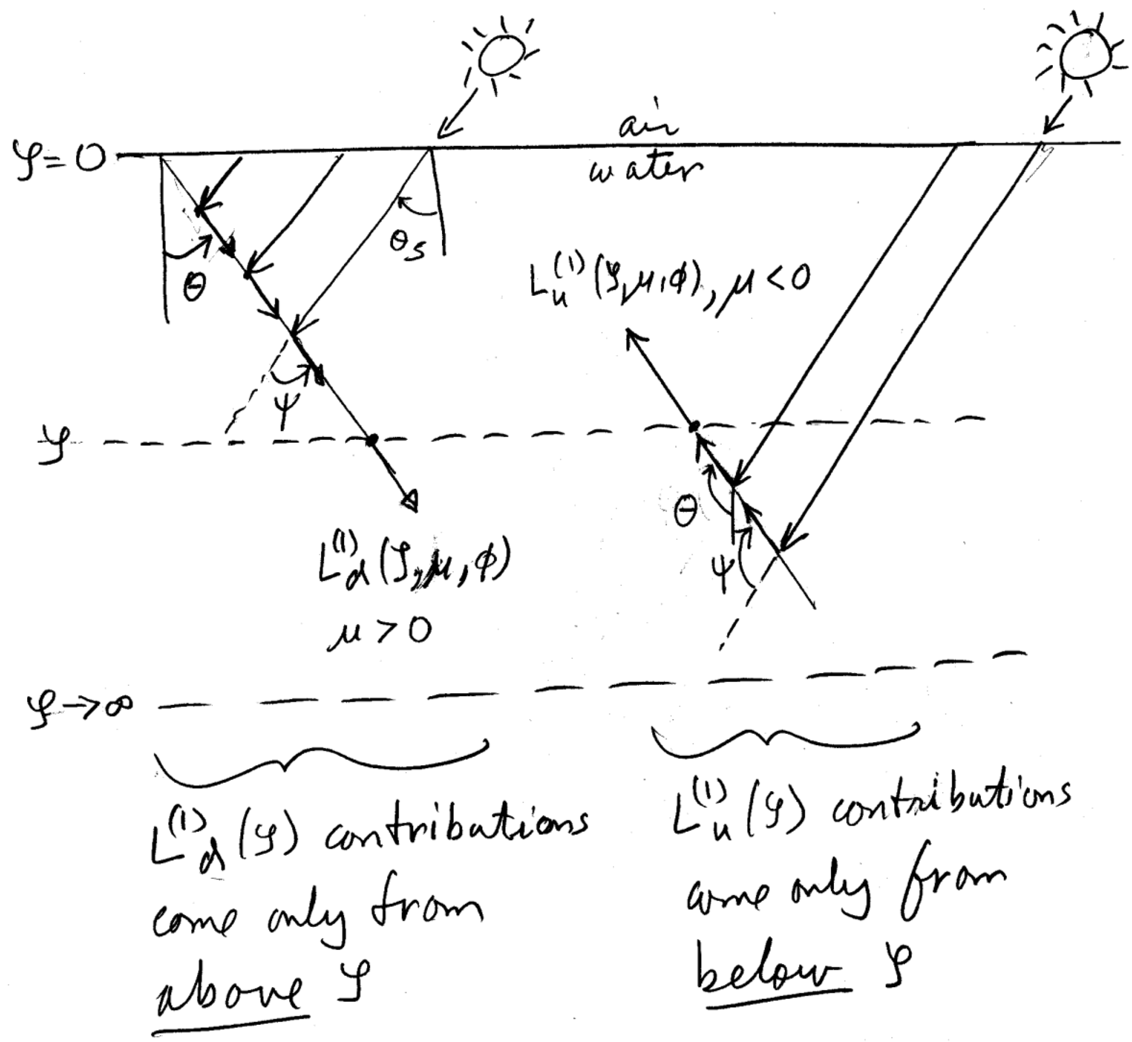
$$\mu \frac{dL^{(1)}}{d\zeta} + L^{(1)} = \underbrace{E_d(0)\,\tilde{\beta}(\mu_s, \phi_s \to \mu, \phi)\, e^{-\zeta/\mu_s}}_{\text{known}} \qquad (1a)$$

This is an ODE (in ζ) with constant coefficients, so we can solve it. The Bnd Conds are

$L^{(1)}(0, \mu, \phi) = 0 \Rightarrow$ no <u>scattered</u> light from sun

$L^{(1)}(\zeta, \mu, \phi) \to 0$ as $\zeta \to \infty$

To integrate (1a), break $L^{(1)}$ into downwelling and upwelling parts:



Eq (1a) has the same form seen in L&W p256.
For $L_d^{(1)}$ we

multiply by $\frac{1}{\mu} e^{\varsigma/\mu}$ (integrating factor)

$$\frac{1}{\mu} e^{\varsigma/\mu}\left[\mu \frac{dL_d^{(1)}}{d\varsigma} + L_d^{(1)}\right] = \frac{1}{\mu} e^{\varsigma/\mu}\left[E_{\perp}\tilde{\beta}\, e^{-\varsigma/\mu_s}\right]$$

$$\frac{d}{d\varsigma}\left[L_d^{(1)} e^{\varsigma/\mu}\right] = \frac{E_{\perp}\tilde{\beta}}{\mu}\, e^{\left(\frac{1}{\mu} - \frac{1}{\mu_s}\right)\varsigma}$$

Now integrate from 0 to ζ:

$$\int_{L_d^{(1)}(0)}^{L_d^{(1)}(\zeta)} d\left[L_d^{(1)} e^{\zeta/\mu}\right] = \frac{E_\perp \tilde{\beta}}{\mu} \int_0^{\zeta} e^{\left(\frac{1}{\mu}-\frac{1}{\mu_s}\right)\zeta'} d\zeta'$$

using $\int e^{ax} dx = \frac{1}{a} e^{ax}$,

$$L_d^{(1)}(\zeta)\, e^{\zeta/\mu} - \underbrace{L_d^{(1)}(0)}_{\substack{\| \\ 0 \text{ by B.C.}}} = \frac{E_\perp \tilde{\beta}}{\mu} \frac{1}{\left(\frac{1}{\mu}-\frac{1}{\mu_s}\right)} \left[e^{\left(\frac{1}{\mu}-\frac{1}{\mu_s}\right)\zeta} - 1\right]$$

mult by $e^{-\zeta/\mu}$ to get

$$L_d^{(1)}(\zeta) = \frac{E_\perp \tilde{\beta}}{\cancel{\mu}} \frac{1}{\frac{\mu_s - \mu}{\cancel{\mu}\mu_s}} \left[\cancel{e^{-\zeta/\mu}}\, \cancel{e^{\zeta/\mu}}\, e^{-\zeta/\mu_s} - e^{-\zeta/\mu}\right]$$

$$= E_\perp \mu_s \tilde{\beta} \frac{1}{\mu_s - \mu} \left[e^{-\zeta/\mu_s} - e^{-\zeta/\mu}\right]$$

For a collimated beam, $E_d(0) = E_\perp(0)\mu_s = E_\perp(0) \cos\theta_s$, so

$$\boxed{L_d^{(1)}(\zeta, \mu, \phi) = E_d(0)\, \tilde{\beta}(\mu_s, \phi_s \to \mu, \phi) \frac{1}{\mu_s - \mu} \left[e^{-\zeta/\mu_s} - e^{-\zeta/\mu}\right]}$$

for $\mu \neq \mu_s$.

This is Eq. (1.32b) in SCP p 14.

For the special case of $\mu = \mu_s$ we get

$$\int d\left[L_d^{(1)} e^{\varsigma/\mu_s}\right] = \frac{\varepsilon_0 \tilde{\beta}(\mu_s, \phi_s \to \mu_s, \phi)}{\mu_s} \underbrace{\int_0^{\varsigma} \underbrace{e^{\left(\frac{1}{\mu_s} - \frac{1}{\mu_s}\right)\varsigma'}}_{1} d\varsigma'}_{\varsigma}$$

note $\mu = \mu_s$ but $\phi \neq \phi_s$, so $\psi \neq 0$

$$\boxed{L_d^{(1)}(\varsigma, \mu_s, \phi) = \varepsilon_0 \tilde{\beta}(\mu_s, \phi_s \to \mu_s, \phi) \frac{\varsigma}{\mu_s} e^{-\varsigma/\mu_s}}$$

Comments:

1. If you didn't need to know this stuff in order to read the literature, I wouldn't be giving this lecture.

2. Now you know why they told you to take differential equations as an undergraduate.

For the upwelling radiance, integrate (1a) from ζ to ∞. Note also that $\mu < 0$ since $\theta > \pi/2$.

$$\int_{\zeta}^{\infty} d\left[L_u^{(1)} e^{\zeta/\mu}\right] = \frac{E_I \tilde{\beta}}{\mu} \int_{\zeta}^{\infty} e^{\left(\frac{1}{\mu} - \frac{1}{\mu_s}\right)\zeta'} d\zeta'$$

$$\underbrace{L_u^{(1)}(\infty)}_{0} \underbrace{e^{-\infty/|\mu|}}_{0} - L_u^{(1)}(\zeta) e^{\zeta/\mu} = \frac{E_I \tilde{\beta}}{\mu} \frac{1}{\left(\frac{1}{\mu} - \frac{1}{\mu_s}\right)} e^{\left(\frac{1}{\mu} - \frac{1}{\mu_s}\right)\zeta'} \Big|_{\zeta}^{\infty}$$

by B.C.

$$-L_u^{(1)}(\zeta) e^{\zeta/\mu} = E_I \tilde{\beta} \frac{\mu_s}{\mu_s - \mu} \left[e^{\left(\frac{1}{\mu} - \frac{1}{\mu_s}\right)\infty} - e^{\left(\frac{1}{\mu} - \frac{1}{\mu_s}\right)\zeta} \right]$$

<0 <0

so $e^{-\infty} = 0$

so we get

$$\boxed{L_u^{(1)}(\zeta) = E_d(0)\, \tilde{\beta}(\mu_s, \phi_s \to \mu, \phi) \frac{1}{\mu_s - \mu} e^{-\zeta/\mu_s}}$$

This is SCP Eq. (1.32a) on p 14.

We now have the radiance distribution in the SSA:

$$L^{(SSA)} = L^{(0)} + \omega_0 L^{(1)}$$

which is

$$L_d^{(SSA)} = \frac{\omega_o E_d(0)\tilde{\beta}(\psi)}{\mu_s - \mu}\left[e^{-\zeta/\mu_s} - e^{-\zeta/\mu}\right]$$

for $\mu > 0$ and $\mu_s \neq \mu$ [Note: $L_d^{(0)} = 0$ if $\mu \neq \mu_s$]

$$L_d^{(SSA)} = L^{(0)}(0, \mu_s, \phi_s)\, e^{-\zeta/\mu_s} \quad \text{if } \mu = \mu_s \text{ and } \phi = \phi_s$$

$$L_d^{(SSA)} = \omega_o E_d(0)\,\tilde{\beta}(\mu_s, \phi_s \to \mu_s, \phi)\,\frac{\zeta}{\mu_s}\, e^{-\zeta/\mu_s}$$

if $\mu = \mu_s$ and $\phi \neq \phi_s$

Eq. 2 of Mobley, Sundman & Boss

$$L_u^{(SSA)} = \frac{\omega_o E_d(0)\tilde{\beta}(\psi)}{\mu_s - \mu}\, e^{-\zeta/\mu_s} \quad \text{(Note: } \mu < 0\text{)}$$

But remember: we get this solution only for a collimated input beam. The δ-function made the source-term evaluation easy. Any other boundary condition [even a uniform $L^{(0)}(0, \mu, \phi)$] makes the source-term evaluation very difficult. Also, the infinitely deep water removed any bottom effects.

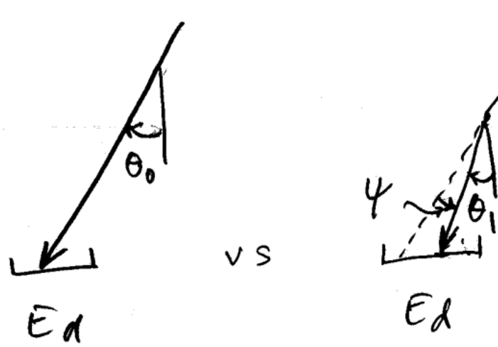
A fun spring-break project:

Program the SSA and compare its predictions of the radiance with those of HydroLight (which includes multiple scattering) for highly absorbing water, highly scattering water, etc.

See when the SSA works well, and when it doesn’t.

The Quasi-Single-Scattering Approximation (QSSA)

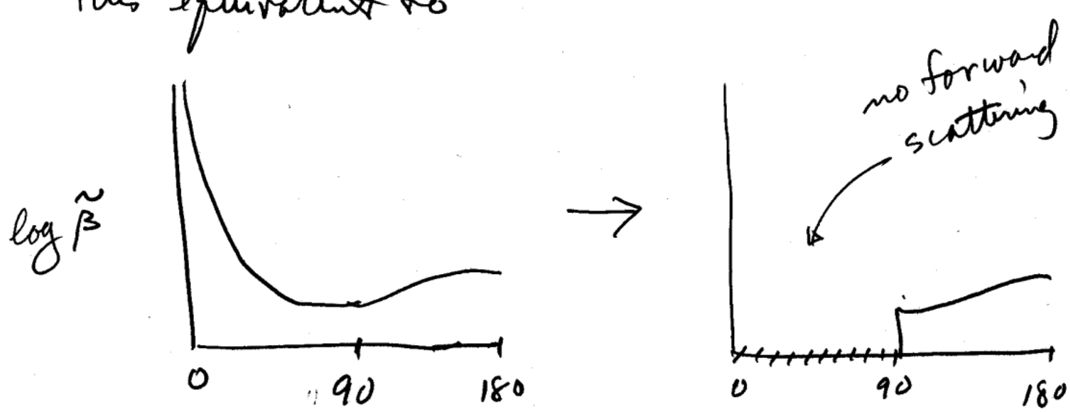
Argument: For highly peaked oceanic phase functions, most of the scattering is through very small angles ψ. But scattering through a small angle is almost the same as no scattering at all, e.g.



almost the same E_d since $\theta_0 \approx \theta_1$

The QSSA says: "Use the formulas from the SSA, but treat forward scattering as no scattering at all."

This equivalent to



or $\boxed{c = a + b = a + \underset{\overset{\parallel}{0}}{b_f} + b_b = \boxed{a + b_b}}$

To be consistent with the approximation

$$c \approx a + b_b$$

we have

$$\omega_0 \equiv \frac{b}{c} = \frac{\overset{\approx 0}{b_f} + b_b}{a + \underset{0}{b_f} + b_b} \approx \frac{b_b}{a + b_b}$$

and

$$\varphi = cz = (a+b)z \approx (a + b_b)z$$

So the QSSA says: use the SSA equations, but make the replacements

$$c \longrightarrow c^* = a + b_b$$

$$\omega_0 \longrightarrow \omega_0^* = \frac{b_b}{a + b_b}$$

$$\varphi \longrightarrow \varphi^* = (a + b_b)z$$

This is an example of a <u>similarity transformation</u>:

Solve one problem, then <u>rescale</u> the variables to get the solution of another problem.

Now let's compute R_{rs} in the QSSA.

From the SSA solution for $L_u^{(SSA)}$ on page (10),

$$R_{rs} \equiv \frac{L_w(\text{air})}{E_d(\text{air})} \sim \frac{L_u(\text{water})}{E_d(\text{water})}$$

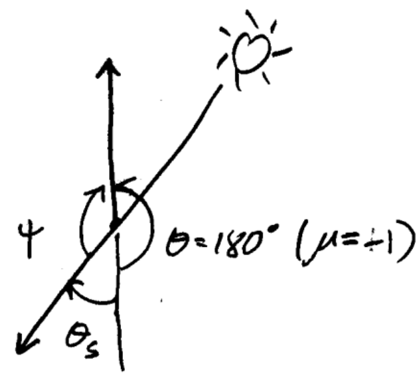
don't worry about factors of 2

$$R_{rs}^{(SSA)} \sim \frac{L_u^{(SSA)}(\text{water})}{E_d(0)} = \frac{\omega_0 \tilde{\beta}(\psi)}{\mu_s - \mu} e^{-cz/\mu_s}$$

but $\omega_0 \tilde{\beta}(\psi) = \frac{b}{c} \tilde{\beta}(\theta_s, \phi_s \to \theta, \phi)$

$$\xrightarrow[\text{QSSA}]{} \omega_0^* \tilde{\beta}(\psi) = \frac{b_b}{a+b_b} \tilde{\beta}$$

for $\psi > 90°$ (take $\theta = 180°$)



So

$$R_{rs}^{(QSSA)} \sim \frac{b_b}{a+b_b} \underbrace{\frac{\tilde{\beta}(\psi)}{\mu_s - \mu}}_{\substack{\text{constant}\\ \text{for given}\\ \mu_s, \mu}} \underbrace{e^{-cz/\mu_s}}_{\substack{\approx 1 \text{ near the}\\ \text{surface, } z \approx 0}}$$

At z = 0, gives Eq. 3 of Mobley, Sundman & Boss

So we are left with the functional dependence

$$\boxed{R_{rs}^{(QSSA)} \sim \frac{b_b}{a+b_b}}$$

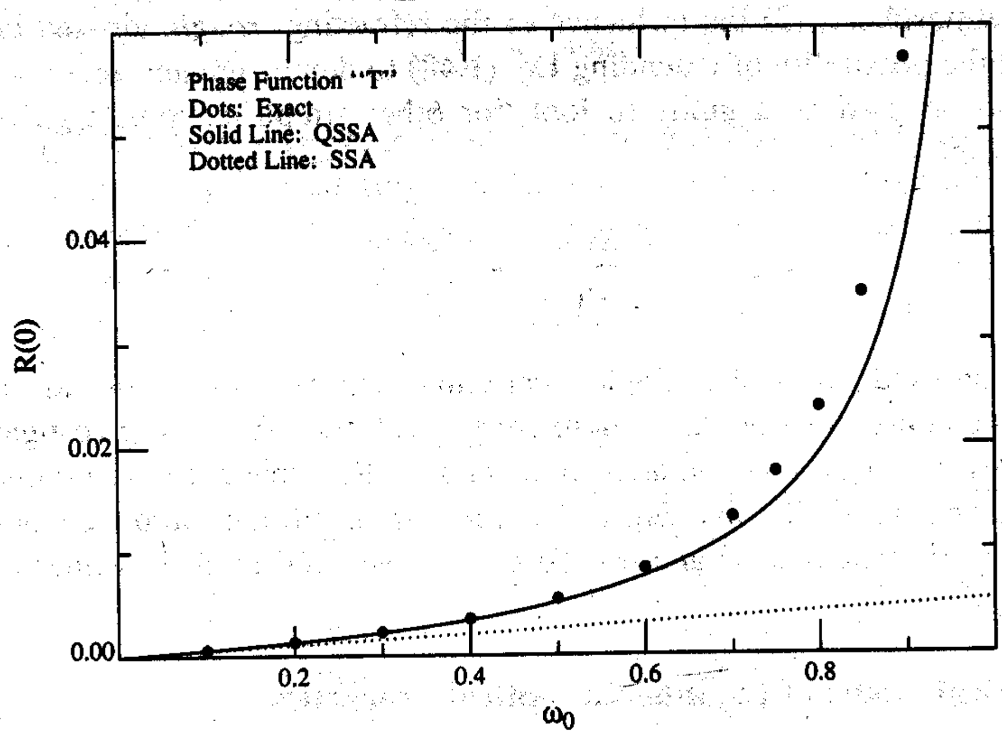


Fig. 1-3. Comparison between exact, quasi-single scattering approximation, and single scattering approximation computations of $R(0)$.

from H. Gordon in Spinrad, Carder, and Perry, op. cit.

A fun summer-vacation project:

Program R_{rs} as given by the QSSA (keeping the factors of 2) and compare its predictions of R_{rs} with those of HydroLight for highly absorbing water, highly scattering water, etc.

See when the QSSA works well, and when it doesn't.

You can go another step in the SSA:

Table 2-6.2. Downwelling and upwelling radiance algorithms for the multiple-scattering hybrid radiometric model for nonconservative scattering ($w_0 \neq 1$)

$$L_d(\tau^*, \hat{s}) = w_0^* F_0 \mu_0 \frac{p_w(\hat{s}\cdot\hat{s}_0)}{4\pi} \frac{\exp(-\tau^*/\mu_0) - \exp(-\tau^*/\mu)}{\mu_0 - \mu}$$

$$+ \left\{ \begin{aligned} & \frac{w_0^* \mu_0}{4\pi}(A + a)\frac{\exp(-\tau^*/\mu_0) - \exp(-\tau^*/\mu)}{\mu_0 - \mu} \\ & + \frac{w_0^*}{4\pi}(B + b)\frac{\exp[\tau^*(\kappa_1 + \kappa_2)] - \exp(-\tau^*/\mu)}{1 + \mu(\kappa_1 + \kappa_2)} \\ & + \frac{w_0^*}{4\pi}(C + c)\frac{\exp[\tau^*(\kappa_1 - \kappa_2)) - \exp(-\tau^*/\mu)}{1 + \mu(\kappa_1 - \kappa_2)} \end{aligned} \right\}$$

$$L_u(\tau^*, \hat{s}) = w_0^* F_0 \mu_0 \frac{p_w(\hat{s}\cdot\hat{s}_0)}{4\pi} \exp(-\tau^*/\mu_0) \frac{1 - \exp[-(\tau_0^* - \tau^*)(1/\mu_0 + 1/(-\mu))]}{\mu_0 + (-\mu)}$$

$$+ \left\{ \begin{aligned} & \frac{w_0^* \mu_0}{4\pi}(A + a)\exp(-\tau^*/\mu_0)\frac{1 - \exp\{-(\tau_0^* - \tau^*)[1/\mu_0 + 1/(-\mu)]\}}{\mu_0 + (-\mu)} \\ & + \frac{w_0^*}{4\pi}(B + b)\exp[\tau^*(\kappa_1 + \kappa_2)]\frac{1 - \exp\{-(\tau_0^* - \tau^*)[-(\kappa_1 + \kappa_2) + 1/(-\mu)]\}}{1 - (\kappa_1 + \kappa_2)(-\mu)} \\ & + \frac{w_0^*}{4\pi}(C + c)\exp[\tau^*(\kappa_1 - \kappa_2))]\frac{1 - \exp[-(\tau_0^* - \tau^*)[-(\kappa_1 - \kappa_2) + 1/(-\mu)]}{1 - (\kappa_1 - \kappa_2)(-\mu)} \end{aligned} \right\}$$

from Walker, Marine Light Field Statistics
Wiley, 1994, 675 pages.

Table 2-6.1. Integration constants for the hybrid radiometric model

$$A = F_0\mu_0 \frac{\tfrac{1}{2} w_0^*}{(\mu_0 - \bar{\mu}_d) - \tfrac{1}{2} w_0^* \mu_0 \left[1 + \dfrac{\mu_0 - \bar{\mu}_d}{\mu_0 + \mu_u}\right]} \qquad w_0^* \neq 1$$

$$= -F_0\mu_0 (1 + 2\mu_0) \qquad w_0^* = 1$$

$$B = -A \frac{\gamma \exp[\tau_0^* (\kappa_1 - \kappa_2)] - \alpha \exp(-\tau_0^* / \mu_0)}{\gamma \exp[\tau_0^* (\kappa_1 - \kappa_2)] - \beta \exp[\tau_0^* (\kappa_1 + \kappa_2)]}$$

$$C = -A \frac{\beta \exp[\tau_0^* (\kappa + \kappa_2)] - \alpha \exp(-\tau_0^* / \mu_0)}{\beta \exp[\tau_0^* (\kappa_1 + \kappa_2)] - \gamma \exp[\tau_0^* (\kappa_1 - \kappa_2)]}$$

$$D = -F_0\mu_0 \frac{(1 + 2\mu_0) + (1 - 2\mu_0) \exp(-\tau_0^* / \mu_0)}{1 + \tau_0^*}$$

$$E = F_0\mu_0 (1 + 2\mu_0)$$

$$a = A \frac{\mu_0 - \bar{\mu}_d}{\mu_0 + \mu_u} = A\alpha \qquad b = B \frac{(1 - \tfrac{1}{2} w_0^*) + \bar{\mu}_d(\kappa_1 + \kappa_2)}{\tfrac{1}{2} w_0^*} = B\beta$$

$$c = C \frac{(1 - \tfrac{1}{2} w_0^*) + \bar{\mu}_d(\kappa_1 - \kappa_2)}{\tfrac{1}{2} w_0^*} = C\gamma \qquad d = D \qquad e = D + E$$

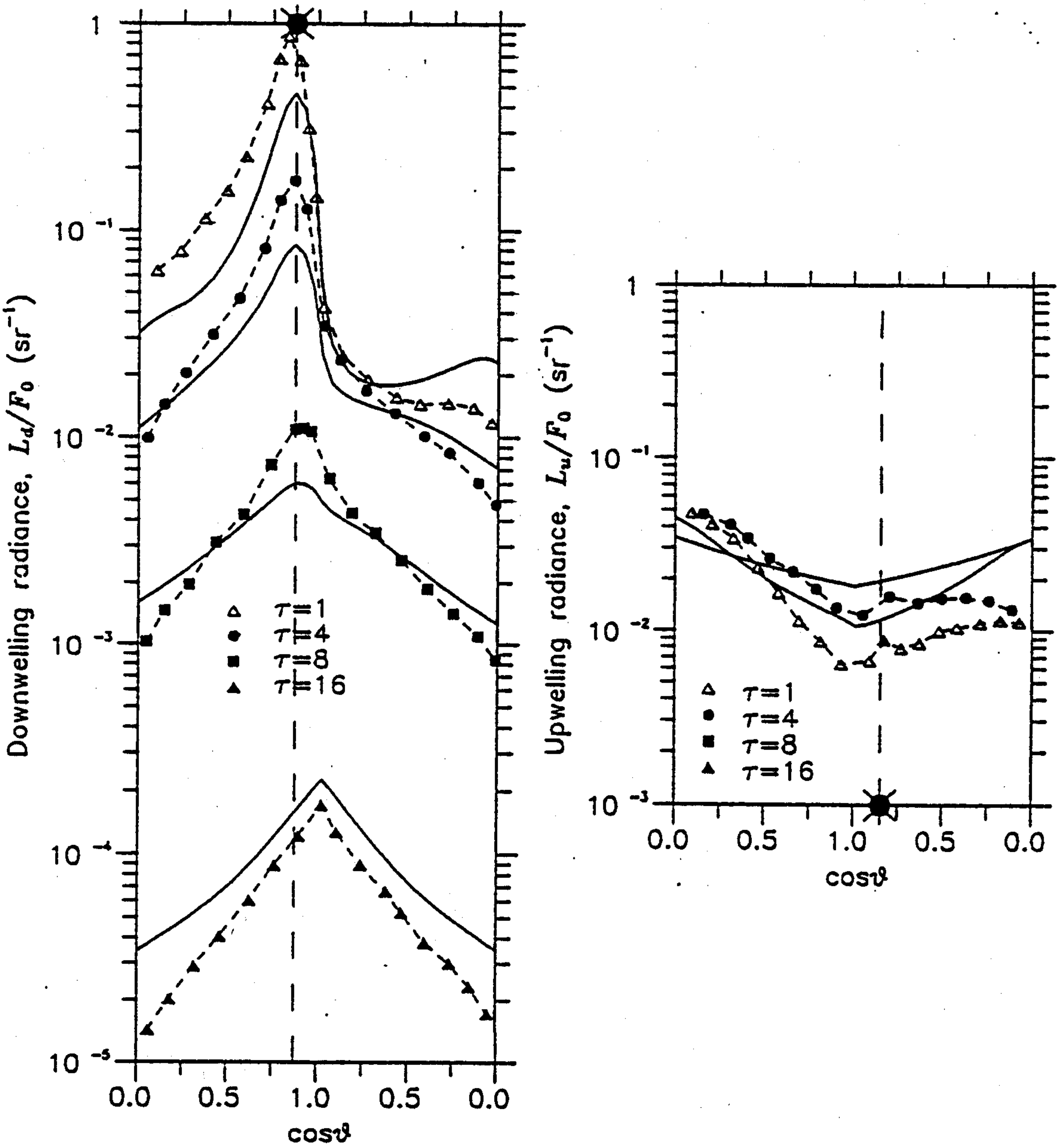


Fig. 2-6.2. Top and bottom radiances for a haze-L scattering medium of finite optical thickness. Symbols are from Kattawar, Plass, and Hitzfelder (1976).

Going any further with the SSA becomes massively complicated, and beyond usefulness. Just use a numerical solution method!