2013 Summer Course

on Optical Oceanography, Remote Sensing, Radiative Transfer Theory, and HydroLight

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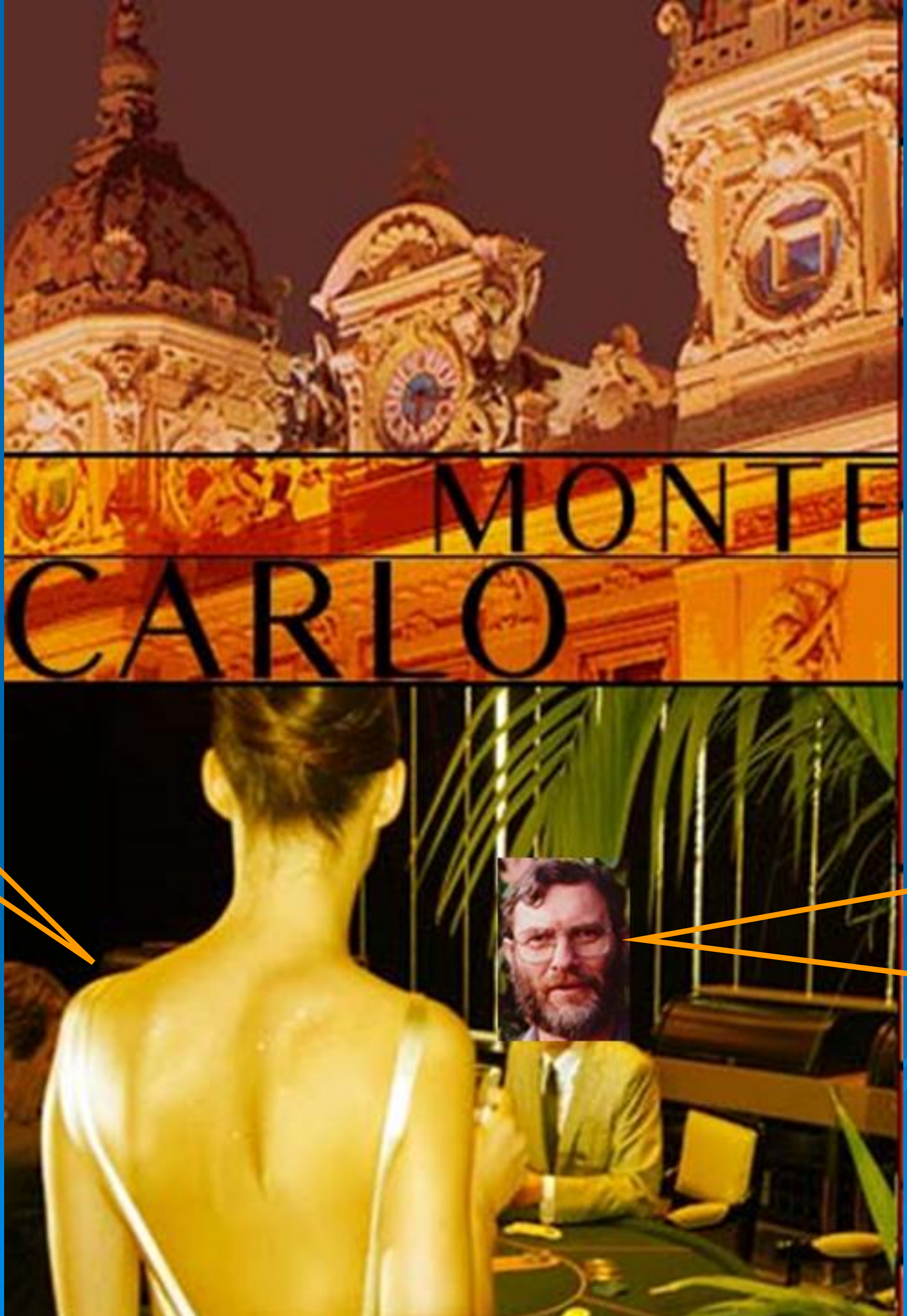
Monte Carlo Simulation

Delivered at the Darling Marine Center,
University of Maine
July 2013



Hey Curt, wanna go to my place and, uh, talk about radiative transfer theory?

Not tonight. I'm still debugging my new Monte Carlo code

Monte Carlo Techniques for Solving the RTE

- The basic idea is to mimic nature in the generation and tracing of photons

- Build up a solution to the RTE one photon at a time

- The tools for doing this are basic probability theory and a random number generator

Points to be covered:

- PDFs and CDFs

- Random number generators

- Using CDFs to randomly select distances, angles, etc.

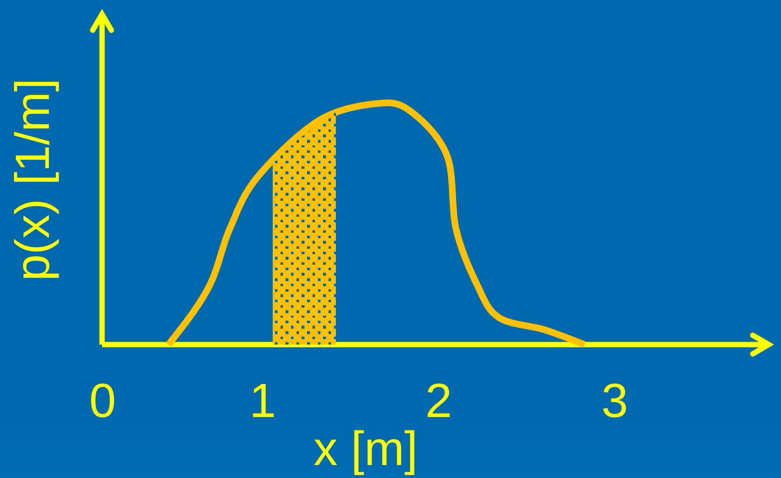
- Monte Carlo noise

There are web book pages on Monte Carlo techniques starting at http://www.oceanopticsbook.info/view/radiative_transfer_theory/level_2/monte_carlo_techniques_introduction

Probability Density Functions

A *probability density function* (PDF) is a non-negative function *p*(*x*) such that the probability that its variable *x* is between *x* and *x*+*dx* is *p*(*x*)*dx*.

Example: *x* = height of adult humans



Prob that a person selected at random from all humans is between 1.0 and 1.3 m tall is

$$\int_{1.0}^{1.3} p(x)dx$$

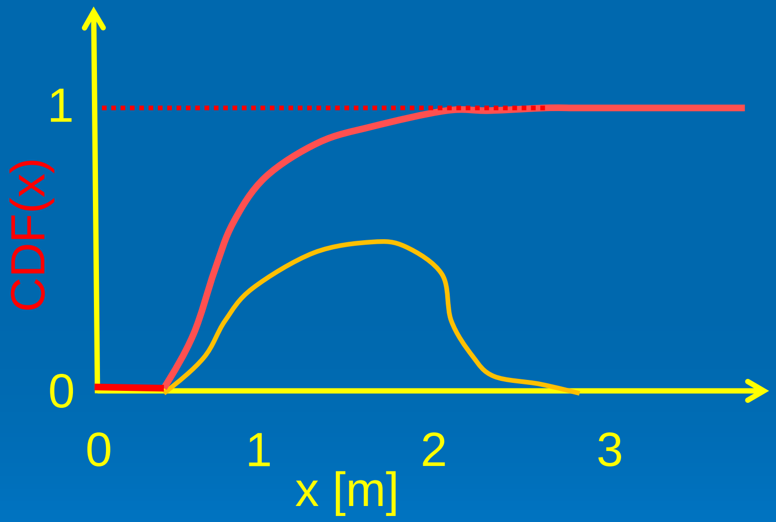
Normalization: $\int_0^\infty p(x)dx = 1$ That is, the prob is one that a person will have some height between 0 and ∞

Units of *p*(*x*) are always 1/[*x*]

Cumulative Distribution Functions

A *cumulative distribution function* (CDF) is a non-negative function *CDF*(*x*) such that the probability that its variable has a value <= *x* is CDF(*x*). For the human height example,

$$CDF(x) = \int_0^x p(x')dx'$$



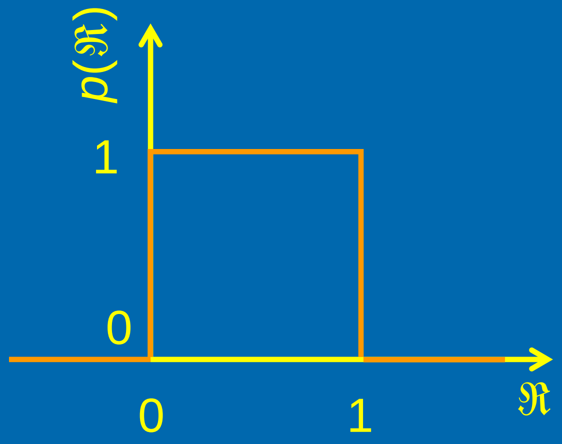
Prob that a person selected at random from all humans is between 1.0 and 1.3 m tall is

CDF(1.3) – CDF(1.0)

Note that CDF(∞) = 1. That is, the prob is one that a person will have some height less than ∞

U(0,1) Random Number Generators

A Uniform 0-1 random number generator is anything (usually a computer program) that when called returns a number $\Re$ between 0 and 1 with equal probability of returning any value $0 < \Re < 1$. $\Re \sim U(0,1)$



0.6314325330
0.2641695440
0.7653187510
0.3009850980
0.9278188350
0.0138932914
0.3010187450
0.1198131440
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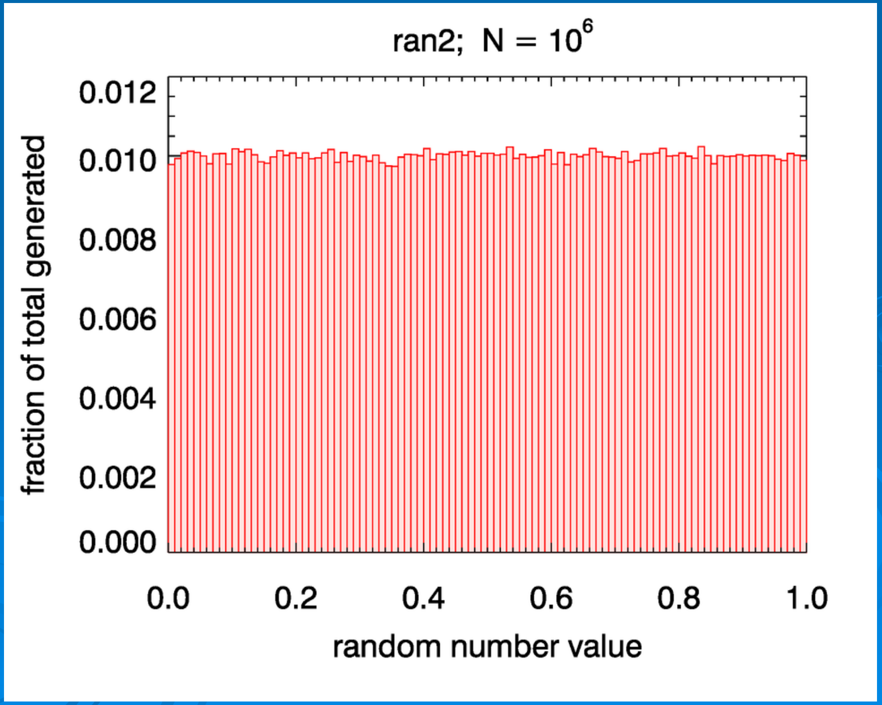
0.1382016390
0.1065650730



Random Determination of Photon Path Lengths

Recall Beer's law (for a collimated beam in a dark ocean):

$L(r) = L(0) \exp(-cr) = L(0) \exp(-\tau)$

The exponential decay of radiance can be explained if the individual photons have a probability of being absorbed or scattered out of the beam between τ and $\tau + d\tau$ that is

$p(\tau)d\tau = \exp(-\tau)\, d\tau \Rightarrow p(\tau) = \exp(-\tau)$

We want to use our U(0,1) random number generator to randomly determine photon path lengths τ that obey the pdf $p(\tau) = \exp(-\tau)$. Going from $\Re$ to τ is a change of variables:

$$p(\Re)d\Re = p(\tau)d\tau$$

$$\int_0^{\Re} p(\Re')d\Re' = \int_0^{\tau} p(\tau')d\tau'$$

$$\Re = \mathrm{CDF}(\tau) = 1 - \exp(-\tau)$$

Random Determination of Photon Path Lengths

Solving

$\Re = 1 - \exp(-\tau)$ for τ

gives

$\tau \;\; = -\ln(1- \Re) = -\ln \Re$

Draw a U(0,1) random number $\Re$, and then the corresponding photon path is

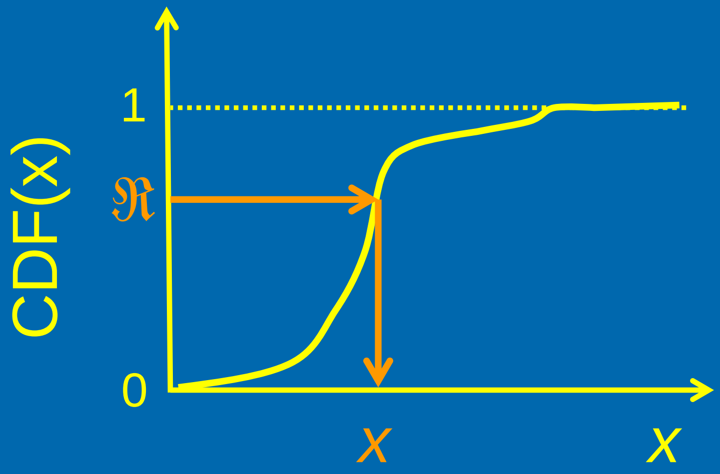
$\tau \;\; = -\ln \Re$

or

$r = -(1/c) \ln \Re$ for distances *r* in meters.

Fundamental Principle of MC Simulation

The equation $\Re$ = CDF(*x*) uniquely determines *x* such that *x* obeys the corresponding pdf *p*(*x*)



General procedure:

1. Figure out the pdf *p*(*x*) that governs the variable of interest, *x*
2. Compute the corresponding CDF(*x*)
3. Draw a U(0,1) random number $\Re$
4. Solve $\Re$ = CDF(*x*) for *x*
5. Repeat steps 3 and 4 many, many, many times to generate a sample of *x* values that reproduces the behavior of *x* in nature

Photon Mean Free Path

The pdf for the distance a photon travels is $p(\tau) = \exp(-\tau)$. What is the average distance $\langle \boldsymbol{\tau} \rangle$ that a photon travels? Called the mean free path.

$$\langle \tau \rangle \equiv \int_0^\infty \tau \, p(\tau) d\tau = \int_0^\infty \tau \, e^{-\tau} \, d\tau = 1$$

or, since τ = cr,

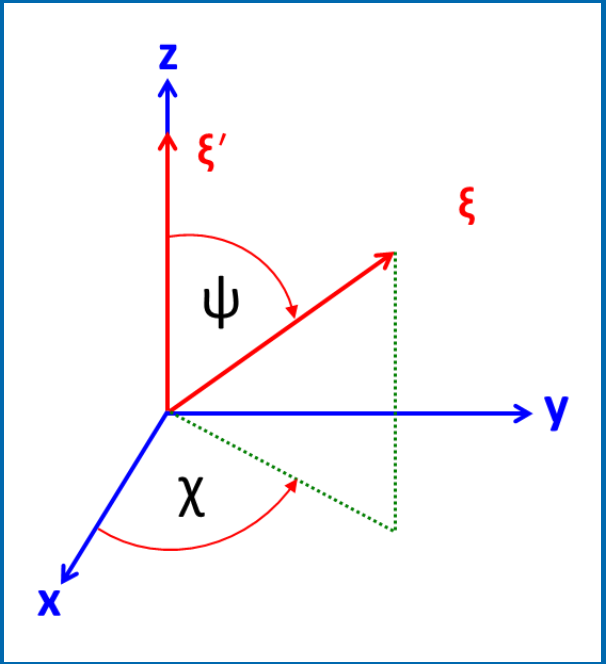
$\langle \boldsymbol{r} \rangle$ = 1/c (meters)

What is the variance about the mean distance traveled?

$$\sigma^2(\tau) \equiv \int_0^\infty [\tau - \langle \tau \rangle]^2 e^{-\tau} d\tau = 1$$

so the standard deviation is also 1/c (meters)

Random Determination of Scattering Angles



Scattering is inherently 3D:

ψ is polar scattering angle

χ is azimuthal scattering angle

$$\int_{4\pi} \tilde{\beta}(\psi', \chi' \to \psi, \chi)\, d\Omega(\psi, \chi) = 1$$

phase functions can be interpreted as pdfs for scattering from (ψ′, χ′) to (ψ, χ)

$$d\Omega(\psi, \chi) = \sin\psi\, d\psi\, d\chi$$

Random Determination of Scattering Angles

For isotropic media and unpolarzed light, ψ and χ are independent, so the bivariate pdf is the product of 2 pdfs:

$$\tilde{\beta}(\psi, \chi)\ \sin\psi\, d\psi\, d\chi = p_\Psi(\psi)\, d\psi\, p_X(\chi)\, d\chi$$

Any azimuthal angle $0 \leq \chi < 2\pi$ is equally likely:

$$p_X(\chi) = 1/(2\pi); \quad CDF_X(\chi) = \chi/(2\pi); \quad \chi = 2\pi\mathfrak{R}$$

$$p_\Psi(\psi) = 2\pi\tilde{\beta}(\psi)\sin\psi$$

$$2\pi\int_0^\pi \tilde{\beta}(\psi)\sin\psi\, d\psi = 1$$ √

$$\mathfrak{R} = CDF(\psi) = 2\pi\int_0^\psi \tilde{\beta}(\psi')\ \sin\psi'\, d\psi'$$

solve for ψ
(usually must solve numerically)

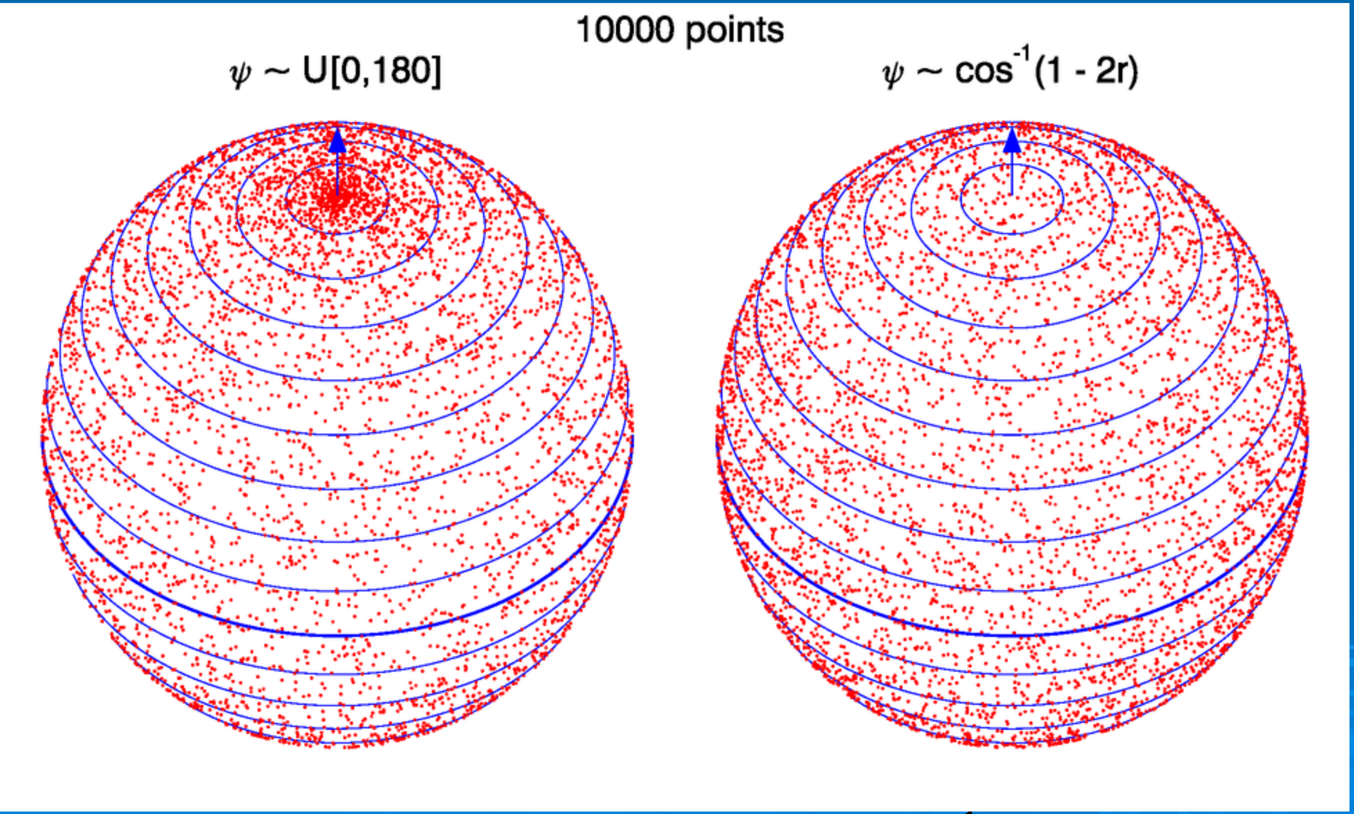
Example: Isotropic Scattering

For isotropic scattering,

$$\tilde{\beta}(\psi) = \frac{1}{4\pi}$$

which gives

$$\psi = \cos^{-1}(1 - 2\mathfrak{R})$$



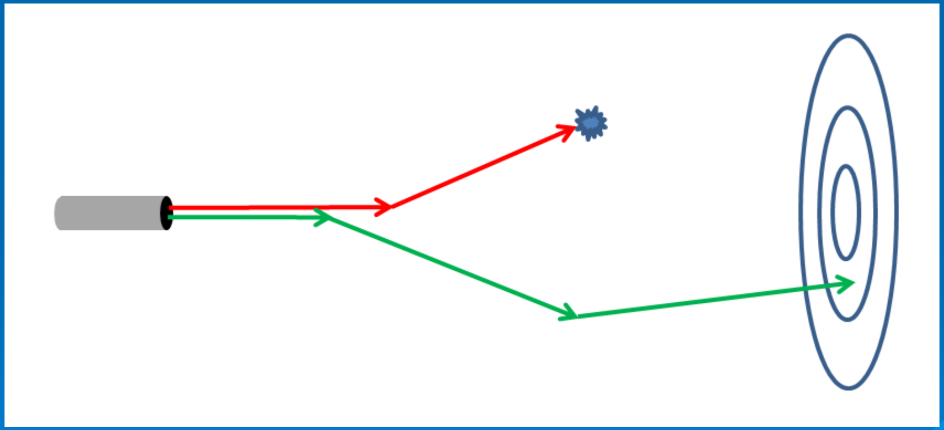
Isotropic means equally likely to scatter into any element of solid angle, not equally likely to scatter through any polar scattering angle ψ

Tracing Photon Packets

The albedo of single scattering, $\omega_o = b/c$, is the probability that a photon will be scattered, rather than absorbed in any interaction

What nature does:

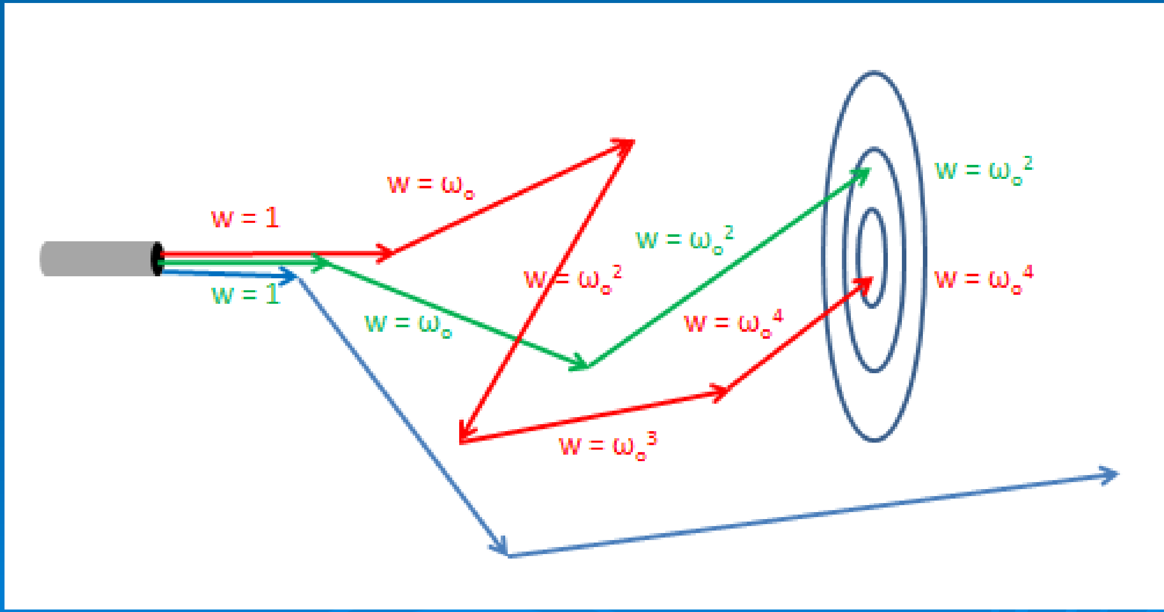
- draws a random number and gets the distance
- draws another random number and compares with ω_o :
  - if $\Re > \omega_o$ the photon is absorbed; start another one
  - if $\Re \leq \omega_o$ the photon is scattered; compute the scattering angles



Any photon that is absorbed never contributes to the answer and is wasted computation. Nature can afford to waste photons; scientists can't.

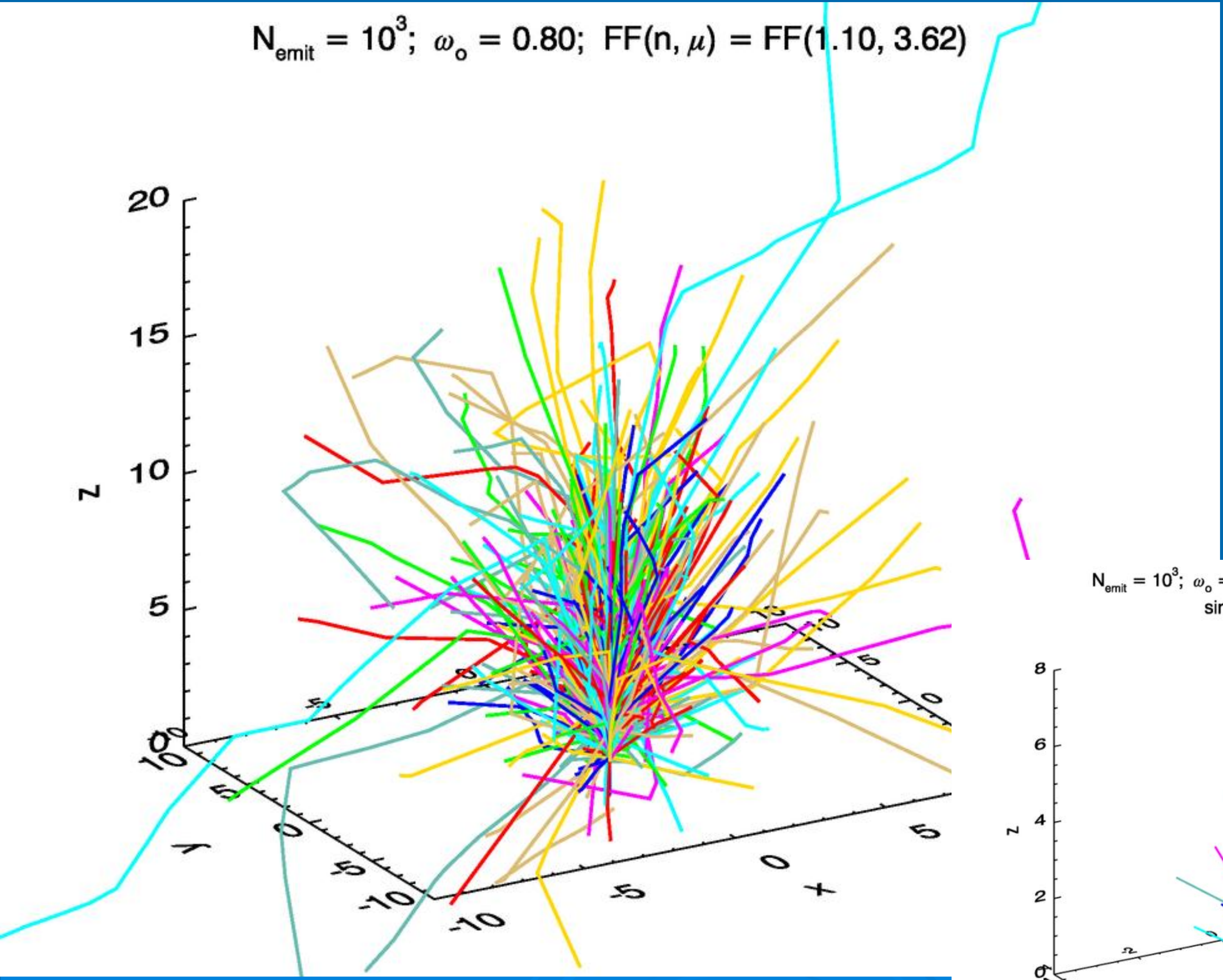
Tracing Photon Packets

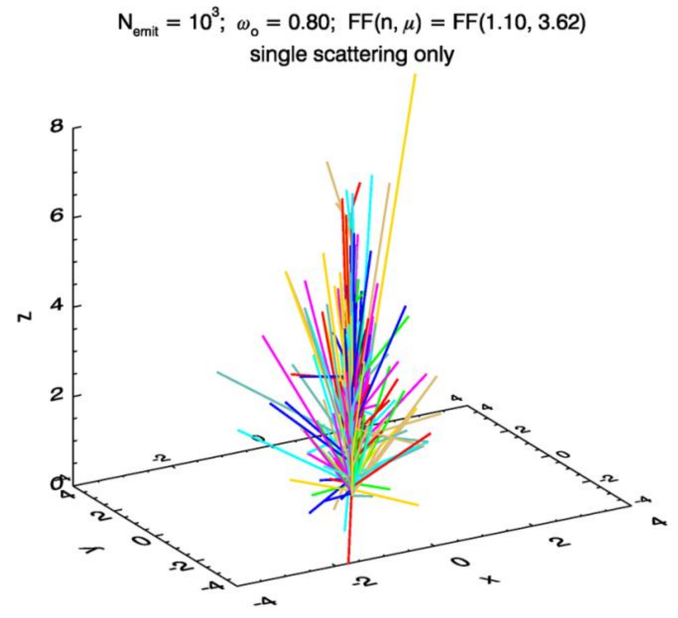
Rather that lose some photons to absorption, consided each "photon" to be a packet of many photons starting with power *w* = 1 W. At each interaction, multiply the current packet weight *w* by ω_o to account for loss of some of the original power to absorption. This increases the number of photon packets that contribute to the answer (although some may still miss the target).



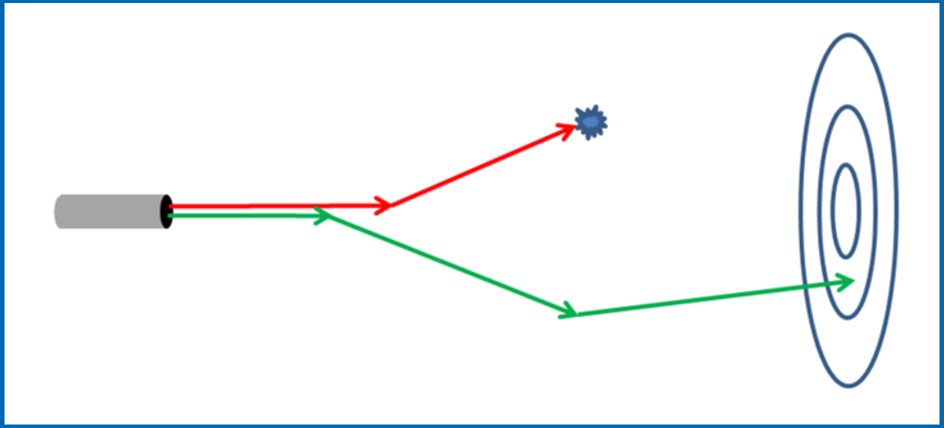
Usually kill the photon packet when w < 10^{-8}, for example, if it hasn't hit the target.

Visualizing Photon Paths

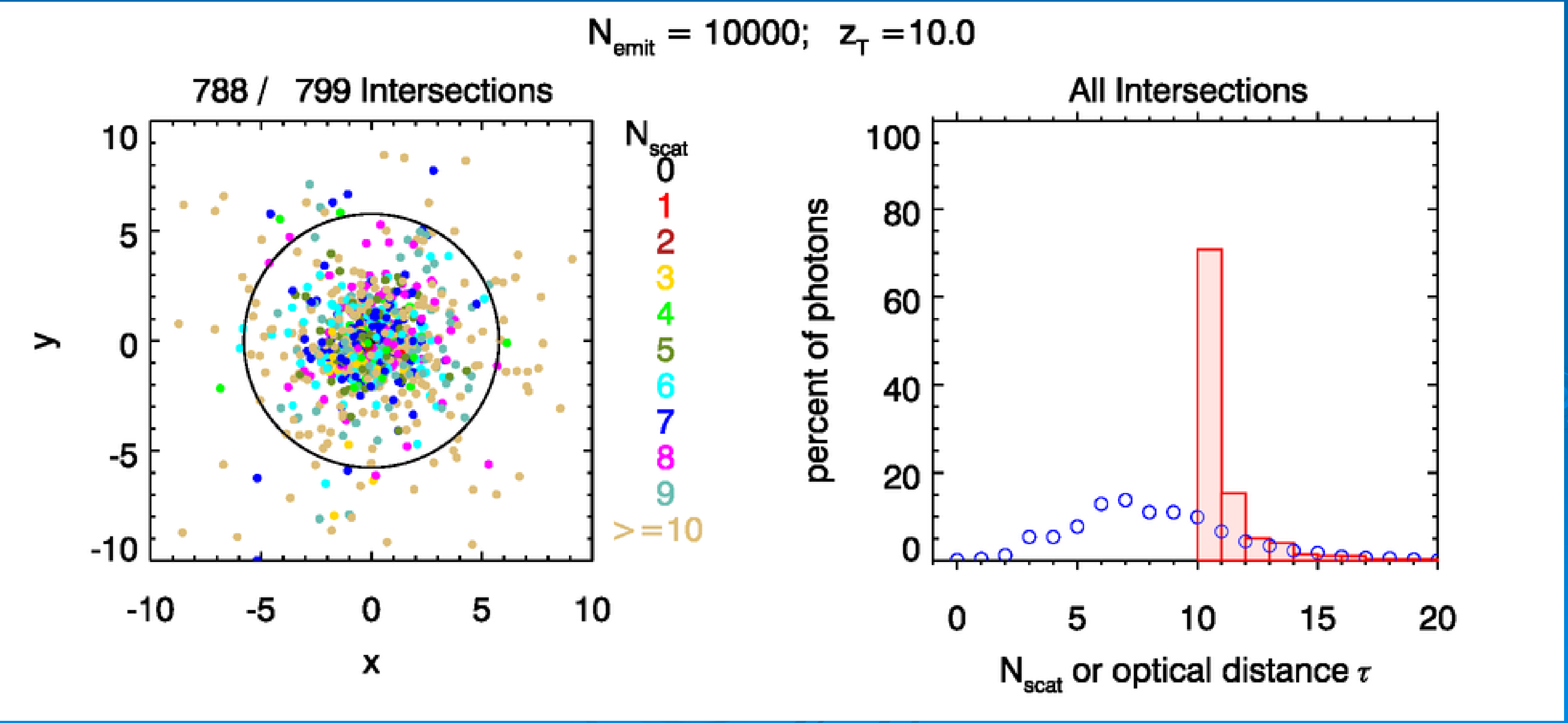




Visualizing Photon Paths

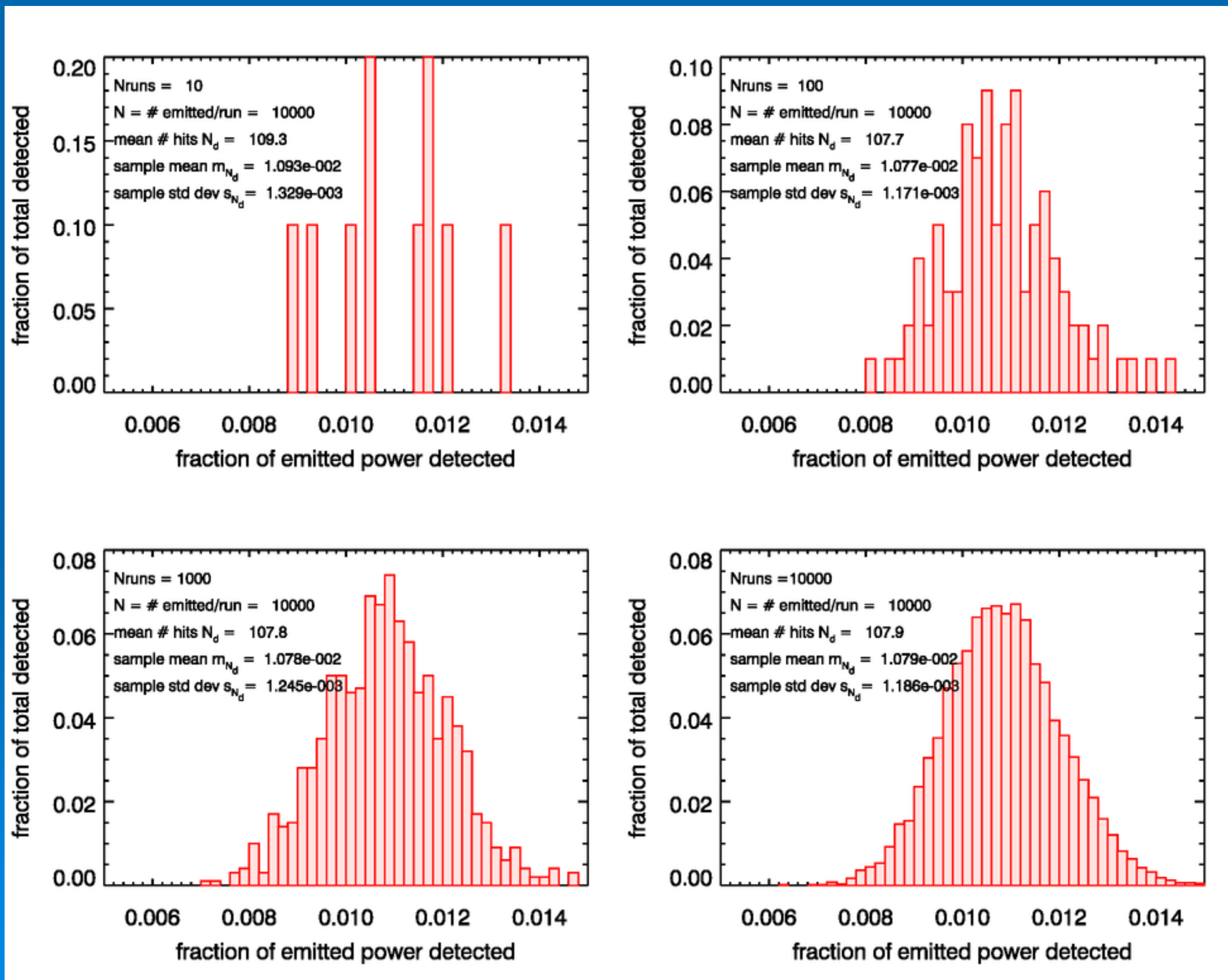


Monte Carlo simulation gives understanding at the photon level, which can't be obtained from radiance (e.g., from HydroLight)



Statistical Noise

The answer you get depends on random numbers and on the number of photons collected, so has statistical noise, aka Monte Carlo noise.



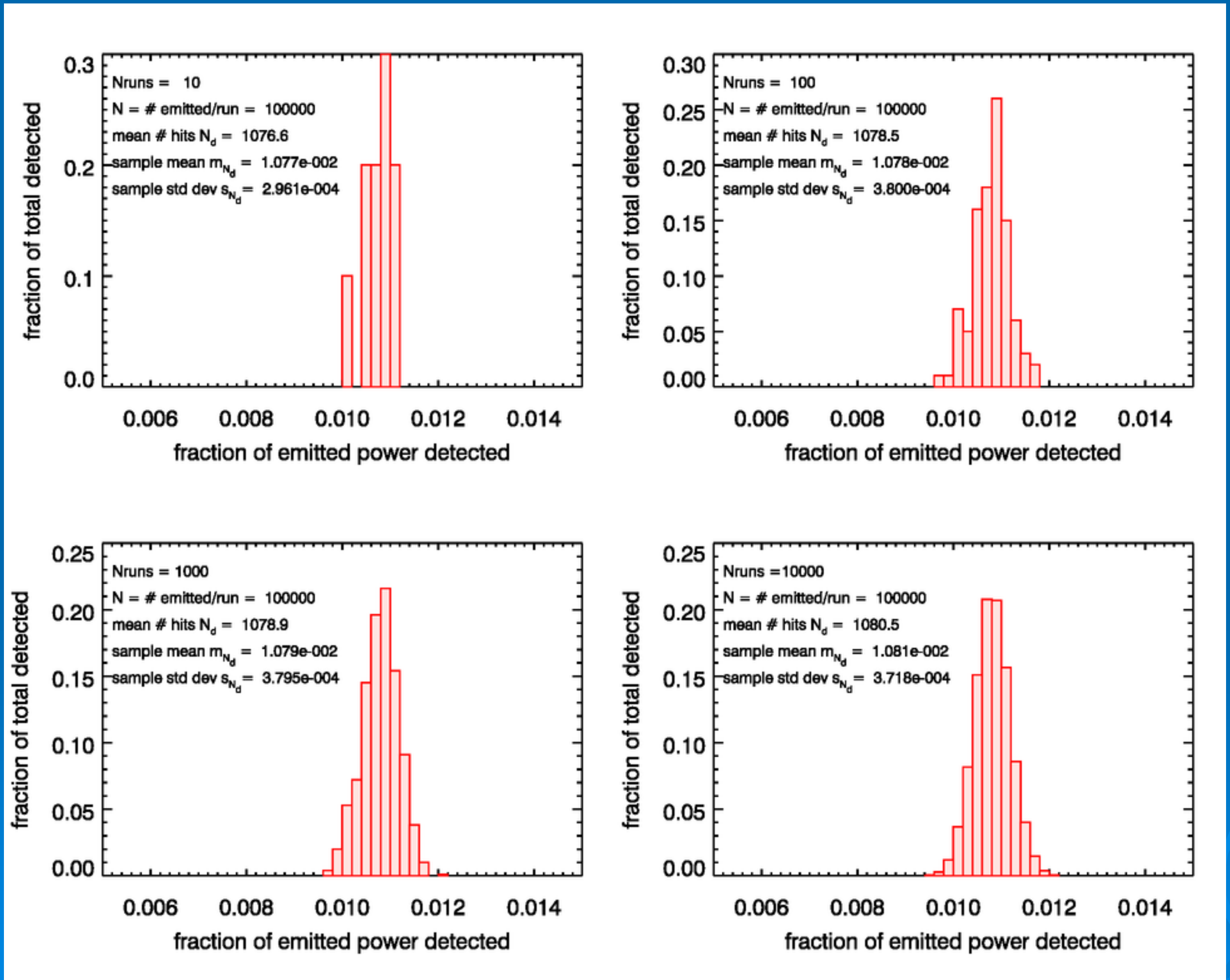
Repeated runs (different sequences of random numbers) with the same number of photons per run.

Note that as more runs are done, the distribution of computed values approaches a Gaussian:

The Central Limit Theorem in action

Statistical Noise

Standard error of the mean too large? Trace more photons...



The same numbers of runs , but with more photons per run.

The variance in the computed values is $\propto 1/N$, N = number of photons *detected*

std dev $\propto 1/\mathrm{sqrt}(N)$

To reduce the std dev of the estimate by a factor of 10, must detect 100 times more photons

Variance Reduction

We now know enough to do the Monte Carlo lab.

However, before writing a MC code to do extensive simulations, read about other ways to get more photons onto the target without more computer time. These are generally called “variance reduction” techniques, and there are many (“Importance sampling,” “Backward ray tracing”, “forced collisions”,...)

In general:

- First, figure out how to simulate what nature does

- Then figure out how to redo the calculations to maximize the number of photons detected (i.e., solve a different problem that has the same answer as the original problem—variance reduction)

- The goal (seldom attained) is to Never Waste a Photon

Following Marco Polo Across the Silk Road in Western China