2015 Summer Course

on Optical Oceanography and Ocean Color Remote Sensing

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The Radiative Transfer Equation

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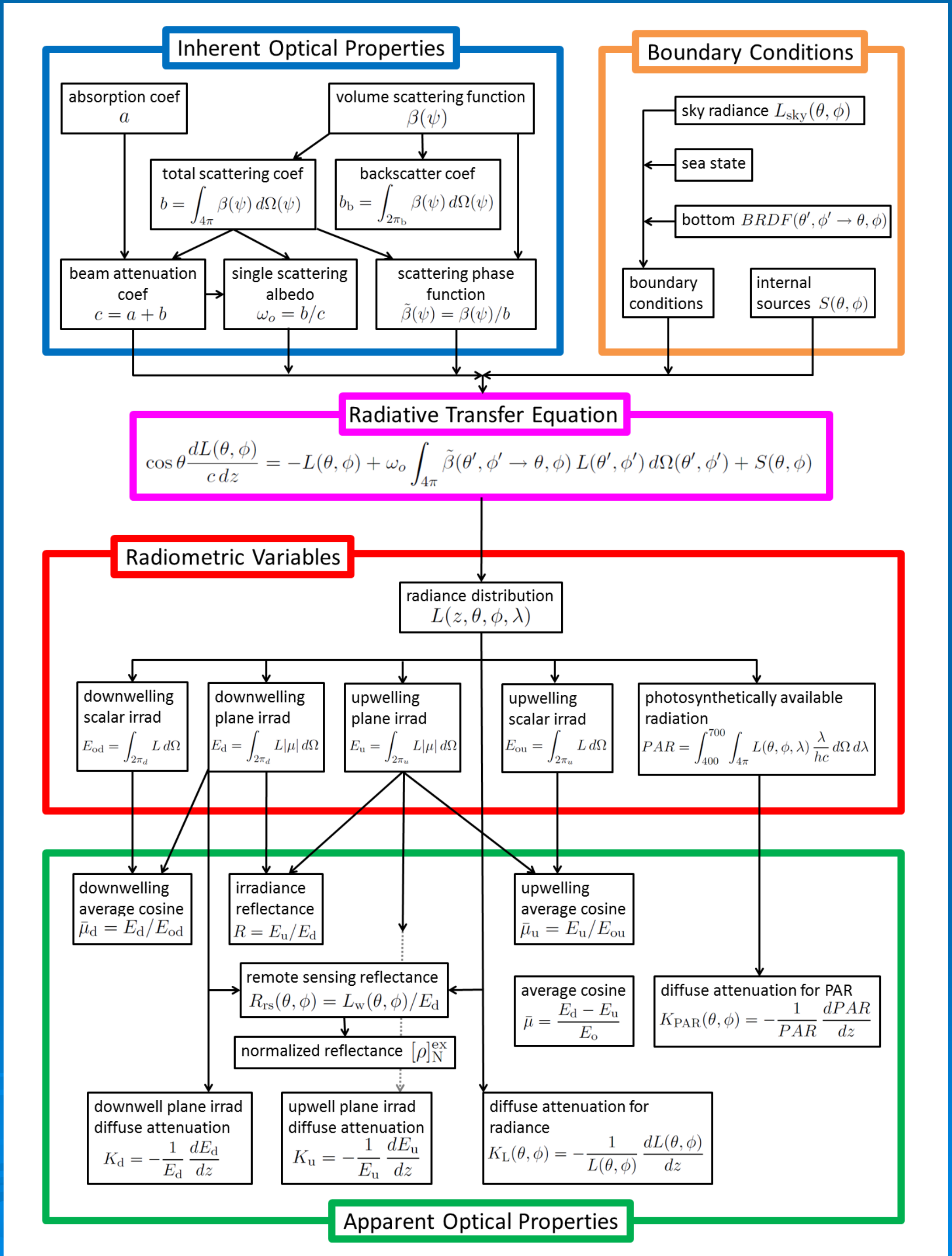
The Radiative Transfer Equation (RTE)

- expresses conservation of energy in terms of the radiance

- connects the IOPs, boundary conditions, and light sources to the radiance

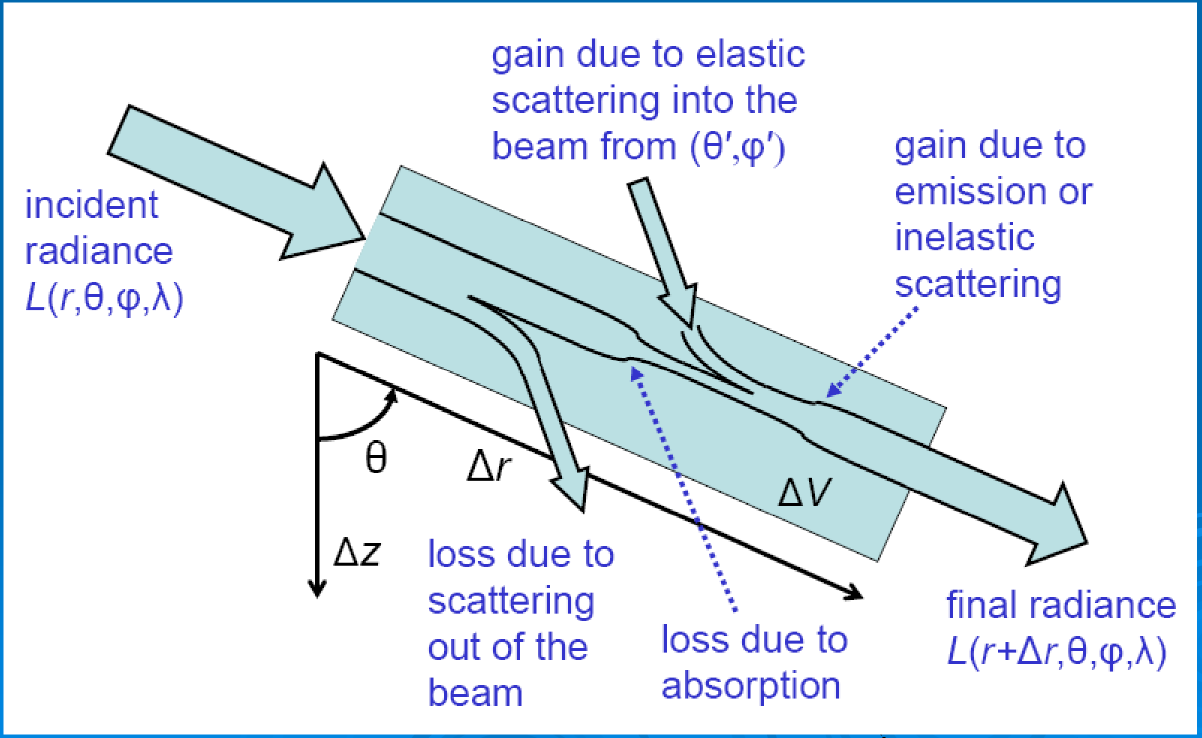
All other radiometric variables (irradiances) and AOPs can be derived from the radiance.

If you know the radiance, you know everything there is to know about the light field



Derivation of the RTE

To derive the time-independent RTE for horizontally homogeneous water, we consider the radiance at a given depth *z*, traveling in a given direction (θ, ϕ), at a given wavelength λ. We then add up the various ways the radiance $L(z,\theta,\phi,\lambda)$ can be created or lost in a distance Δr along direction (θ, ϕ), going from depth *z* to $z+\Delta z$



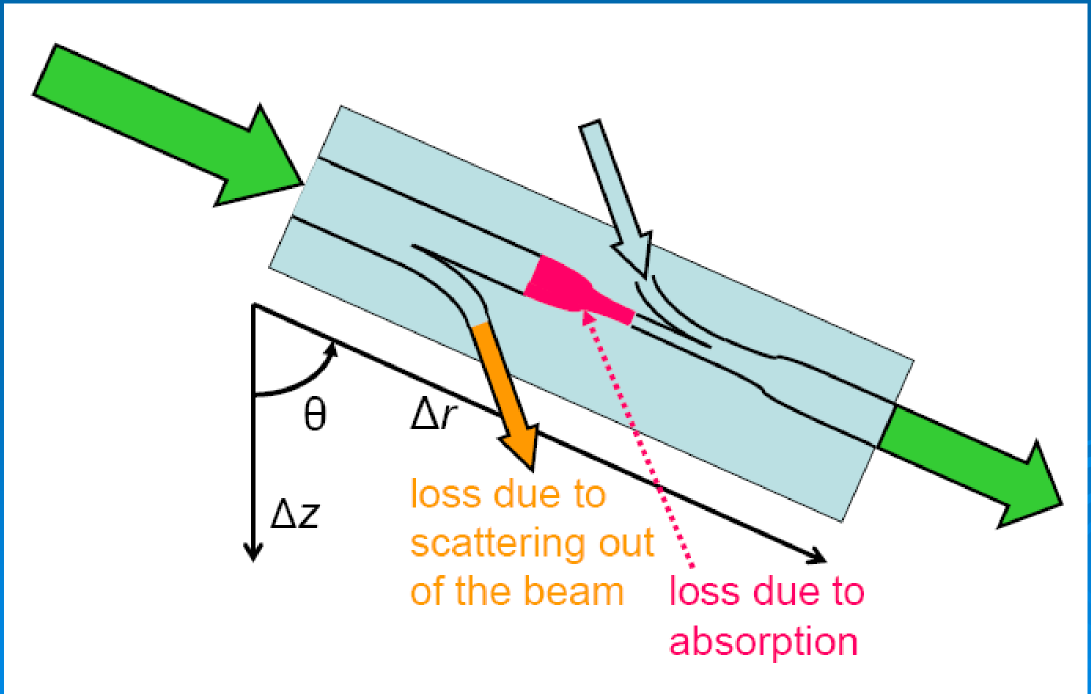
Losses of Radiance

The loss due to absorption is proportional to how much radiance there is:

$$\frac{dL(z,\theta,\phi,\lambda)}{dr} = -\,a(z,\lambda)\,L(z,\theta,\phi,\lambda)$$

Likewise for loss of radiance due to scattering out of the beam:

$$\frac{dL(z,\theta,\phi,\lambda)}{dr} = -\,b(z,\lambda)\,L(z,\theta,\phi,\lambda)$$

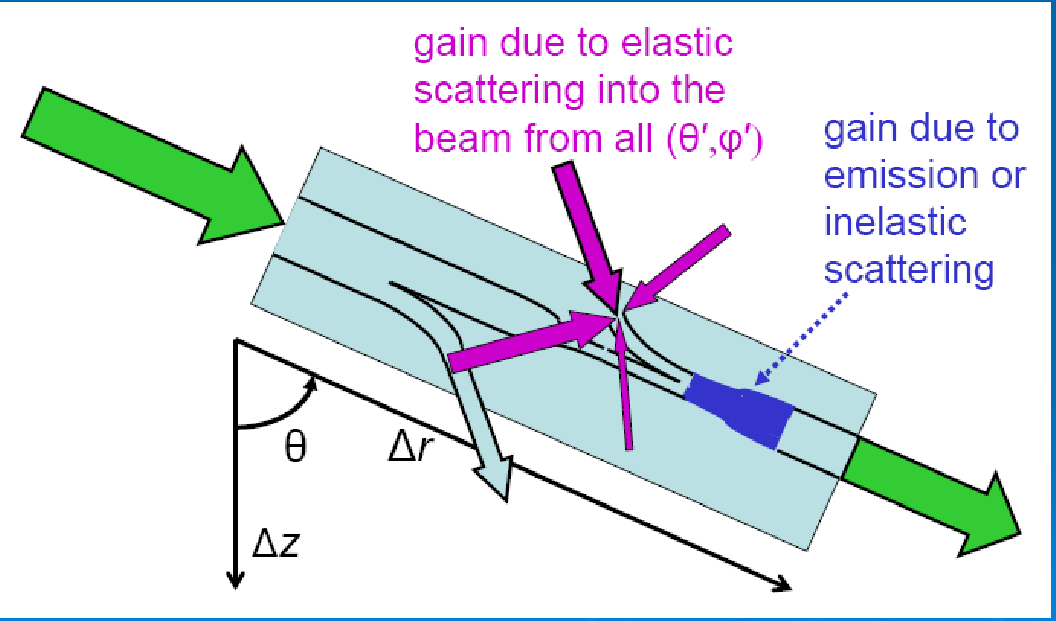


Sources of Radiance

Scattering into the beam from all other directions increases the radiance:

$$\frac{dL(z,\theta,\phi,\lambda)}{dr} = \int_{4\pi} L(z,\theta',\phi',\lambda)\,\beta(z;\,\theta',\phi' \rightarrow \theta,\phi\,;\lambda)\,d\Omega'$$

See www.oceanopticsbook.info/view/radiative_transfer_theory/deriving_the_radiative_transfer_equation for more detail



There can be internal sources of radiance $S(z,\theta,\phi,\lambda)$, such as bioluminescence

$$\frac{dL(z,\theta,\phi,\lambda)}{dr} = S(z,\theta,\phi,\lambda)$$

Add up the Losses and Sources

$$\frac{dL(z,\theta,\phi,\lambda)}{dr} = -\, a(z,\lambda)\, L(z,\theta,\phi,\lambda)$$

$$-\, b(z,\lambda)\, L(z,\theta,\phi,\lambda)$$

$$+ \int_{4\pi} L(z,\theta',\phi',\lambda)\, \beta(\mathrm{z};\, \theta',\phi' \rightarrow \theta,\phi;\, \lambda)\, d\Omega'$$

$$+\, S(z,\theta,\phi,\lambda)$$

Finally, note that $a + b = c$ and that $dz = dr \cos\theta$ to get

The 1D RTE, Geometric-depth Form

$$\cos\theta \, \frac{dL(z,\theta,\phi,\lambda)}{dz} = -\, c(z,\lambda)\, L(z,\theta,\phi,\lambda)$$

$$+ \int_{4\pi} L(z,\theta',\phi',\lambda)\, \beta(\mathrm{z};\, \theta',\phi' \rightarrow \theta,\phi;\, \lambda)\, d\Omega'$$

$$+ S(z,\theta,\phi,\lambda)$$

This is the RTE that HydroLight solves.

The VSF $\beta(\mathrm{z};\, \theta',\phi' \rightarrow \theta,\phi;\, \lambda)$ is usually written as $\beta(\mathrm{z}, \psi, \lambda)$ in terms of the scattering angle ψ, where

$$\cos\psi = \cos\theta' \cos\theta + \sin\theta' \sin\theta \cos(\phi' - \phi)$$

The 1D RTE, Optical-depth Form

Define the increment of dimensionless optical depth ζ as $d\zeta = c\,dz$
and write the VSF as *b* times the phase function, $\tilde{\beta}$,
and recall that $\omega_o = b/c$ to get

$$\cos\theta \, \frac{dL(\zeta,\theta,\phi,\lambda)}{d\zeta} = - L(\zeta,\theta,\phi,\lambda)$$

$$+ \omega_o \int_{4\pi} L(\zeta,\theta',\phi',\lambda)\, \tilde{\beta}(\zeta;\, \theta',\phi' \to \theta,\phi;\, \lambda)\, d\Omega'$$

$$+ S(\zeta,\theta,\phi,\lambda)/c(\zeta,\lambda)$$

Can specify the IOPs by *c* and the VSF β, or by ω_o and the phase function $\tilde{\beta}$ (and also *c*, if there are internal sources)

Note that a given geometric depth *z* corresponds to a different optical depth $\zeta(\lambda) = \int_0^z c(z',\lambda)\, dz'$ at each wavelength

The 1D RTE, Geometric-depth Form

$$\cos\theta \, \frac{\underline{dL(z,\theta,\phi,\lambda)}}{dz} = -\,\underline{c(z,\lambda)}\;\underline{L(z,\theta,\phi,\lambda)}$$

$$+ \int_{4\pi} \underline{L(z,\theta',\phi',\lambda)}\;\underline{\beta(z;\theta',\phi'\to\theta,\phi;\lambda)}\; d\Omega'$$

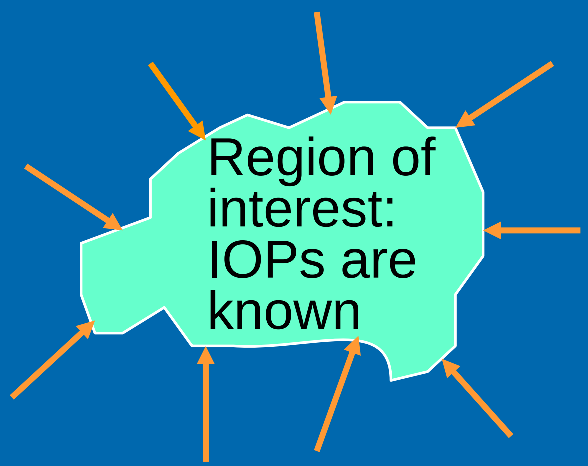
$$+ \underline{S(z,\theta,\phi,\lambda)}$$

NOTE: The RTE has the TOTAL *c* and TOTAL VSF. Only oceanographers (not light) care how much of the total absorption and scattering are due to water, phytoplankton, CDOM, minerals, etc.

The RTE is a linear (in the unknown radiance), first-order (only a first derivitive) integro-differential equation. Given the green (plus boundary conditions), solve for the red. This is a two-point (surface and bottom) boundary value problem.

Solving the RTE

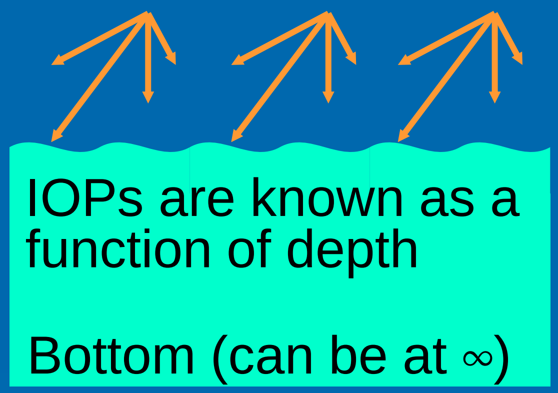
A unique solution of the RTE requires:



Radiance incident onto all boundaries from outside the region is known

Stretch out the region to make a horizontally homogeneous ocean

Radiance incident onto sea surface is known



A 3-D problem

A 1-D problem

Given the IOPs within the region and the incident radiances, we can solve for the radiance within and leaving the region

Solving the RTE: The Lambert-Beer Law

A trivial solution:

• homogeneous water (IOPs do not depend on z)
• no scattering (VSF β = 0, so c = a + b = a)
• no internal sources (S = 0)
• infinitely deep water (no radiance coming from the bottom boundary, so L → 0 as z → ∞)
• incident radiance *L*(z=0) is known just below the sea surface

$$\cos\theta \, \frac{dL(z,\theta,\phi,\lambda)}{dz} = -\,a(\lambda)L(z,\theta,\phi,\lambda)$$

$$\int_{L(z=0,\theta,\phi,\lambda)}^{L(z,\theta,\phi,\lambda)} \frac{dL}{L} = -\int_0^z \frac{a\,dz}{\cos\theta}$$

$$L(z,\theta,\phi,\lambda) = L(z=0,\theta,\phi,\lambda)\,e^{-az/\cos\theta}$$

Note that this *L* satisfies the RTE, the surface boundary condition, and the bottom boundary condition *L*(z=∞) = 0.

Solving the RTE: Gershun's Law

Start with the 1D, source-free, RTE.

$$\begin{aligned}\cos\theta\frac{dL(z,\theta,\phi,\lambda)}{dz} &= -\,c(z,\lambda)\,L(z,\theta,\phi,\lambda)\\ &+\int_0^{2\pi}\int_0^{\pi} L(z,\theta',\phi',\lambda)\,\beta(z,\theta',\phi'\to\theta,\phi,\lambda)\sin\theta' d\theta' d\phi'\end{aligned}$$

Integrate over all directions. The left-hand-side becomes

$$\begin{aligned}\int_0^{2\pi}\int_0^{\pi}\cos\theta\frac{dL(z,\theta,\phi)}{dz}d\Omega(\theta,\phi) &= \frac{d}{dz}\int_0^{2\pi}\int_0^{\pi} L(z,\theta,\phi)\cos\theta d\Omega(\theta,\phi)\\ &= \frac{d}{dz}[E_{\mathrm{d}}(z)-E_{\mathrm{u}}(z)]\end{aligned}$$

Solving the RTE: Gershun's Law

The – *cL* term becomes

$$\begin{aligned}\iint -c(z)L(z,\theta,\phi)d\Omega(\theta,\phi) &= -\,c(z)\iint L(z,\theta,\phi)d\Omega(\theta,\phi)\\ &= -\,c(z)E_{\mathrm{o}}(z)\end{aligned}$$

The elastic-scatter path function becomes

$$\begin{aligned}&\iint \left[\iint L(z,\theta',\phi')\,\beta(z,\theta',\phi'\rightarrow\theta,\phi)d\Omega(\theta',\phi')\right] d\Omega(\theta,\phi)\\ =\ &\iint L(z,\theta',\phi')\left[\iint \beta(z,\theta',\phi'\rightarrow\theta,\phi)d\Omega(\theta,\phi)\right] d\Omega(\theta',\phi')\\ =\ &b(z)\iint L(z,\theta',\phi')d\Omega(\theta',\phi')\\ =\ &b(z)E_{\mathrm{o}}(z)\end{aligned}$$

Solving the RTE: Gershun's Law

Collecting terms,

$$\frac{d}{dz}\left[E_{\mathrm{d}} - E_{\mathrm{u}}\right] \;=\; -\,cE_{\mathrm{o}} + bE_{\mathrm{o}}$$

or

$$a(z,\lambda) \;=\; -\,\frac{1}{E_{\mathrm{o}}(z,\lambda)}\,\frac{d}{dz}\left[E_{\mathrm{d}}(z,\lambda) - E_{\mathrm{u}}(z,\lambda)\right]$$

Gershun's law can be used to retrieve the absorption coefficient from measured in-water irradiances (at wavelengths where inelastic scattering effects are negligible).

This is an example of an explicit inverse model that recovers an IOP from measured light variables.

Water Heating and Gershun's Law

The rate of heating of water depends on how much irradiance there is and on how much is absorbed:

$$\frac{\partial T}{\partial t} = \frac{1}{\rho c_v} a\, E_o = \frac{1}{\rho c_v} \frac{\partial (E_d - E_u)}{\partial z} \qquad \left[\frac{\text{deg C}}{\text{sec}} \right]$$

ρ = 1025 kg m^{-3} is the water density
c_v = 3900 J (kg deg C)$^{-1}$ is the specific heat of sea water

This is how irradiance is used in a coupled physical-biological-optical ecosystem model to couple the biological variables (which, with water, determine the absorption coefficient and the irradiance) to the hydrodynamics (heating of the upper ocean water)

Solving the RTE

Exact analytical (i.e., pencil and paper) solutions of the RTE can be obtained only for trivial situations, such as no scattering. There is no function that gives

$$L(z,\theta,\phi,\lambda) = f(a, VSF, \text{sun angle, bottom reflectance, etc.})$$

even for very simple situations such as homogenous water with isotropic scattering.

Even the extremely simple geometry of an isotropic point light source in an infinite homogeneous ocean is unsolved. This is because of the complications of scattering (which don't exist for problems like the gravitational field around a point mass or the electric field around a point charge).

Solving the RTE: Approximate Analytical Methods

Approximate analytical solutions can be obtained for idealized situations such as single scattering in a homogeneous ocean. These solutions are seldom used today, but they were very important in the early (pencil and paper) days of remote sensing (used by Gordon in many CZCS-era papers).

Quick outline: SOS → SSA → QSSA
(successive order of scattering; single-scattering approximation; quasi-single-scattering approximation)

Assume:

- The water is homogeneous: the IOPs do not depend on depth;
- The water is infinitely deep;
- The sea surface is level (zero wind speed);
- The sun is a point source is a black sky, so that the incident radiance onto the sea surface is collimated;
- There are no internal sources or inelastic scattering.

Solving the RTE: The SOS Approximation

The RTE is then

$$\mu \frac{dL(\zeta,\mu,\phi)}{d\zeta} = -L(\zeta,\mu,\phi) + \omega_o \int_0^{2\pi} \int_{-1}^{1} L(\zeta,\mu',\phi')\, \tilde{\beta}(\mu',\phi' \to \mu,\phi)\, d\mu'\, d\phi'$$

Now write radiance = unscattered + scattered once + scattered twice + …

$$L(\zeta,\mu,\phi) = L^{(0)}(\zeta,\mu,\phi) + \omega_o\, L^{(1)}(\zeta,\mu,\phi) + \omega_o^2\, L^{(2)}(\zeta,\mu,\phi) + \cdots$$

This leads to a sequence of solvable equations:

$$\mu \frac{dL^{(0)}}{d\zeta} = -L^{(0)}$$

$$\mu \frac{dL^{(1)}}{d\zeta} = -L^{(1)} + \int_0^{2\pi} \int_{-1}^{1} L^{(0)}\, \tilde{\beta}(\mu',\phi' \to \mu,\phi)\, d\mu'\, d\phi'$$

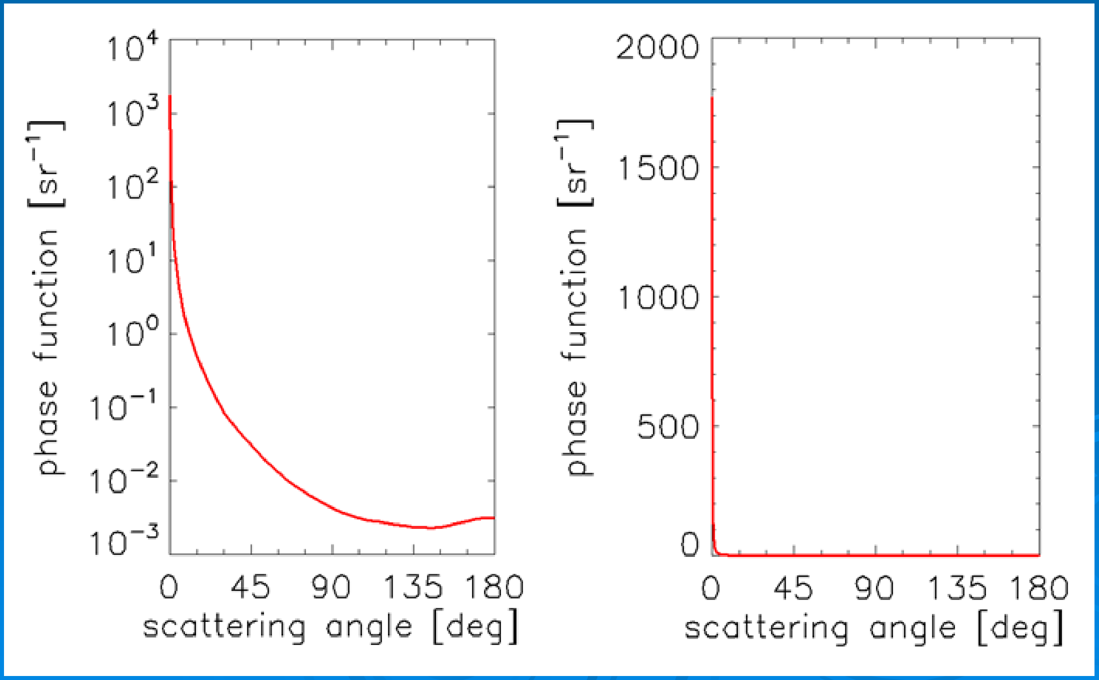
$$\mu \frac{dL^{(2)}}{d\zeta} = -L^{(2)} + \int_0^{2\pi} \int_{-1}^{1} L^{(1)}\, \tilde{\beta}(\mu',\phi' \to \mu,\phi)\, d\mu'\, d\phi'$$

Solving the RTE: The SSA Approximation

Now must integrate these equations with the appropriate boundary conditions at the sea surface and the bottom. See the Web Book page www.oceanopticsbook.info/view/radiative_transfer_theory/level_2/the_single scattering_approximation
for the calculus.

The SSA stops after one scattering term.

Now recall the shape of the VSF:



Solving the RTE: The QSSA Approximation

The QSSA treats forward scattered radiance in beam *c* as unscattered:

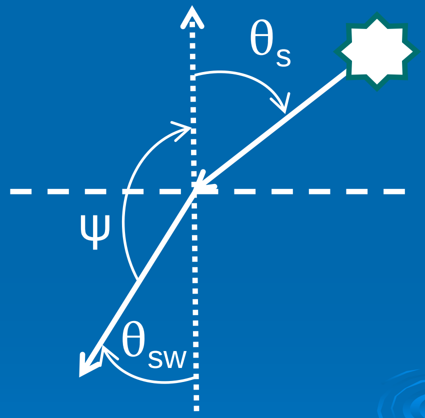
$$c = a + b = a + b_f + b_b \approx a + b_b$$

$$\zeta = \; c\,z \approx (a + b_b)z$$

$$\omega_o = \frac{b}{c} \approx \frac{b}{a + b_b}$$

Eventually get simple formulas for L_u, R_{rs}, & K_d: e.g.

$$R_{rs} = \frac{b_b}{a + b_b}\,\frac{\tilde{\beta}(\mu_{sw}, \phi_{sw} \to \mu = -1, \phi = 0)}{B}\,\frac{1}{\mu_{sw} + 1}$$

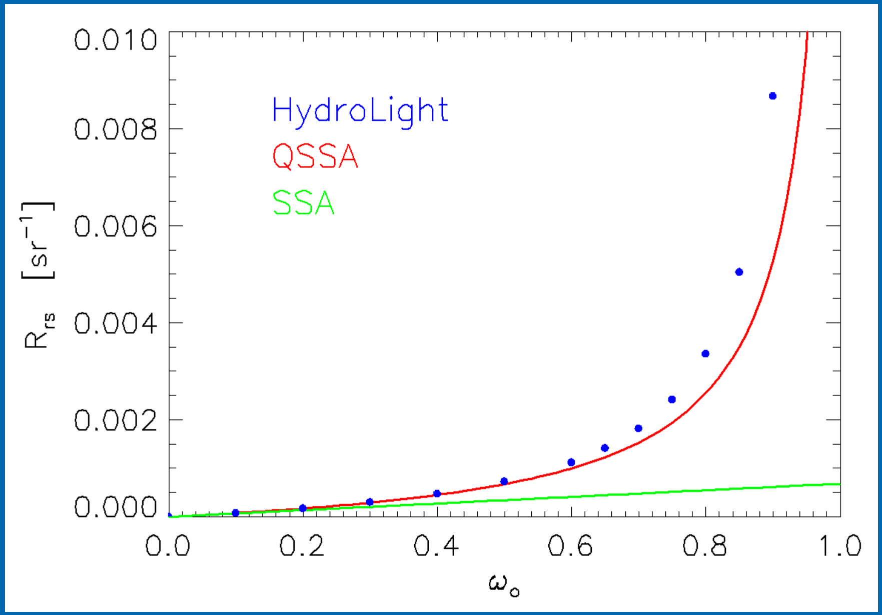


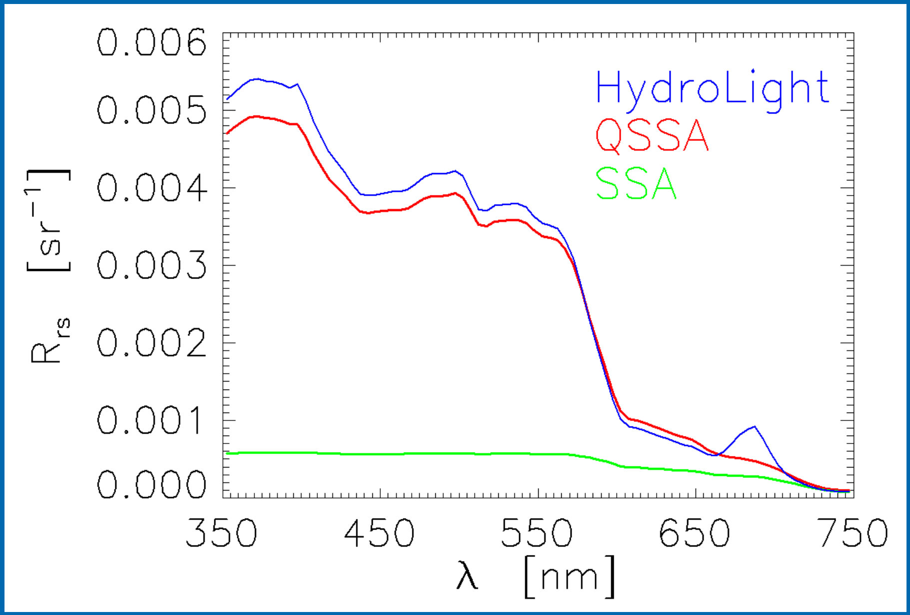
This is where $R_{rs} \sim b_b/(a + b_b)$ comes from.

See www.oceanopticsbook.info/view/radiative_transfer_theory/level_2/the_quasisinglescattering_approximation for the math.

Solving the RTE: The QSSA Approximation

The QSSA works surprisingly well for R_{rs} (and K_d):





Why does the QSSA work so well?

Why does the QSSA work better than the SSA, when it is even simpler?

Solving the RTE: The QSSA Approximation

Recall the path radiance term of the RTE:

$$\mu \frac{dL(\zeta,\mu,\phi)}{d\zeta} = -L(\zeta,\mu,\phi) + \omega_o \int_0^{2\pi}\int_{-1}^{1} L(\zeta,\mu',\phi')\,\tilde{\beta}(\mu',\phi' \to \mu,\phi)\,d\mu'\,d\phi'$$

Example: For $b = 4a$ and $B = b_b/b = 0.02$ we have

$$\omega_o\,\tilde{\beta} = \frac{b}{c}\,\tilde{\beta} = \frac{b}{a+b}\,\tilde{\beta} = 0.8\,\tilde{\beta}$$

In the QSSA this becomes

$$\omega_o\,\tilde{\beta} = \frac{b}{c}\,\tilde{\beta} = \frac{b}{a+b_b}\,\tilde{\beta} = \frac{b}{a+Bb}\,\tilde{\beta} = 3.7\,\tilde{\beta}$$

The QSSA accounts for the "missing" multiple scattering in the single-scattering formulation by artificially increasing the amount of single scattering. The QSSA parameterizes multiple scattering within the single-scattering mathematical framework by artificially making the albedo of single scattering greater than 1. See the Web Book QSSA page for the details.

Solving the RTE: Numerical Methods

The solution of the RTE for any realistic conditions of scattering or geometry must be done numerically. Three widely used exact numerical methods are seen in the literature (in RT theory, “exact” means that we don’t make approximations such as single scattering. Given accurate inputs and enough computer time, you can get the correct answer as closely as you wish.)

- Discrete ordinates: Widely used in atmospheric optics
  - highly mathematical
  - difficult to program
  - doesn’t handle highly peaked phase functions well
  - most codes need a level sea surface
  - models the medium as homogeneous layers
  - fast for irradiances and homogeneous systems
  - slow for radiances and inhomogeneous systems
  - therefore, not much used in oceanography

Solving the RTE: Numerical Methods

- Monte Carlo: Widely used
  - simple math, easy to program
  - can solve 3D, time-dependent problems
  - easy to implement polarization
  - run time increases exponentially with optical depth
  - have to trace many photons to get accurate radiance estimates (solutions have statistical noise)
  - very long run times for radiances and/or great depths
  - more useful for irradiance computations and/or shallow depths

- Invariant Imbedding: What Hydrolight uses
  - highly mathematical (see *Light and Water*, Chaps 7 and 8)
  - difficult to program
  - 1D (depth dependence) problems only
  - run time increases linearly with optical depth
  - computes radiances accurately (no statistical noise)
  - extremely fast and accurate even for radiances and large depth

Sea Kayaking in SE Greenland

photo by Curtis Mobley