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Computer based underwater imaging analysis

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ABSTRACT

We have recently developed a model of the optics of the underwater environment that includes the effects of scattering and absorption. The model is used to predict the image degradation that occurs in water. The model is applicable to standard light source illumination, range-gated system illumination and laser line scan illumination. The model takes as input standard pictures and produces as output degraded images with the effect of scattering, absorption and detector noise added in. The model allows us to test against realistic images the performance of various image recovery and enhancement techniques and to compare the various active imaging systems that could be used in underwater identification systems. Using a previously developed sea surface and sky irradiance model developed at DREV, we are currently extending the model to handle underwater imaging from airborne platforms by a multi-pulse laser bathymetry system and by a range gated laser-imaging system.

Keywords: simulation, imaging, propagation, scattering, underwater.

1. INTRODUCTION

Some time ago we built a range-gated imager with a matched pulse laser illuminator[1] (LUCIE) which demonstrated a performance increase in the imaging range by a factor of five over conventional imaging systems. Extensive sea trials were carried out on the West Coast of Canada. During these trials a near-forward angle nephelometer[2] (NEARSCAT) was used to measure the intrinsic optical properties of the surrounding water. These, to our knowledge unique, simultaneous measurements of imaging along with the optical properties allow us to reliably compare model images simulated by using in-situ data on the optical properties of water along with the actual underwater images. In this paper we present the basic framework of a computer model of the underwater light propagation which can produce simulated images. This framework is based on a new realistic approximation of the phase function[3-4] and includes analytic expressions for the mean square angle, the beam spread function (BSF), as well as the modulation transfer function (MTF)

2. SUMMARY DESCRIPTION OF THE UNDERWATER IMAGING MODEL

The imaging model is based on McGlamery's approach[5]which follows a formalism of conventional imaging systems. In that formalism, the effects of imperfections of the components of an imaging system on the quality of the image are accounted for by a point spread function (psf). This function is an impulse response of the optical system, i.e. an image of a point source of light. By convolving the perfect image of an object with the psf, one obtains the actual image, degraded by imperfections of the system.

In position-shift-invariant systems, with the psf being the same for all points in the image, the convolution can be efficiently calculated through the use of two-dimensional Fourier transform. McGlamery employed this route. However, the dominating psf term due to light scattering by seawater depends on the distance (in water) between a point in the object and a conjugate point in the image. Based on experimental data of Voss[6], the variability of the light-scattering psf term within the image may range from about 2:1 (normal viewing) to 20:1 (oblique viewing) for an imaging system with a field view of 1 rad.
The present model generalizes McGlammery's approach by removing the condition of shift-invariance in regard of the effect of light scattering. Accordingly, the image irradiance is calculated through a direct convolution that permits accounting for the dependency of the light-scattering-based psf on distance.

The model has been devised as a design environment for underwater imaging systems rather than a stand-alone underwater imaging simulation program. Accordingly, the environment was given a modular structure which permits the substitution of various recipes (as C++ modules) for the calculation of light scattering effects on the image smoothing and also allows an easy selection (via a component database) of various image conversion components, such as specific CCD's, image intensifiers, photomultipliers, and current-to-voltage amplifiers.

The imaging module for the environment has been designed to allow examination of image generation using three imaging system types: a parallel system, a range-gated system, and a serial system. The parallel system is one in which a camera views the entire object simultaneously (each pixel in parallel) under a steady illumination. The range-gated system is a parallel system that utilizes pulsed illumination with a pulse width significantly shorter than the time of travel of light between the imaged object and the system. Finally, the serial system builds an image by synchronously scanning the object with a narrow but steady illumination beam and a narrow-field receiver.

In this paper we will focus on the application of the model to simulate a range-gated imaging system, as represented by LUCIE. An extended description of the image generation model is available elsewhere[7]. In the imaging module the image of a water and object scene is always time dependent because different portions of the scene image are created by photons with different time histories. In the case of a range-gated system the time-dependency of the image is modeled by using several discrete image time layers. Each layer is the image received within time interval t_i , t_{i+1} , for i = 1 to n, where t_{i+1} - t_i is the time gate width. In modeling range-gated systems, any non-linear effects related to the increase in the instantaneous power of the pulsed illumination and other effects of the short time scale are neglected. In the case of a parallel system, images on all time layers are added to form the actual image.

The image is calculated by using the concept of reverse projection: each image pixel is projected onto the scene, cutting a diverging rod of water. All light reaching the pixel must originate within that rod. This simplifies the calculations of contributions to the image from the scene. The time-dependent image in a range-gated system is determined by separately calculating images for each individual slab of the scene water. These slabs are "cut" parallel to the object plane and have their thickness smaller than the system's time gate width. The scene water is considered homogeneous. The placement of an image of a particular rod's section (an intersection of the rod with the scene slab or with the object plane) in an image time layer depends on the time of travel of light from the light source, via the rod section, to the camera.

The image of a slab is calculated in a three-step process. This process applies to each slab and (with a minor change that will be discussed in a moment) to the object itself. For this reason, we will refer to the water slabs and the object as scene layers. First, the irradiance distribution at a scene layer is calculated using the radiance distribution of the light source, the system geometry (location and orientation of the light source in respect of the object), the beam spread function of the scene water, and the absorption coefficient of the scene water. Second, the intensity of light within the scene rod's solid angle is obtained. For a slab layer, this intensity is calculated by using the scattering function of the scene water. For the object layer, the intensity is calculated by assuming that the object is a Lambertian diffuser and accounting for the average reflectivity within the scene rod footprint on the object. The assumption about the Lambertian diffuser is an important one because it permits the usage of the simple concept of the point spread function to account for the smoothing of the object's image due to the light scattering by sea water. Third, the layer's perfect image is convoluted with the point spread function (PSF) due to light scattering by the scene water between the scene layer and the camera, and psf's due to de-focusing, and the lens. The difference between the PSF and psf is explained in the following paragraph.

Thus, the effect of the scene water on the image quality is determined by the following major optical parameters of sea water: the scattering function (or equivalently by the phase function and the scattering coefficient), the attenuation coefficient, and the Beam (or equivalently, see Gordon[8], Point) Spread Function of sea water. The spread functions depend on the phase function, scattering and absorption coefficients, and distance in water. Some confusion may occur in the use of the term point spread function. In an in-air, imaging system, a spread function, which is referred to as the point spread function (psf), is in fact the beam spread function (BSF) in the meaning defined in this work. This point spread function is denoted by lowercase psf.

The BSF of a scattering medium is defined as a spatial distribution of irradiance (as measured with an irradiance meter) at a surface of a sphere at the center of which is located a uni-directional source of light (a collimated beam). This distribution, relative to the power transmitted, is brought about by redirection of part of the beam power through light scattering. The PSF is an angular distribution of radiance (relative to the source intensity) resulting from the scattering of light emitted by a Lambertian point source. The complementarity of these two functions stems from the analogies between the elements of their definitions: the radiance meter and the point source in the definition of the PSF correspond to the beam and the irradiance meter in the definition of the BSF respectively. In the imaging model, the BSF is calculated by using a small-angle scattering approximation.

3. SMALL ANGLE PARAMETERS USING A NEW PHASE FUNCTION

As outlined in the previous section, the phase function of scene water is a key optical parameter of an underwater imaging model. Based on a simplification of the single particle phase function due to Chen[9] and by assuming an inverse power law (Junge) for the particle size distribution we have been able to derive a simple analytic form for both the phase function and its cumulative angular integral[3-4]. This form was successfully compared to experimental phase functions data sets with the broadest reported angular and wavelength coverage[10-11]. Given a particle size distribution of the following form,

$$F(r) = \frac{C}{r^{\mu}}$$

we obtain the resulting unity-normalized scattering phase function:

$$b(\theta) = \frac{1}{4\pi}\frac{1}{(1-\delta)^2\delta^{v}}\left(\left[v(1-\delta)-(1-\delta^{v})\right]+\frac{4}{u^2}\left[\delta(1-\delta^{v})-v(1-\delta)\right]\right) \quad (1)$$

where

$$v = \frac{3-\mu}{2}\quad,\quad \delta = \frac{u^2}{3(n-1)^2}\quad,\quad u = 2\sin(\theta/2)$$

The normalization conditions is:

$$B(\lambda,\mu,n) = C\frac{\pi}{\cos(\pi\mu/2)}\left(\frac{2\pi(n-1)}{\lambda}\right)^{\mu-3} \quad (2)$$

where $B(\lambda,\mu,n)$ is the total scattering coefficient. This present phase function approximation differs from that discussed in our previous work[3] because it does not include explicitly the polarization dependence. Such dependence prevents one from obtaining a simple analytic normalization condition. The effect of the polarization dependence is only noticeable at large angles θ, where the contribution from small particles dominates the phase function. The polarization dependence can be introduced as a correction term at a later stage. By defining the normalized cumulative probability distribution of scattering as:

$$w(\theta) = 2\pi\int_0^{\theta} b(\theta')\sin(\theta')\,d\theta' \quad (3)$$

we obtain:

$$w(\theta) = \frac{1}{(1-\delta)\delta^{v}}\left[(1-\delta^{v+1})-\frac{u^2}{4}(1-\delta^{v})\right] \quad (4)$$

In order to include polarization effects without destroying the simple normalization properties of the phase function formula, one should first subtract from equation 1 a constant equal to $b(\pi)$ and then add back a polarization term with the same angle integrated value of $4\pi\, b(\pi)$. This procedure obviously conserves the normalization of the total phase function.

We thus obtain the following final formulae:

$$b_f(\theta) = b(\theta) + b(\pi)\frac{[3\cos^2(\theta) - 1]}{4} \tag{5}$$

$$w_f(\theta) = w(\theta) + b(\pi)\frac{\pi\cos(\theta)\sin^2(\theta)}{2} \tag{6}$$

Equations 2, 5 and 6 are the final formulae that are used in the analysis[4] of the near-forward angle nephelometer data (NEARSCAT) and they serve as a basis for further work. Using these equations, it is possible to obtain an analytic form for the mean of the square of the angle of scattering. This parameter defines the angular dispersion behavior of underwater light sources[12]. From this point on, since:

$$\theta \to 0, \quad u = 2\sin(\theta/2) \to \theta$$

we will use u as the small angle scattering parameter. This does not change anything in the region where θ is small but it has the enormous convenience of allowing an analytic solution for the mean square of the scattering angle. After a considerable amount of algebra one obtains the following relations:

$$\langle 4\sin^2(\theta/2)\rangle = 2\pi\int_0^\pi b(\theta)\, 4\sin^2(\theta/2)\sin(\theta)\, d\theta$$

$$\begin{aligned}\langle\theta^2\rangle \approx \langle 4\sin^2(\theta/2)\rangle = 4\alpha^\nu &\left\{\frac{{}_2F_1[1, 2-\nu; 3-\nu; 1/\alpha]}{2-\nu} - \frac{{}_2F_1[1, 1-\nu; 2-\nu; 1/\alpha]}{1-\nu}\right\} \\ &+ 4\alpha + 4\alpha^2\left(\frac{\alpha-1}{\alpha}\right)\ln\left[\frac{\alpha-1}{\alpha}\right]\end{aligned} \tag{8}$$

with

$$\alpha = \frac{3(n-1)^2}{4}.$$

Using the fact that α is very small and that therefore the argument of the Gaussian hypergeometric functions is large, we can perform an asymptotic expansion and obtain the following simple expression:

$$\langle\theta^2\rangle \approx 4\alpha^\nu \sum_{m=1}^{\infty} \frac{\alpha^m}{(\nu-2+m)(\nu-1+m)} + 4\alpha + 4\left(\alpha^2 - \alpha\right)\left\{\ln\left[\frac{1-\alpha}{\alpha}\right] - \pi\cot[\pi\nu]\right\} \tag{9}$$

The sum in equation 9 converges extremely quickly since α is such a small parameter. Equations 8-10 correspond to the mean square of the scattering angle obtained by integrating the first term of the phase function as given in equation 5. This term includes the effect of large angle scattering due to small particles that gives rise to a uniform scattering background which can be eliminated in imaging detectors by a contrast stretch. The effect of this term can be subtracted from the results of equation (9) in order to obtain a better representation of the small angle component of scattering.

$$\langle \theta^2 \rangle_f = \langle \theta^2 \rangle - 8\pi b(\pi) \tag{11}$$

4. EXPERIMENTAL INSTRUMENTATION AND RESULTS

One of our fully instrumented experimental data set consists of images of underwater targets taken in Patricia Bay on the West Coast of Canada. This inlet is on the East Side of Vancouver Island. These tests, previously described elsewhere[1], were carried out on two different days. The visibility on these two days represents the extremes of the full range of conditions that we measured over several weeks of trials in two different years. The tests were carried out at a mean depth of 35 m. A frame supporting two targets was first lowered to the bottom. The targets were white on a black background. One target was a standard television resolution pattern framed by an hexagonal border. The other target was a set of vertical bars of various widths. That target can be used to measure the square wave modulation transfer function. These images were taken with the LUCIE active imaging system.

As we mentioned previously, during each dive a narrow forward angle transmissometer-nephelometer[2] (NEARSCAT) was also lowered to within 2 meters of the bottom. This multi-spectral near-forward-angle nephelometer[2] was designed to perform essentially simultaneous measurements at many wavelengths. NEARSCAT uses four angular channels: a central channel, extending from 0 to 4.7 mrad, to measure both a transmitted and scattered signal, and three concentric annular rings to measure the scattered signal out to an angle of 26 mrad. Because well over one third to one half of the integrated scatter in seawater occurs in the first 30 mrad, it is possible, by judicious extrapolation with a sound theoretical phase function, to accurately estimate the total scatter. The data from the four previously mentioned channels is used to calculate the extinction coefficient and to estimate the scattering coefficient using a fit to a two-parameter phase function (μ, n). The absorption coefficient is then given by the difference between the extinction and scattering coefficients. For practical reasons, the instrument's path length is limited to 50 cm. With the instrument, attenuation lengths up to 50 m can be reliably measured.

From elementary considerations, the amount of light collected by each of the probe's detectors can be calculated using the single scattering paraxial approximation. Further details of the computations can be found elsewhere[4]. The scattering term is obtained for all wavelengths simultaneously by using a modified version of equation 2.

$b(\lambda) = b(\lambda_r)$

$$B(\lambda, \mu, n) = B_r(\mu, n)\left(\frac{\lambda_r}{\lambda}\right)^{\mu-3} \tag{7}$$

λ_r is an arbitrary reference wavelength and $B_r(\mu, n)$ is the value of the total scattering coefficient at this reference wavelength. For the sake of convenience we will use 530 nm as the reference wavelength in the remainder of this work.
Table 1 shows the absorption and scattering coefficients at 530 nm and the power law exponent and mean index of refraction on the 2 days when the experimental images were obtained by the LUCIE camera.

Table 1. Experimental water conditions in Patricia Bay

Optical parameters	12 January 1992	13 January 1992
Absorption coefficient (m^{-1})	.21 (m^{-1})	.21 (m^{-1})
Scattering coefficient (m^{-1})	.27 (m^{-1})	.44 (m^{-1})
Power law exponent	3.56	3.63
Mean index of refraction	1.09	1.09
Mean square angle of scattering ($radians^2$)	.118	.141
Mean square large angle scattering ($radians^2$)	.039	.049
Net mean square small angle scattering ($radians^2$)	.079	.092

The large variability in scattering shows the imperative necessity of simultaneous measurements of the optical properties for any significant comparison of experimental results with computer imaging models.

5. FEYNMAN PATH INTEGRAL MODEL RESULTS

In this section, we will compute the beam spread function (BSF) and the medium modulation transfer function (MTF) using the parameters derived above from the high precision near-forward angle nephelometer system (NEARSCAT). Given the fact that the phase function is highly forward peaked and that the mean of the square of the scattering angle is a small quantity, several approaches can be taken to approximate the behavior of the scattered light field[12-14]. By modifying the Feynman path integral approach[14] the effect of significant absorption such as that encountered in coastal waters can be included in a simple model. In this approach we obtain the following expression for the probability distribution of photons scattered from a beam propagating in the z direction through a distance l from its source:

$$P(X,Y,\theta_x,\theta_y,l)=e^{-aS(X,Y,\theta_x,\theta_y,l)}p(X,Y,\theta_x,\theta_y,l) \tag{8}$$

P is the probability of finding a photon at a distance (X,Y,l) from the beam source and scattering angles (θ_x,θ_y) with respect to that beam axis. S is the mean photon path length in a medium with an absorption coefficient of a. p is the corresponding probability of finding a scattered photon in a medium with no absorption and can be expressed as follows:.

$$p(X,Y,\theta_x,\theta_y,l)=\frac{12}{\pi^2h^2}\exp\left\{\frac{-4}{hl}\left[3(X-\theta_x l/2)^2+3(Y-\theta_y l/2)^2+(\theta_x l/2)^2+(\theta_y l/2)^2\right]\right\} \tag{9}$$

where

$$h=b\langle\theta^2\rangle l^2 \tag{10}$$

The mean photon path length can be expressed as follow:

$$S(X,Y,\theta_x,\theta_y,l)=\int_0^l\sqrt{1+\left(\frac{dx}{dz}\right)^2+\left(\frac{dy}{dz}\right)^2}\,dz\approx l+\left(\frac{1}{2}\right)\int_0^l\left(\frac{dx}{dz}\right)^2dz+\left(\frac{1}{2}\right)\int_0^l\left(\frac{dy}{dz}\right)^2dz \tag{11}$$

The full expression for the photon path length leads to a complex set of elliptic integrals. For small values of the path derivatives, which is always the case for small scattering angles, the binomial expansion can be used and it leads to the expression on the far right of equation 11. From the path integral formulation the path derivatives are given by:

$$\left(\frac{dx}{dz}\right)=\frac{2}{l}(3X-\theta_x l)\left(\frac{z}{l}\right)+\frac{3}{l}(\theta_x l-2X)\left(\frac{z}{l}\right)^2$$
$$\left(\frac{dy}{dz}\right)=\frac{2}{l}(3Y-\theta_y l)\left(\frac{z}{l}\right)+\frac{3}{l}(\theta_y l-2Y)\left(\frac{z}{l}\right)^2 \tag{12}$$

Substituting equations 9-12 into equation 8 and integrating over the appropriate variables one obtains the following expression for the probability distribution of photon scattered from a beam as a function of distance from the beam source:

$$P(r^2,l)=\frac{12(1-e^{-Bl})e^{-al}}{(h/l)(ah/15+4)}\exp\left[-\frac{3(a^2h^2/80+13ah/15+4)}{(h/l)(ah/15+4)}\left(\frac{r^2}{l^2}\right)\right]d\left(\frac{r^2}{l^2}\right) \tag{13}$$

with

$$r^2 = X^2 + Y^2 \quad , \quad \theta^2 = \theta_x^2 + \theta_y^2$$

The probability of finding a scattered photon as a function of distance from the source is given by the integral of equation 13 over all values of the radius r.

$$P(l) = \frac{4(1-e^{-Bl})\, e^{-al}}{(a^2h^2/80 + 13ah/15 + 4)} \tag{14}$$

Equation 13 can be used to compute the shape and amplitude of the illuminating laser pulse at the target plane. The result is given by the convolution of the probability distribution of scattered photons with the angular intensity distribution of the illuminating laser beam. For a gaussian laser beam of unit beam power and with a half angle divergence of θ_b we have:

$$I = \frac{1}{\pi\theta_b^2} \exp\left[-\frac{\theta^2}{\theta_b^2}\right] \tag{15}$$

After performing the convolution of equation 15 with equation 13 we obtain the total scattered term. The total light intensity on target is then adding the scattered contribution with the remaining unscattered component.

$$I\left(\frac{r^2}{l^2}, l\right) = \left\{ \frac{(1-e^{-Bl})4\, e^{-al}}{[\theta_b^2 + \theta_s^2](a^2h^2/80 + 13ah/15 + 4)} \exp\left[-\frac{(r^2/l^2)}{(\theta_b^2 + \theta_s^2)}\right] + \frac{e^{-(a+B)l}}{\theta_b^2} \exp\left[-\frac{(r^2/l^2)}{\theta_b^2}\right] \right\} d\left(\frac{r^2}{l^2}\right) \tag{16}$$

with

$$\theta_s^2 = \frac{(h/l)(ah/15+4)}{3(a^2h^2/80 + 13ah/15 + 4)}$$

Equation 16 can be used to predict the illumination pattern at the target. In the case of a gaussian diverging beam it consists of the sum of a gaussian irradiance distribution due to the scattered photons and of the original beam of unscattered photons. If the total scattering coefficient is large, the second term becomes negligible at large distances from the source. Given the illumination pattern, one can multiply the image from the range-gated camera by its inverse to produce a uniform illumination picture.

Light scattering by water also severely modifies the image received by the detector. Using the point spread function it is possible to derive an analytic form of the modulation transfer function. By using formula 8 multiplied by the fraction of all particles scattered out and by integrating over all angles we can approximate the point spread function (PSF) as follows:

$$psf(\theta, l) = (1-e^{-Bl}) \int_{-\infty}^{\infty} P(l\theta_x^0, l\theta_y^0, \theta_x - \theta_x^0, \theta_y - \theta_y^0, l)\, d\theta_x^0\, d\theta_y^0 \tag{17}$$

This procedure gives:

$$psf(\theta, l) = \frac{3(1-e^{-Bl})e^{-al}}{\pi(h/l)(23ah/120+7)} \exp\left[-\frac{3(a^2h^2/80+13ah/15+4)}{4(h/l)(23ah/120+7)}\theta^2\right] \tag{18}$$

The modulation transfer function of the scattered component is obtained from the point spread function by the following integral.

$$mtf(\omega) = 2\pi \int_0^\infty J_0(\omega\theta) psf(\theta, l)\theta \, d\theta$$

The total modulation transfer function must include the effect of the photons that have not been scattered. In order properly account for these photons we must add their contribution.

$$MTF(\omega) = (1 - e^{-Bl}) e^{-al} \, 2\pi \int_0^\infty J_0(\omega\theta) psf(\theta, l) \, \theta \, d\theta + e^{-(a+B)l} \tag{19}$$

where B is the total scattering coefficient. Using formula 18 in equation 19 we obtain finally:

$$MTF(\omega) = \frac{4(1 - e^{-Bl}) e^{-al}}{(a^2h^2/80 + 13ah/15 + 4)} \exp\left[-\frac{(h/l)(23ah/120 + 7)}{3(a^2h^2/80 + 13ah/15 + 4)} \omega^2 \right] + e^{-(a+B)l} \tag{20}$$

Equation 20 is the complete normalized MTF of a medium which both scatters and absorbs light. As the spatial frequency in the object plane approaches zero, the MTF approaches the probability of finding a photon at a given distance from the source. This probability is the same as that given in equation 14. If there is no absorption that probability becomes unity. This is the correct normalization of the MTF when no light absorption is present. At high spatial frequencies, the MTF becomes a constant equal to the amplitude of the unscattered radiative component that is given by the second term on the right hand side of equation 20. In other words, at high spatial frequencies only the unscattered photons remain. Given the MTF one can apply a Wiener filter to the Fourier transform of the range-gated images and recover their lost high frequency components. Because of its simple analytic form and the absence of zeros or poles, equation 20 is one of the few that is robust enough to be used successfully in such inverse filtering schemes.

6. CONCLUSIONS AND FUTURE WORK

Using a new phase function, we have developed a model of the optics of the underwater environment and derived an analytic form for the laser beam illumination pattern and the modulation transfer function. Data obtained with the NEARSCAT nephelometer validate the analytic model approach taken in this work. Our next tasks will include evaluation of the underwater imaging model results against actual pictures taken with the LUCIE system, evaluation of various image enhancement algorithms, as well as evaluation of the potential analytic simplifications of the model and their regimes of applicability. We also plan to generalize the model to properly handle airborne imaging problem.

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Angular dispersion results

$$P(\theta^2, l) = \frac{4(1-e^{-Bl})e^{-al}}{(h/l)(ah/5+4)} \exp\left[-\frac{(a^2h^2/80+13ah/15+4)}{(h/l)(ah/5+4)}(\theta^2)\right] d(\theta^2)$$

$$\sigma_a^2 = \sigma^2 \frac{(ah/5+4)}{(a^2h^2/80+13ah/15+4)}$$